

C3.1 Algebraic Topology

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Please be aware there are likely typos in these notes: comments/corrections are welcome!

Course Book

- **Hatcher, Algebraic Topology** – Chp. 2 & 3

This is also freely available from the author's website.

Expectations

- You are expected to read chapters 2 & 3 of Hatcher
- You should read the technical remarks about orientation signs in these notes: we will likely not have time for those in lectures.
- This course will not discuss intersection numbers rigorously. The notes often mention these in order to develop your intuition. The books by Bott & Tu and Guillemin & Pollack discuss these ideas rigorously.

Other references

- Ulrike Tillmann's C3.1 notes – see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

Other books

Massey, A basic course in Algebraic Topology

(Not "Algebraic Topology: An Introduction" ← does not treat homology)

James W. Vick, Homology Theory

MORE BASIC but full of ideas:

Fulton, Algebraic Topology: a first course.

MORE ADVANCED:

May, A concise course in Algebraic Topology

Davis & Kirk, Lecture Notes in Algebraic Topology

Bredon, Topology and Geometry

Classics by Spanier, Dold, also see references in May's book

Bott & Tu, Differential forms in Algebraic Topology

Guillemin & Pollack, Differential Topology

CONTENTS

0. OVERVIEW OF THE COURSE

Motivation, category theory, functors H_* and H^* : some computations
why functors are useful: Invariance of dimension, Brouwer fixed pt thm

1. ALGEBRAIC PRELIMINARIES: CHAIN COMPLEXES

Graded abelian groups

chain complexes, chain maps, subcomplex, quotient complex

chain map induces map on homology

exact sequence, snake lemma: SES induces LES on H_* , naturality of LES

5-Lemma, SES splits \Leftrightarrow direct sum

2. Δ -COMPLEXES AND SIMPLICIAL HOMOLOGY

Δ^n , n -simplices, Δ -complex (structure), simplicial cx, triangulation

simplicial chain complex, $H_*^\Delta(S^n)$, $H_*^\Delta(T^2)$, remark about orientations

$H_*^\Delta(\sqcup \text{conn.comp.}) \cong \bigoplus H_*^\Delta(\text{conn.comp.})$, $H_0^\Delta(X) \cong \mathbb{Z}^{\#\text{conn.comp.}}$

3. SINGULAR HOMOLOGY

Motivation, singular chain cx

naturality / functoriality, H_* (point)

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

chain homotopy, prism operator

homotopic maps $f \simeq g$ (relative A), homotopy equivalent spaces $X \simeq Y$

contractible space, homotopic maps induce equal maps on homology

homotopy equivalence induces iso on H_* , $H_*(\mathbb{R}^n) = H_*(\mathbb{D}^n) = H_*(pt)$

pairs of spaces, relative homology $H_*(X, A)$, LES in H_* for pair

reduced homology $\tilde{H}_*(X)$, LES for \tilde{H}_* , $H_*(\mathbb{D}^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})$

naturality of LES for pairs

5. EXCISION THEOREM AND QUOTIENTS

retractions, deformation retractions, excision thm, quotients,

good pairs $\Rightarrow H^*(X, A) \cong \tilde{H}^*(X/A)$, generator of $H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$

another remark about orientations

6. MAYER-VIETORIS SEQUENCE

MV LES, $H_*(S^n)$

wedge sum $X \vee Y$, cone CX , suspension ΣX , connected sum $X \# Y$

7. DEGREE OF MAPS OF SPHERES

degree, application to tangent vector fields on sphere, hairy ball theorem
local degree, proof of fundamental thm of algebra

8. CELLULAR HOMOLOGY

CW complexes, cellular complex, $\text{rank } H_n^{CW} \leq \#n\text{-cells}$

$H_*^{CW}(D^1 \times D^1)$, $H_*^{CW}(\mathbb{R}P^n)$, $H_*^{CW}(S^n)$, $H_*^{CW}(\Sigma g)$

$\Delta\text{-cx} \Rightarrow \text{CW cx}$, $H_*^{CW}(X) \cong H_*^\Delta(X) \cong H_*(X)$, Axioms for homology

9. COHOMOLOGY

cochains, cohomology, $H^*(X)$, $H_{CW}^*(X)$, $H_\Delta^*(X)$, $H^*(\mathbb{R}P^3)$

functoriality, homotopy invariance, cochain homotopy, dual of a SES
excision, LES, Mayer-Vietoris for H^* , axioms for cohomology

10. CUP PRODUCT

Cup product, $H^*(X)$ unital graded-commutative ring, pull-back is ring hom,
examples: $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory

11. KÜNNETH FORMULA AND PRODUCT SPACES

Tensor products of R -mods, tensor product of chain cxes,
algebraic Künneth thm, product spaces $X \times Y$, Euler characteristic χ
CW-cx for product space, Künneth thm, $H^*(S^n \times S^m)$, $H^*(T^n)$

12. UNIVERSAL COEFFICIENTS THEOREM

Universal coeff. thm, Background on Ext groups and free resolutions
(Co)homology with coefficients in a ring/field/module, $H^*(\mathbb{R}P^2; \mathbb{Z}/2)$
univ. coeff. thm for PID R , Duality $H^*(X; \mathbb{F}) \cong H_*(X; \mathbb{F})$ over fields
Structure thm for f.g. mods M over PID R , $\text{Ext}_R^1(M; R)$, torsion shift H_* to H^{*+1}

13. MANIFOLDS: POINCARÉ-LEFSCHETZ DUALITY

Local orientation, orientation, fundamental class, degree, P. duality, L. duality,
Locally finite homology H_*^{lf} , cohomology with compact supports H_c^* , Cap product and P.D.,
Alexander duality, Knot complements, Jordan curve thm

0. OVERVIEW OF THE COURSE

Motivation

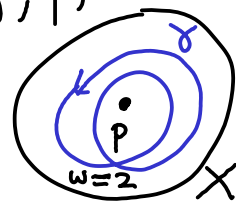
Space X $\xrightarrow{\text{associate}}$ Algebraic object $A(X)$
 like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y \implies$ Isomorphism $A(X) \cong A(Y)$

PAY-OFF Can tell spaces apart:
 compute $A(X), A(Y) \rightsquigarrow$ if $A(X) \neq A(Y)$ then $X \neq Y$

Examples

- 1) Set $X \longrightarrow A(X) = \#X \in \mathbb{N} \cup \{\infty\}$
 (bijection $X \rightarrow Y$) \implies same size
 - 2) Vector space $X \longrightarrow A(X) = \dim X \in \mathbb{N} \cup \{\infty\}$
 (linear iso $X \rightarrow Y$) \implies same dim
 - 3) Topological Space X
 - $\longrightarrow \# \pi_0(X) = \# \text{ path components} \in \mathbb{N} \cup \{\infty\}$
 - $\longrightarrow \# \text{ connected components} \in \mathbb{N} \cup \{\infty\}$
 - $\longrightarrow \chi(X) = \text{Euler characteristic} \in \mathbb{Z}$
 - for $X \subseteq \mathbb{R}^2$
 - Function $X \times \widetilde{\mathcal{L}X} \longrightarrow \mathbb{Z} \cup \{\infty\}$
 $(p, \gamma) \longmapsto w(\gamma; p)$
 loops = $C^0(S^1, X)$
 Winding number of γ around p .
- (Homeomorphism $X \rightarrow Y$) $\longrightarrow A(X) = A(Y)$



CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, " \cong " means homeomorphism

"id" = identity map

All diagrams commute unless we say otherwise, e.g. $A \xrightarrow{\alpha} B$ means $\beta \circ \alpha = \delta \circ \gamma$

Category theory is the best language to phrase all this

Algebraic topology builds functors

such as



We will not use much category theory, just basic terminology:

Def A category C consists of the data:

$Ob(C)$ = a collection of objects

$Hom(A, B)$ = a set of morphisms between any $A, B \in Ob C$ ("arrows")

- with composition rule $Hom(B, C) \times Hom(A, B) \xrightarrow{\circ} Hom(A, C)$
which is associative.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$\underbrace{\hspace{10em}}_{g \circ f}$$

- with identity morphs $id_A \in Hom(A, A)$ s.t. $f \circ id_A = id_B \circ f = f$
 $\forall (f: A \rightarrow B) \in Hom(A, B)$

Example Sets = { sets with all maps between sets }

Top = { topological spaces with continuous maps }

Gps = { groups with group homs }

Def A (covariant) functor $F: C_1 \rightarrow C_2$ is the data:

- an assignment $(A \in Ob C_1) \mapsto (F(A) \in Ob C_2)$

- an assignment $(A \xrightarrow{f} B) \mapsto (F(A) \xrightarrow{F(f)} F(B))$
 $Hom_{C_1}^{\uparrow}(A, B) \quad Hom_{C_2}^{\uparrow}(F(A), F(B))$

Compatible with identities and compositions.

$$F(id_A) = id_{F(A)} \quad F(g \circ f) = F(g) \circ F(f)$$

A contravariant functor is defined similarly except it reverses the

direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in Hom(F(B), F(A))$

(so $F(g \circ f) = F(f) \circ F(g)$ reverses order of compositions)

Examples

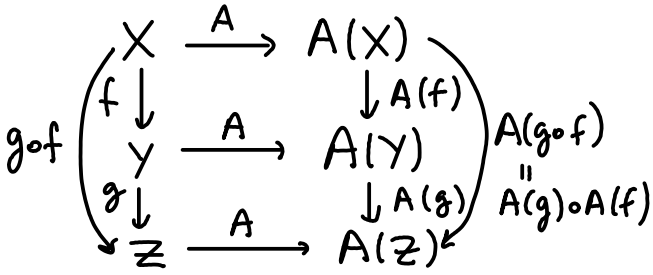
- 1) $F: \text{Top} \rightarrow \text{Sets}$, $A \mapsto A$, $f \mapsto f$ "forget the topology and continuity"
- 2) $F: \text{Sets} \rightarrow \text{Gps}$, $A \mapsto$ free abelian group generated by A

$$\mathbb{Z}\langle A \rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i : a_i \in A, n_i \in \mathbb{Z} \right\}$$

$$(A \xrightarrow{f} B) \mapsto \left(F(f) : \mathbb{Z}\langle A \rangle \rightarrow \mathbb{Z}\langle B \rangle \right)$$

$$\sum n_i \cdot a_i \mapsto \sum n_i \cdot f(a_i)$$

When we say a construction is natural we mean functorial:



$A: (\text{a category of spaces}) \rightarrow (\text{a cat. of algebraic objects})$
 The algebraic objects we assigned are assigned compatibly with maps of spaces, and the compatibility maps $A(f)$ are also compatible w.r.t. composition.
 So we made compatible choices in constructing A .

Not to be confused with natural transformations of functors (later) which is about relating two such constructions A_1, A_2 in a compatible way

Example of a functor in algebraic topology (see B3.5 Topology and Groups course)

$$\pi_1(X, p) = \text{Fundamental group} = \left\{ \gamma \in \mathcal{L}X \mid \gamma(1) = p \right\} / \text{Continuous deformations of loops based at } p$$

↑
topological space

↙
 $p \in X$

Group multiplication: concatenate loops $\gamma_1 * \gamma_2$ (each travelling twice as fast)

Examples

$\pi_1(\mathbb{R}^n) = 0$ ← (for basepoint $= 0 \in \mathbb{R}^n$: deform: $h: S^1 \times [0,1] \rightarrow \mathbb{R}^n$, $h(t,s) = (1-s)\gamma(t)$)

$\pi_1(S^1) \cong \mathbb{Z}$ ← total # times wind around circle

$\pi_1(S^n) \cong 0$ $n \geq 2$ (not obvious)

$\pi_1(\text{torus}) \cong \mathbb{Z}^2$ ← those loops generate π_1

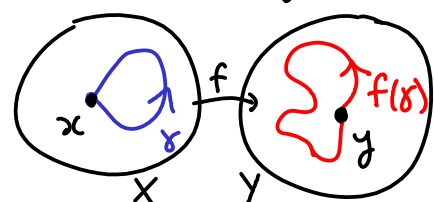
FUNCTOR

Based Top = { Topological spaces with choice of basepoint, and continuous basepoint-preserving maps } $\xrightarrow{\pi_1}$ Gps

$$(X, p) \mapsto \pi_1(X, p)$$

$$\left((X, x) \xrightarrow{f} (Y, y) \right) \mapsto \left(\pi_1(X, x) \xrightarrow{f \circ \gamma} \pi_1(Y, y) \right)$$

$\gamma \mapsto f \circ \gamma$



Lemma Functors map isomorphisms to isomorphisms (iso. means \exists inverse w.r.t. composition)

Pf $A \xrightarrow{f} B \xrightarrow{g} A \Rightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FA$, similarly for $B \xrightarrow{g} A \xrightarrow{f} B$. \square

\xrightarrow{id} $\xrightarrow{F(id)=id}$ \xrightarrow{id}

Def Natural transformation $\alpha: F \rightarrow G$ between functors $C_1 \xrightarrow{F} C_2$ is an association $(A \in \text{Ob } C_1) \mapsto (\alpha_A: F(A) \rightarrow G(A))$

such that $(A \xrightarrow{f} B) \Rightarrow \begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$ $\in \text{Hom}_{C_2}(F(A), G(A))$ (commutes)

It is called a natural isomorphism if each α_A is an isomorphism in C_2

Example of a natural transformation in algebraic topology

Let $H_1(X, p) = \text{abelianisation of } \pi_1(X, p)$ (want to identify $ab=ba$)
 \Rightarrow natural trans. $(\text{Based Top } \xrightarrow{\pi_1} \text{Gps}) \xrightarrow{\alpha} (\text{Based Top } \xrightarrow{H_1} \text{Gps})$ commutators

which associates $(X, p) \in \text{Based Top} \mapsto (\alpha_{(X,p)}: \pi_1(X, p) \xrightarrow{\text{quotient}} H_1(X, p))$

Cultural Rmk higher homotopy groups $\pi_n(X, p) = \left\{ \begin{array}{l} S^n \xrightarrow{\text{cts}} X \\ \text{basept} \mapsto p \end{array} \right\} / \text{cts deform}^n$

FACT abelian for $n \geq 2$, but hard: e.g. $\pi_k(S^n)$ not all known.

We will not study these in this course.

We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) \exists natural transformation $\pi_n \rightarrow H_n$

which will make sense at the end of course:

$f: S^n \xrightarrow{\text{cts}} X$ gives rise to a class $f_*[S^n] \in H_n(X)$.

Exercise to practice these notions from category theory:

Summarise your undergraduate linear algebra as follows:

1) \exists functor $F: \underbrace{\left\{ \begin{array}{l} \mathbb{R}^n \text{ for } n \in \mathbb{N} \\ \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) = \left\{ \begin{array}{l} m \times n \\ \text{matrices} \end{array} \right\} \end{array} \right\}}_{\text{Mat}} \rightarrow \underbrace{\left\{ \begin{array}{l} \text{finite dimensional real vector spaces} \\ \text{and linear maps} \end{array} \right\}}_{\text{Vect}}$

2) A choice of basis for each vector space V determines a functor $G: \text{Vect} \rightarrow \text{Mat}$

3) Construct natural isomorphisms $G \circ F \xrightarrow{\alpha} \text{Id}_{\text{Mat}}$, $F \circ G \xrightarrow{\beta} \text{Id}_{\text{Vect}}$

When functors satisfying such natural isos exist, the categories are called equivalent (not isomorphic). So Mat, Vect are equivalent categories.

Aim of the course: build a functor

HOMOLOGY $H_*: \text{Top} \longrightarrow \text{Graded abelian groups}$

$$\begin{array}{ccc} X & \longmapsto & H_*(X) \\ (X \rightarrow Y) & \longmapsto & (H_*(X) \rightarrow H_*(Y)) \end{array}$$

← grading $* \in \mathbb{Z}$
(grading preserving hom)

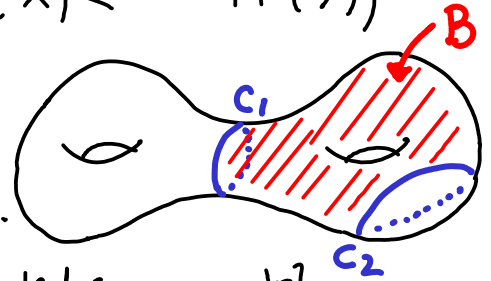
and a contravariant functor

COHOMOLOGY $H^*: \text{Top} \longrightarrow \text{Graded rings}$

$$\begin{array}{ccc} X & \longmapsto & H^*(X) \\ (X \rightarrow Y) & \longmapsto & (H^*(X) \longleftarrow H^*(Y)) \end{array}$$

Rough idea:

$H_* X$ is generated by "nice" subspaces $C \subseteq X$ which have no boundary: $\partial C = \emptyset$, modulo identify C_1, C_2 if $C_1 \cup C_2$ arises as a boundary ∂B . Call such C_1, C_2 homologous.



Facts

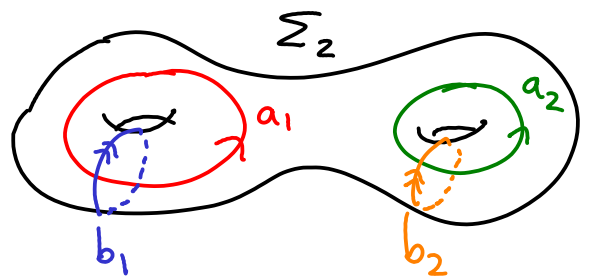
- $H_0(X) \cong \bigoplus_{\pi_0 X} \mathbb{Z}$ ← $\pi_0 X = \{\text{path-connected components}\}$
generated by a point in each path-comp. (two points connected by a path are homologous)
- $X = \sqcup X_i$ path-components $\Rightarrow H_*(X) \cong \bigoplus H_*(X_i)$
- $\chi(X) = \sum_{d \geq 0} (-1)^d \text{rank } H_d(X)$
↑ max # \mathbb{Z} -linearly independent elements

Euler characteristic

Example: compact surfaces

$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

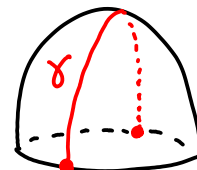
↑ orientable surface
genus g
 $\chi = 2 - 2g$



We will show that those 4 loops generate $H_1(\Sigma_2)$

$$H_*(N_h) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{h-1} & * = 1 \\ 0 & \text{else} \end{cases}$$

↑ non-orientable surface
 S^2 with h Möbius bands attached
 $\chi = 2 - h$



$$N_1 = \mathbb{RP}^2 = S^2 / \pm \text{Id}$$

Notice γ is a loop. It generates $H_1(N_1) = \mathbb{Z}_2$

(Notice " 2γ " is homologous to 0: )

Examples of homology calculations

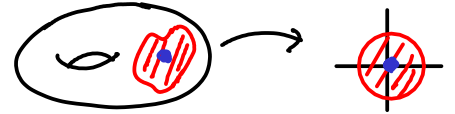
$$H_*(\mathbb{R}^n) \cong H_*(\mathbb{D}^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$$

n-dimensional ball
 $\mathbb{D}^n = \{x : \|x\| \leq 1\} \subseteq \mathbb{R}^n$

$$H_*(S^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$$

$\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ n-dim sphere

Hausdorff top. space
 s.t. each pt has an open
 neighbourhood homeo
 to an open ball in \mathbb{R}^n

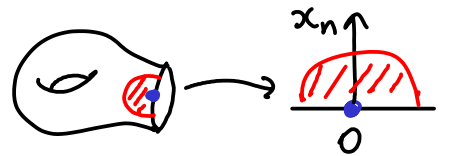


$$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \text{ for } \underline{n\text{-dimensional manifolds}} \\ \mathbb{Z} & \text{for } * = n \text{ for connected } \underline{\text{orientable compact manifold}} \\ 0 & \text{for } * = n \text{ for } \begin{cases} \underline{\text{non-orientable}} \\ \underline{\text{non-compact}} \\ \underline{\text{connected manifolds with boundary } \neq \emptyset} \end{cases} \end{cases}$$

boundary point has an open nbhd homeo to open
 nbhd of $0 \in \underline{\text{half-space}} : \{x \in \mathbb{R}^n : x_n \geq 0\}$

Rmk M compact connected
 n-mfd

$$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k \text{ some } k \geq 0 & \text{if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & \text{non-orientable} \end{cases}$$



$$H_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & \underline{\text{odd}} * = 1, 3, 5, \dots < n \\ \mathbb{Z} & * = n \text{ if } \underline{n \text{ odd}} \\ 0 & \text{else} \end{cases}$$

$S^n / \pm \text{id}$
real projective space

(equivalently: space of real lines
 through 0 in \mathbb{R}^{n+1})

$\mathbb{R}P^n$ orientable $\Leftrightarrow n$ odd
 (e.g. $\mathbb{R}P^1 \cong S^1$)

$$H_*(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{even } * = 0, 2, 4, \dots, 2n \\ 0 & \text{else} \end{cases}$$

e.g. $\mathbb{C}P^1 \cong S^2$
 stereographic projection

space of complex
 lines through $0 \in \mathbb{C}^{n+1}$

Complex projective space
 $\cong (\mathbb{C}^{n+1} \setminus 0) / \mathbb{C}^* \text{-rescaling}$
 $= \{ [z_0, z_1, \dots, z_n] : z_j \in \mathbb{C} \text{ not all } 0 \} / [z] = [\lambda z] \text{ for } \lambda \in \mathbb{C}^*$

Examples of cohomology calculations

$$H^0(X) = \prod_{\pi_0 X} \mathbb{Z} \quad \leftarrow \text{if } \pi_0 X \text{ finite, then } \cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$$

but if infinite then not: here allow only finite sums

$$H^*(X) \cong \prod H^*(X_i) \quad \leftarrow X_i \text{ path-components of } X$$

FACT

If $H_n(X)$ finitely generated abelian gp, so

$$H_n(X) \cong \mathbb{Z}^{r_n} \oplus T_n \quad \leftarrow T_n = \text{torsion elements} \\ = \text{elements of finite order}$$

Then $H^n(X) \cong \mathbb{Z}^{r_n} \oplus T_{n-1}$ as abelian groups

$\Rightarrow H^*(\Sigma_g), H^*(\mathbb{R}^n), H^*(D^n), H^*(S^n), H^*(\mathbb{C}P^n)$ same as for H_* , but:

$H^*(N_{\mathbb{R}}) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{h-1} & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$	$H^*(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \quad (h=1)$	$H^*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * \text{ even} = 2, 4, \dots \leq n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$
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and $H^n(\text{non-orientable compact } n\text{-mfd}) \cong \mathbb{Z}/2$.

\Rightarrow The interesting feature is the ring structure:

$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/x^{n+1}$ $\mathbb{Z}[x] = \text{polynomials in } x \text{ with } \mathbb{Z}\text{-coefficients}$

grading: $|x| = 2$

$H^*(S^n) \cong \mathbb{Z}[x]/x^2$ $|x| = n$

$H^*(T^n) \cong \wedge[x_1, \dots, x_n]$ $|x_i| = 1$

\parallel
 $S^1 \times \dots \times S^1$
n-torus

exterior algebra generated by symbols x_i with $i_1 < \dots < i_k$
product given by \wedge using relations $x_i \wedge x_j = -x_j \wedge x_i$ (and \wedge is bilinear)
and $x_i \wedge x_i = 0$.

$H^*(\mathbb{R}P^{2n}) \cong \mathbb{Z}[x]/(2x, x^{n+1})$ where $|x| = 2$

$H^*(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}[x, y]/(2x, x^{n+1}, y^2, xy)$ $|y| = 2n+1$

$H^*(\Sigma_g) \cong \wedge[a_1, \dots, a_g, b_1, \dots, b_g] / \langle a_i b_j \text{ for } i \neq j, a_i b_i - a_j b_j, a_i a_j, b_i b_j \rangle$

$|a_i| = |b_i| = 1$ \leftarrow exterior alg. instead of poly. alg since $a_i b_i = -b_i a_i$

Why more information?

\leftarrow connected sum: remove a ball in each, glue along ∂ ball

$S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ have same $H_* = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & * = 2 \\ \mathbb{Z} & * = 4 \end{cases}$

but the rings H^* are not iso, hence $S^2 \times S^2 \not\cong \mathbb{C}P^2 \# \mathbb{C}P^2$.

Example of why such functors are useful

Suppose $\exists F_* : \text{Top} \rightarrow \text{Gps}$ functors s.t.

- ① $F_*(S^n) \neq 0 \iff * = n$ and
- ② $F_*(D^n) = 0$ all $*$

Rmk We'll build such an F_* : reduced homology \tilde{H}_*
 s.t. $\tilde{H}_* = H_*$ for $* \neq 0$, and $\tilde{H}_0 \cong \mathbb{Z}^{\#(\text{path-components})-1}$

Theorem Invariance of dimension

(Brower ~1910)

$$\begin{aligned} S^n \cong S^m &\iff n=m \\ \mathbb{R}^n \cong \mathbb{R}^m &\iff n=m \end{aligned}$$

by ①

"homeomorphisms preserve dimension"
 Non-trivial result because there are space-filling curves. e.g. Peano (1890)
 \exists cts surjection $[0,1] \rightarrow [0,1]^2$
 interval \parallel square
 The theorem implies this is not injective. (cts. bij. compact \rightarrow Hausdorff) \implies homeo

Pf Lemma $\implies F_n(S^n \xrightarrow{\cong} S^m)$ is iso $F_n(S^n) \xrightarrow{\cong} F_n(S^m)$ of gps.

If $\mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m$, then can extend φ to the one-point compactifications: $S^n \cong \mathbb{R}^n \cup \{\infty\} \xrightarrow{\cong} \mathbb{R}^m \cup \{\infty\} \cong S^m$, $\infty \mapsto \infty$. \square

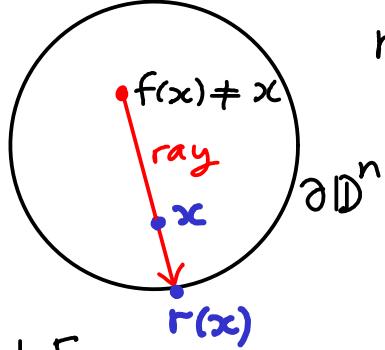
↑ ("Alexandroff extension")
 stereographic projection $(x_0, \dots, x_n) \mapsto \frac{(x_1, \dots, x_n)}{1-x_0}$

Rmk new open neighbourhoods at ∞ are $\{\infty\} \cup (\mathbb{R}^n \setminus C)$ where C is (closed & compact).
 The extended map is cts since $\varphi^{-1}(C)$ is (closed & compact) since φ^{-1} is homeo.

Theorem Brower fixed point thm by ① & ②

$f : D^n \rightarrow D^n$ continuous $\implies f$ has a fixed point ($f(p)=p$ some p)

Proof Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial D^n$



notice: $r : D^n \rightarrow \partial D^n = S^{n-1}$ continuous

$r|_{\partial D^n} = \text{id}_{S^{n-1}}$

$$S^{n-1} = \partial D^n \xrightarrow{\text{inclusion } i} D^n \xrightarrow{r} S^{n-1}$$

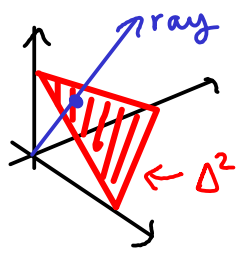
$r \circ i = \text{id}$

apply F_{n-1}
 $\implies F_{n-1}(r) \circ F_{n-1}(i) = \text{id} \implies F_{n-1}(i)$ injective $F_{n-1}(S^{n-1}) \rightarrow F_{n-1}(D^n) \xrightarrow{\neq 0} \xrightarrow{=0} \mathbb{Z} \square$

Example $A = n \times n$ matrix, $A_{ij} > 0$ real $\implies \exists$ real evale $\lambda > 0$ with real evector (v_1, \dots, v_n) with $v_i \geq 0$

(Brower)

Hint



$X = \{\text{rays in "positive octant"}\} \leftarrow x \in \mathbb{R}^n : x_i \geq 0 \forall i$
 notice $AX \subseteq X$
 notice $X \cong \Delta^n = \{x \in \text{octant} : \sum x_i = 1\} \cong D^n$
 ray \mapsto ray $\cap \Delta^n$

I. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

Graded abelian groups

Def A \mathbb{Z} -graded abelian group C is an abelian group together with a direct sum decomposition

$$C = \bigoplus_{n \in \mathbb{Z}} C_n \quad \leftarrow \text{abelian group}$$

Convention: always grade by \mathbb{Z} unless say otherwise. $\leftarrow (C_n = 0 \text{ for } n < 0)$

Example $C = \mathbb{Z}[x]$ = integer polynomials in x , $C_n = \mathbb{Z} \cdot x^n$ \leftarrow so grading by degree

A graded ab. gp. A is a graded subgp of C if

- subgp
- $A_n \subseteq C_n$.

A homomorphism $h: C \rightarrow D$ of gr. ab. gps is hom of gps s.t.

$$h(C_n) \subseteq D_n$$

A hom of degree k is hom with

$$h(C_n) \subseteq D_{n+k}$$

Shift by k : \mathbb{Z} -gr. ab. gp. $C[k]$ with

$$C[k]_n = C_{k+n}$$

Notice:

$C[k]_0 = C_k$
is now in degree zero, so shifted down by k

\Rightarrow Can view gr. hom of deg k as a gr. hom

$$h: C \rightarrow D[k]$$

Abelian groups which are finitely generated

recall f.g. means \exists surjection $\mathbb{Z}^m \rightarrow G$ for some m

FACT Finitely generated abelian groups are classified:

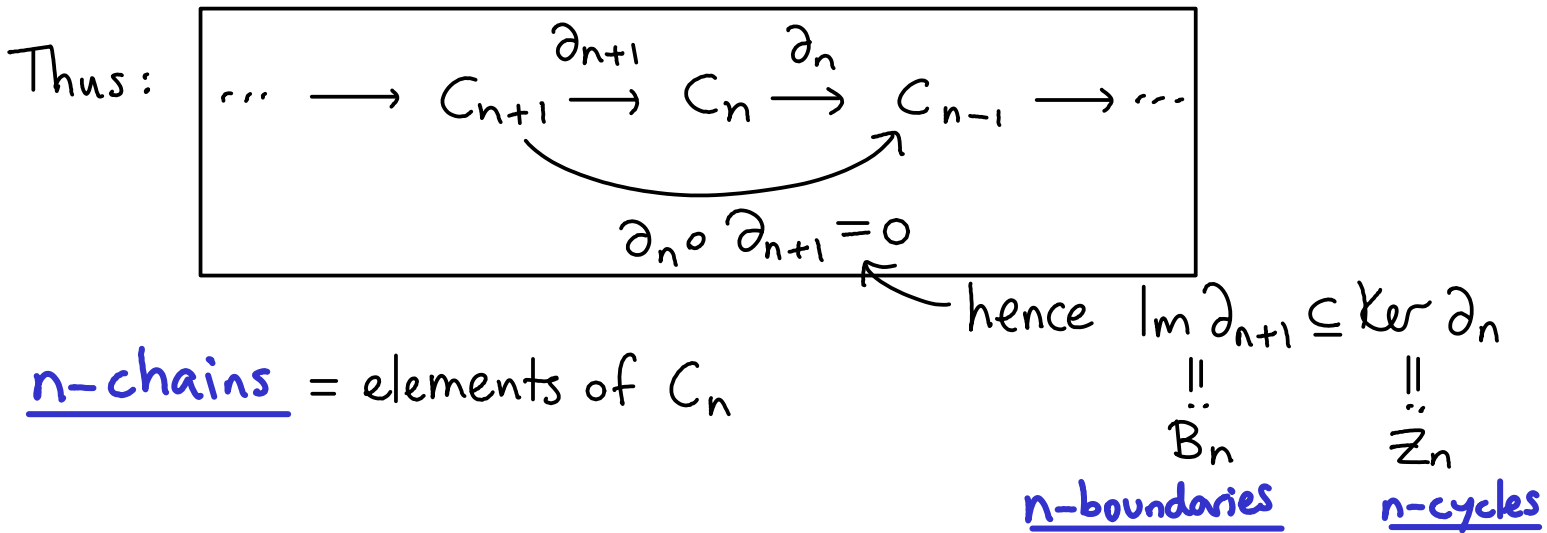
$$G \cong \underbrace{\mathbb{Z}^r}_{\text{free part}} \oplus \underbrace{\mathbb{Z}/p_1^{n_1} \oplus \dots \oplus \mathbb{Z}/p_k^{n_k}}_{\text{called rank } G} \oplus \underbrace{\dots}_{\text{torsion part}}$$

$n_i \neq 0 \in \mathbb{N}$
 p_i primes (possibly not distinct)

compare finite dimensional vector spaces / field \mathbb{F} : $V \cong \mathbb{F}^r \leftarrow r = \dim V$

Chain complexes

Def A chain complex (C_*, ∂_*) is a gr. ab. gp. C together with a hom ∂ of degree -1 such that $\partial \circ \partial = 0$.



n-chains = elements of C_n

n-boundaries n-cycles

Now consider "cycles modulo boundaries":

Def The homology of (C_*, ∂_*) is the gr. ab. gp.

$$H_n(C_*, \partial_*) = \frac{\text{Ker } \partial_n}{\text{Im } \partial_{n+1}}$$

often abbreviate by $H_*(C)$

Two cycles are called homologous if they differ by a boundary

Def A chain map $h : (C_*, \partial_*) \rightarrow (\tilde{C}_*, \tilde{\partial}_*)$

is a graded hom such that $h \circ \partial_* = \tilde{\partial}_* \circ h$

Example A chain subcomplex $C_* \subseteq \tilde{C}_*$ is a graded subgp with $\partial_* = \text{restriction of } \tilde{\partial}_* \text{ to } C_*$ \leftarrow (so require $\tilde{\partial}_*(C_*) \subseteq C_*$)

So the inclusion $\text{incl} : (C_*, \partial_*) \rightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a chain map.

Also get quotient complex \tilde{C}_*/C_* \leftarrow (so cosets $[\tilde{c}] = \{\tilde{c} + c : c \in C_*\}$)

with $\tilde{\partial}_* [\tilde{c}] = [\tilde{\partial}_* \tilde{c}]$ (well-defined: $\tilde{\partial}_* C_* = \partial_* C_* \subseteq C_*$)

Lemma A chain map induces a hom on homology

$$H_*(h) = h_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$$

$$[x] \longmapsto [h(x)]$$

Proof $\text{Ker } \partial_n \xrightarrow{\text{hom}} \text{Ker } \tilde{\partial}_n$
 $x \longmapsto h(x)$ since $\tilde{\partial}(h(x)) = h(\underbrace{\partial x}_{=0}) = 0$

Need $\text{Im } \partial_n \longrightarrow \text{Im } \tilde{\partial}_n$ to get well-defined hom

$$H_n(C) = \text{Ker } \partial_n / \text{Im } \partial_{n+1} \longrightarrow \text{Ker } \tilde{\partial}_n / \text{Im } \tilde{\partial}_{n+1} = H_n(\tilde{C})$$

Proof: $h(b) = h(\partial c) = \tilde{\partial} h(c) \in \text{Im } (\tilde{\partial})$. \square
 $\uparrow b = \partial c \in \text{Im } (\partial)$

The last step was a very simple example of a proof by "diagram chasing"

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \dots \\ & & \downarrow h_{n+1} & & \downarrow h_n & & \downarrow h_{n-1} & & \\ \dots & \longrightarrow & \tilde{C}_{n+1} & \xrightarrow{\tilde{\partial}_{n+1}} & \tilde{C}_n & \xrightarrow{\tilde{\partial}_n} & \tilde{C}_{n-1} & \longrightarrow & \dots \end{array}$$

(Commutativity of this diagram is the definition of h being a chain map)

$$\begin{array}{ccc} c & \xrightarrow{\partial} & \partial c = b \\ h \downarrow & & \downarrow h \\ hc & \xrightarrow{\tilde{\partial}} & \tilde{\partial}(hc) = h\partial c = h(b) \end{array} \quad \square$$

Curiosity (Non-examinable) If C_n free abelian gp $\forall n$, then every graded hom $H_*(C) \rightarrow H_*(\tilde{C})$ arises from a chain map. [see Dold, proposition II.4.6]

Def (C_*, ∂_*) is exact (or acyclic) if $H_*(C) = 0$
 so $\text{Im } \partial_{n+1} = \text{Ker } \partial_n$

In general exact sequence means $\boxed{\text{Im}(\text{previous map}) = \text{Ker}(\text{next map})}$

Example If $0 \rightarrow A \xrightarrow{i} B \rightarrow 0$ is exact then $A \cong B$ via i .

A short exact sequence (SES) is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

Easy exercise

$$(0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0) \text{ exact} \Leftrightarrow \begin{cases} i \text{ injective} \\ \pi \text{ surjective} \\ B/i(A) \cong C \text{ via } [b] \mapsto \pi(b) \end{cases}$$

Examples $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \rightarrow 0$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\text{inclusion}} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\text{project}} \mathbb{Z}/2 \rightarrow 0$$

Note A, C do not determine B.

Snake Lemma A SES of chain complexes and chain maps yields a long exact sequence (LES) on homology:

$$\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*[-1]} \dots$$

(So exact triangle: $H_*(A) \rightarrow H_*(B) \rightarrow H_*(C) \rightarrow H_*(A)[-1]$)

↑ degree -1 map
 $H_*(C) \rightarrow H_*(A)[-1]$
 called connecting map

Pf simplify notation by identifying A with $i(A) \subseteq B$: $a \in A \xrightarrow{i} B$
 $a \equiv i(a) \in B$
 $\partial a \equiv i \partial a = \partial i a$
 \Rightarrow now $A_* \subseteq B_*$ inclusion of subcomplex:

$$0 \rightarrow (A_*, \partial_*) \xrightarrow{\text{incl}} (B_*, \partial_*) \xrightarrow{\pi} (C_*, \tilde{\partial}_*) \rightarrow 0$$

$$\begin{array}{ccccccc} 0 & \rightarrow & A_n & \rightarrow & B_n & \rightarrow & C_n \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A_{n-1} & \rightarrow & B_{n-1} & \rightarrow & C_{n-1} \rightarrow 0 \\ & & & & \exists b & \xrightarrow{\text{surj.}} & \text{cycle } c = \pi(b) \\ & & & & \downarrow & & \downarrow \\ & & & & \partial b & \rightarrow & \tilde{\partial} c = 0 \\ & & & & \uparrow & \text{lifts to } A \text{ by exactness} & \end{array}$$

Define $\delta: H_*(C) \rightarrow H_*(A)[-1]$ (typically b is not in A , so ∂b need not be a bdry in A)
 $c \mapsto \partial b$ \leftarrow where $b \in \pi^{-1}(c)$

Well-defined? $\cdot \pi^{-1}(c) = \{b+a: a \in A\}$ and $\partial(b+a) = \partial b + \underbrace{\partial a}_{\text{boundary in } A}$

- cycle \rightarrow cycle: $\partial(\partial b) = 0 \checkmark$
- boundary \rightarrow boundary: $\exists \beta \xrightarrow{\text{surj.}} x \in C_{n+1}$
 $\downarrow \quad \downarrow$
 $\partial \beta \rightarrow \text{boundary } c = \partial x$
 \downarrow
 0

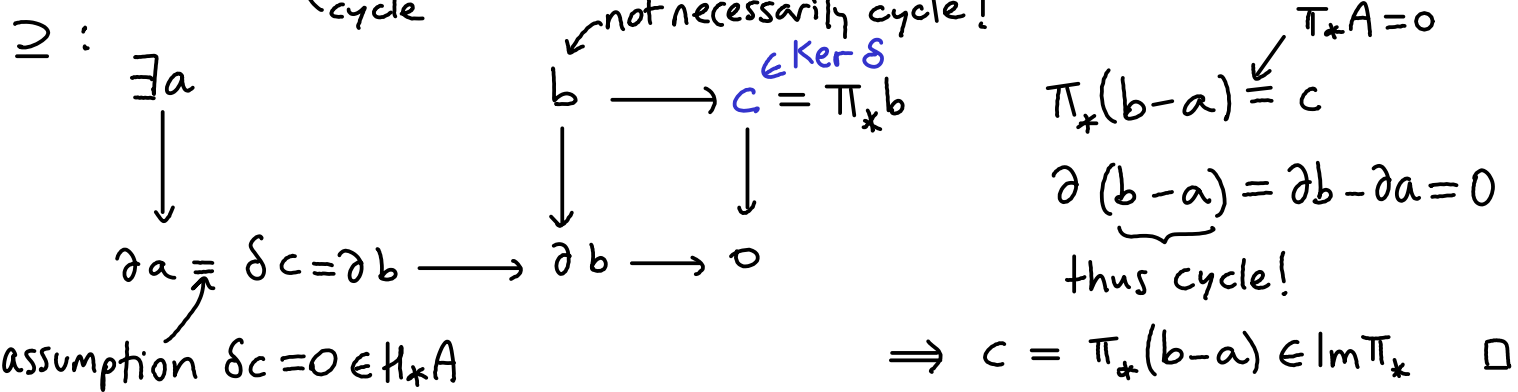
\Rightarrow can pick $b = \partial \beta$
 $\Rightarrow \partial b = \partial \partial \beta = 0 \checkmark$

because if $b' \in \pi^{-1}(c)$ then $\pi(b'-b) = c-c = 0$ so $b'-b \in A$ by exactness

Exactness at $H_n(C)$ (exercise: check exactness at H_*A, H_*B):

Need $\text{Im } \pi_* = \text{Ker } \delta$:

\subseteq : $\delta(\pi_* b) = \partial b = 0 \checkmark$
 \uparrow cycle



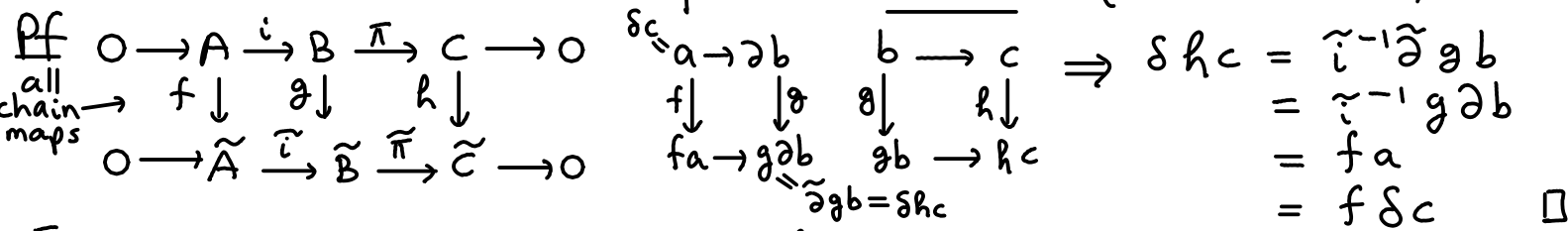
Rmk $0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0$ SES \Rightarrow the connecting map of LES is

$$\delta: H_*(C) \rightarrow H_*(A)[-1]$$

$$c \mapsto i^{-1}(\partial b)$$

$\forall b \in B$ with $\pi(b) = c$.

Lemma The construction of δ is natural (i.e. functorial)



Exercise Deduce the LES is natural, so

$$\begin{array}{ccccccc} \dots & \rightarrow & H_* A & \xrightarrow{i_*} & H_* B & \xrightarrow{\pi_*} & H_* C \xrightarrow{\delta} H_{*-1}(A) \rightarrow \dots \\ & & f_* \downarrow & & g_* \downarrow & & h_* \downarrow & & f_* \downarrow \\ \dots & \rightarrow & H_* \tilde{A} & \rightarrow & H_* \tilde{B} & \xrightarrow{\tilde{\pi}_*} & H_* \tilde{C} \xrightarrow{\delta} H_{*-1}(\tilde{C}) \rightarrow \dots \end{array}$$

5-Lemma

$$\begin{array}{ccccccccc}
 A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & E \\
 \cong \downarrow \alpha & & \cong \downarrow \beta & & \downarrow \gamma & & \cong \downarrow \delta & & \cong \downarrow \varepsilon \\
 A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' & \rightarrow & E'
 \end{array}$$

exact rows $\implies \gamma$ also iso.

Pf exercise (diagram chase) \square

Splitting Lemma

Cor $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ SES of abelian gps

If $B \xrightarrow{\beta} C$ s.t. $\beta \circ \gamma = \text{id}_C$ then the SES splits: $B \cong A \oplus C$

Pf

$$\begin{array}{ccccccc}
 0 & \rightarrow & A & \rightarrow & A \oplus C & \rightarrow & C \rightarrow 0 \\
 \parallel & & \parallel & & \downarrow \alpha + \gamma & & \parallel & & \parallel \\
 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0
 \end{array}$$

converse: if $B \cong A \oplus C$
then define $\gamma: C \subseteq A \oplus C \cong B$

Exercise If $A \xrightarrow{\alpha} B$ s.t. $\mu \circ \alpha = \text{id}_A$ then it splits: $B \xrightarrow[\mu \oplus \beta]{\alpha} A \oplus C$
(and equivalent to existence of γ above)

Exercise If C is a free abelian group ($C \cong \bigoplus_{i \in I} \mathbb{Z}$) then the SES splits.

Rmk A free $\not\Rightarrow$ splits, e.g. $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$

Cultural Rmk Splitting Lemma generalises the rank-nullity theorem from linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\implies \text{Im } \beta \oplus \text{Ker } \beta \cong V$

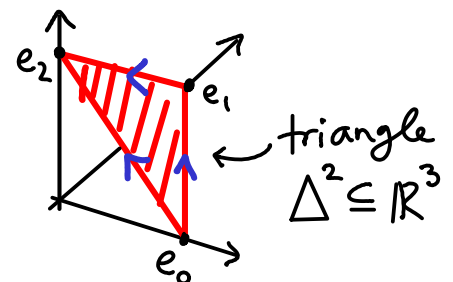
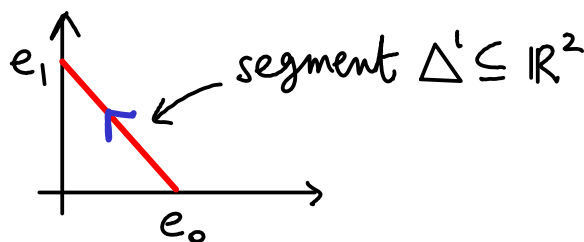
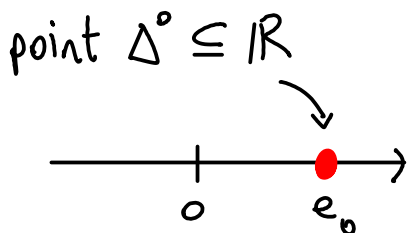
Pf $0 \rightarrow \text{Ker } \beta \xrightarrow{\text{incl}} V \xrightarrow{\beta} \text{Im } \beta \rightarrow 0$ is SES, and splits since $\text{Im } \beta$ free.

2. Δ -COMPLEXES AND SIMPLICIAL HOMOLOGY

Standard n-simplex $\Delta^n = \left\{ \begin{array}{l} (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$

$\sum t_i e_i$ (standard basis of \mathbb{R}^{n+1} e_0, \dots, e_n ($e_0 = (1, 0, \dots, 0), \dots$))

Examples



Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ s.t. \leftarrow any $k \geq 0$

$v_1 - v_0, \dots, v_n - v_0$ \mathbb{R} -linearly independent

$[v_0, \dots, v_n] = \underline{n\text{-Simplex}}$ spanned by v_0, \dots, v_n

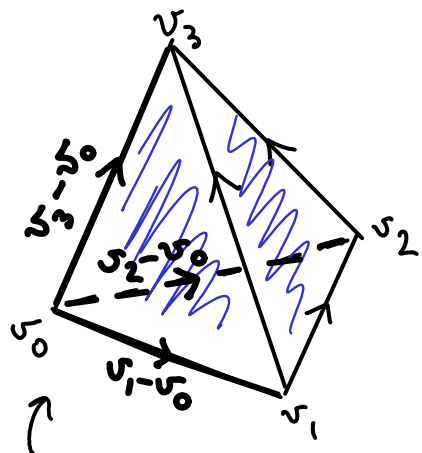
= convex hull of v_0, \dots, v_n

= $\left\{ \sum t_i v_i : t_i \geq 0 \text{ and } \sum t_i = 1 \right\}$

= Image of \mathbb{R} -linear homeo $\sigma: \Delta^n \rightarrow \mathbb{R}^{n+k}$
 $\sigma(e_i) = v_i$

canonical homeomorphism

\leftarrow (homeo onto the image, not onto \mathbb{R}^{n+k})



Will often blur the distinction between map σ and its image,

$$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$$

but the ordering of the v_j will be important (so the map σ is) more precise

We encode this extra data by orienting the edges $v_i \rightarrow v_j$ if $i < j$

Def d-dimensional faces $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

Example 0-dim faces are the vertices v_0, \dots, v_n

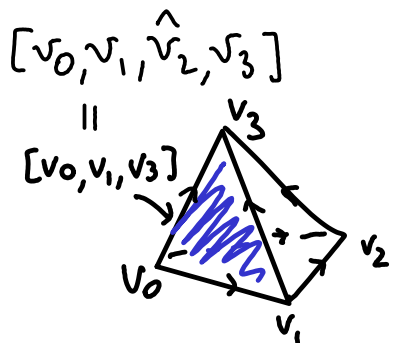
facts = $(n-1)$ -dimensional faces

= $[v_0, \dots, \hat{v}_l, \dots, v_n]$ where we omit v_l

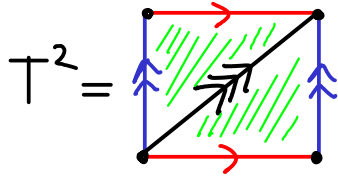
= $\left\{ \sum t_i v_i \in [v_0, \dots, v_n] : t_l = 0 \right\}$

= Image $\sigma|_{\Delta_l^{n-1}} : \Delta_l^{n-1} \rightarrow \mathbb{R}^{n+k}$

" $\{t \in \Delta^n : t_l = 0\}$



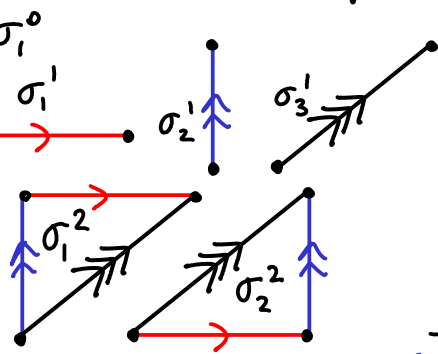
Example Can build a torus out of simplices:



1 0-simplex $\bullet \sigma_i^0$

3 1-simplices $\sigma_1^1 \quad \sigma_2^1 \quad \sigma_3^1$

2 2-simplices $\sigma_1^2 \quad \sigma_2^2$



each facet is associated to another simplex, and we identify them linearly

$T^2 =$ quotient space $\bigsqcup \sigma_i^n /$ canonical homeos associated to the facets

(don't confuse the abstract simplices with their images in $T^2 =$ quotient space)

for example identify facet σ_1^2 of σ_1^2 with σ_2^1 via linear homeo (orientation-preserving)

Def Δ -complex is determined by data

- indexing set I_n , for each $n \in \mathbb{N}$
- choice of n -simplex σ_α^n (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \leq i \leq n$, associate some $\beta(\alpha, i) \in I_{n-1}$
- consistency condition (see later)

The Δ -complex is the quotient space

$$X = \bigsqcup_{\substack{\alpha \in I_n \\ n \in \mathbb{N}}} \sigma_\alpha^n / \begin{array}{l} i\text{-th facet of } \sigma_\alpha^n \text{ is identified with } \sigma_{\beta(\alpha, i)}^{n-1} \\ \text{via the order-preserving canonical linear homeo} \end{array}$$

(quotient topology: $U \subseteq X$ is open $\Leftrightarrow U$ intersects σ_α^n in an open set, $\forall \alpha, n$)

A Δ -complex structure on a top. space Y is a homeo from a Δ -cx $X \cong Y$.

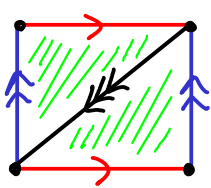
Explicit description of the facet identification

$$\begin{array}{ccc} \{\sum s_j w_j\} = [w_0, \dots, w_{n-1}] & \longrightarrow & [v_0, \dots, v_n] = \left\{ \sum t_j v_j \right\} \\ \uparrow \sigma_{\beta(\alpha, i)}^{n-1} & & \uparrow \sigma_\alpha^n \\ \Delta^{n-1} & \longrightarrow & \Delta_i^{n-1} \subseteq \Delta^n \\ (s_0, \dots, s_{n-1}) & \mapsto & (s_0, \dots, s_{i-1}, 0, s_i, \dots, s_{n-1}) \end{array}$$

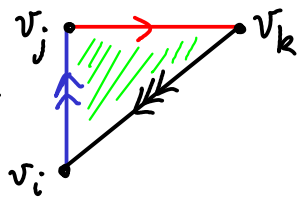
$\sigma_\alpha^n|_{\Delta_i^{n-1}} = \{s_0 v_0 + \dots + s_{i-1} v_{i-1} + s_i v_{i+1} + \dots + s_{n-1} v_n\} = [v_0, \dots, \hat{v}_i, \dots, v_n]$

Non-example

This decomposition for T^2 is not a Δ -complex.



because:



vertices are not totally ordered:

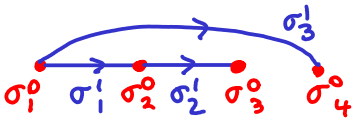
$$i < j < k < i \quad \Rightarrow$$

Consistency condition

We want to additionally ensure that each point of X lies in the interior of exactly one σ_α^n , because we want to avoid unexpected identifications.

Example:

$$X = \triangle$$



then glue $\sigma_1^2 = \triangle$ via $\sigma_3^1 \rightarrow \sigma_2^1$

notice how σ_3^1, σ_4^1 get identified in the quotient, but we only notice this after gluing σ_1^2 (If you try to run the definition of simplicial homology - defined later - you notice that the differential cannot satisfy $\partial_1 \circ \partial_2 = 0$)

Equivalently: the facet gluing maps are compatible under double restriction: $\forall i < j$

$$\begin{array}{ccccccc}
 [v_0, \dots, v_n] & \xrightarrow{\text{facet}} & [v_0, \dots, \hat{v}_i, \dots, v_n] & \xrightarrow{\text{identify}} & [w_0, \dots, w_{n-1}] & \xrightarrow{\text{facet}} & [w_0, \dots, \hat{w}_j, \dots, w_{n-1}] & \xrightarrow{\text{identify}} & [x_0, \dots, x_{n-2}] \\
 & & \xrightarrow{\text{facet}} & & \xrightarrow{\text{identify}} & & \xrightarrow{\text{facet}} & & \xrightarrow{\text{identify}} \\
 & & [v_0, \dots, \hat{v}_j, \dots, v_n] & \xrightarrow{\text{identify}} & [z_0, \dots, z_{n-1}] & \xrightarrow{\text{facet}} & [z_0, \dots, \hat{z}_i, \dots, z_{n-1}] & \xrightarrow{\text{identify}} & [x_0, \dots, x_{n-2}]
 \end{array}$$

this ensures that $[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$ is identified with the same $[x_0, \dots, x_{n-2}]$ whether we first restrict to $t_i = 0$ (omit v_i) or first restrict to $t_j = 0$ (omit v_j).

Another equivalent condition: can define the k -th skeleton of Δ -cx X ,

$X^k =$ quotient space you get by gluing all simplices of dimensions $\leq k$. Consistency is the condition that the boundary of each σ_α^n should map continuously into X^{n-1}

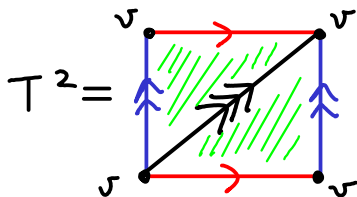
(in the above Example consider the vertex $\triangle = \partial \sigma_1^2$) (more precisely, the "topological realisation" of a simpl. complex)

Rmk (see Part A Topology) A simplicial complex is a Δ -complex in which

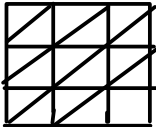
each d -dim face is uniquely determined by d distinct vertices.

A homeo from such a complex to X is a triangulation of X .

Non-example



both 2-simplices have vertices v, v, v

whereas $T^2 =$  is a triangulation.

Harder exercise (Non-examinable) If you apply barycentric subdivision twice to a Δ -cx then you get a simplicial complex. (Hint: After the first subdivision, any given simplex has distinct vertices)

Simplicial chain complex

Def For a Δ -complex X , let $X_n =$ set of n -simplices of X

$C_n^\Delta(X) =$ free abelian group generated by the set X_n

$$= \left\{ \sum_{\alpha \in I_n} c_\alpha \cdot \sigma_\alpha^n : c_\alpha \in \mathbb{Z} \text{ and only finitely many } c_\alpha \neq 0 \right\}$$

differential: $\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha, i)}$ } and extend linearly

so: $\partial_n [v_0, \dots, v_n] = \sum (-1)^i \cdot [v_0, \dots, \hat{v}_i, \dots, v_n]$

will show $\partial \circ \partial = 0$, so get simplicial homology: $H_*^\Delta(X) = H_*(C_*^\Delta, \partial_*)$

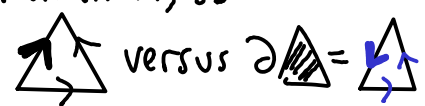
Examples

$\partial_1 (\overrightarrow{v_0 \rightarrow v_1}) = -v_0 + v_1$

$\partial_2 (\text{triangle } v_0, v_1, v_2) = + \overrightarrow{v_1 \rightarrow v_2} - \overrightarrow{v_0 \rightarrow v_2} + \overrightarrow{v_0 \rightarrow v_1}$

$\partial_1 \circ \partial_2 (\text{this}) = +(v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0$

$\partial \circ \partial = 0$ fails for $\overrightarrow{\Delta}$ (not Δ -complex), try!

Later:
The $(-1)^i$ signs keep track of whether the orientation agrees/disagrees with geometric boundary orientation, so


Lemma $\partial \circ \partial = 0$

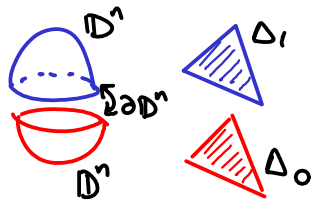
Pf $\partial_{n-1} (\partial_n [v_0, \dots, v_n]) = \sum (-1)^i \partial_{n-1} [v_0, \dots, \hat{v}_i, \dots, v_n]$
 $= \sum_{j < i} (-1)^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$ ← antisymmetric if swap i, j
 $+ \sum_{j > i} (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$
 $= 0 \quad \square$

Example $S^1 = \text{circle}$ Δ -cx: $X_0: 1$ 0-simplex \bullet $e^0 = e_{\beta(1,0)} = e_{\beta(1,1)}$
 $X_1: 1$ 1-simplex \rightarrow e^1

$0 \rightarrow C_1^\Delta \rightarrow C_0^\Delta \rightarrow 0$
 $\parallel \quad \parallel$
 $\mathbb{Z}e \quad \mathbb{Z}v$
 $e \mapsto v - v = 0$

$\Rightarrow H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{else} \end{cases}$

Example Δ -cx structure on S^n :

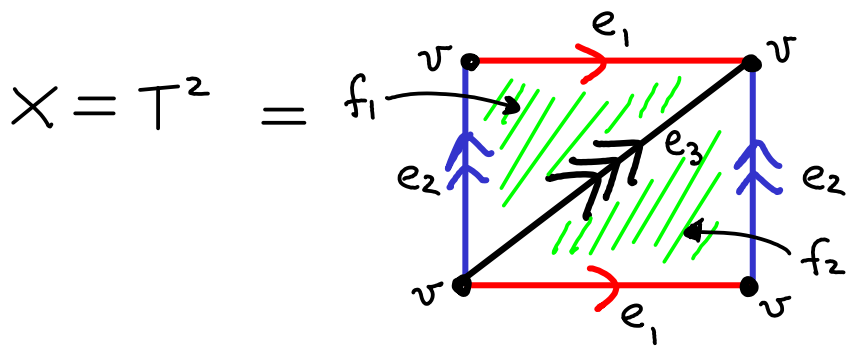


$S^n = \Delta^n \cup \Delta^n$ / glue along $\partial \Delta^n$
 call this Δ_1 ← this Δ_0

One can deduce: $H_* S^n = \begin{cases} \mathbb{Z} \langle \text{point} \rangle \cong \mathbb{Z} & * = 0 \\ \mathbb{Z} \langle \Delta_1 - \Delta_0 \rangle \cong \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ ← pick any vertex

but messy!

Example



$$0 \rightarrow C_2^\Delta \xrightarrow{\quad} C_1^\Delta \xrightarrow{\quad} C_0^\Delta \rightarrow 0$$

$\mathbb{Z}f_1 + \mathbb{Z}f_2 \qquad \qquad \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 \qquad \qquad \mathbb{Z}v$

$$f_1 \mapsto e_1 - e_3 + e_2 \qquad e_1, e_2, e_3 \mapsto v - v = 0$$

$$f_2 \mapsto e_2 - e_3 + e_1$$

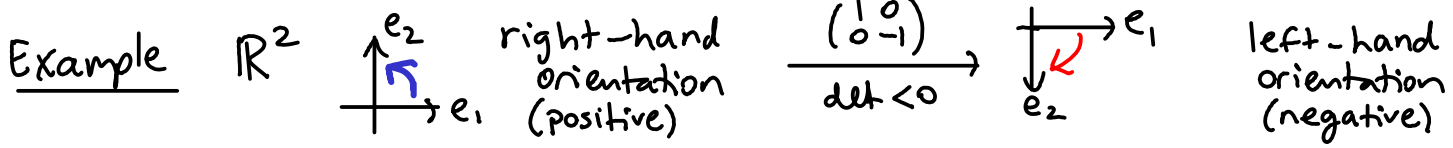
$$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & * = 0 \\ (\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3) / \mathbb{Z}(e_1 - e_3 + e_2) & * = 1 \leftarrow \text{freely generated by } e_1, e_2 \\ \mathbb{Z} \cdot (f_1, -f_2) & * = 2 \\ 0 & \text{else} \end{cases}$$

$$\cong \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^2 & * = 1 \\ 0 & \text{else} \end{cases}$$

Alternative useful method using a trick from algebra:
 Smith normal form of ∂_2 :
 $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col. op.}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
 hence:
 $H_2 \cong \ker \partial_2 \cong \mathbb{Z}$
 $H_1 \cong \text{Coker } \partial_2 \cong \mathbb{Z}^2$
 so after \mathbb{Z} -isos of C_2, C_1 we get $\mathbb{Z}^2 \rightarrow \mathbb{Z}^3, (a,b) \mapsto (a, 0, 0)$

Remark about orientations (see also my B3.2 Geometry of Surfaces notes)

For a vector space an orientation is a choice of basis modulo linear endomorphisms of $\det > 0$



Fact $GL(n, \mathbb{R})$ has 2 path-components $\begin{cases} A: \det A > 0 \\ A: \det A < 0 \end{cases}$ so can always continuously deform a basis to another within same orientation

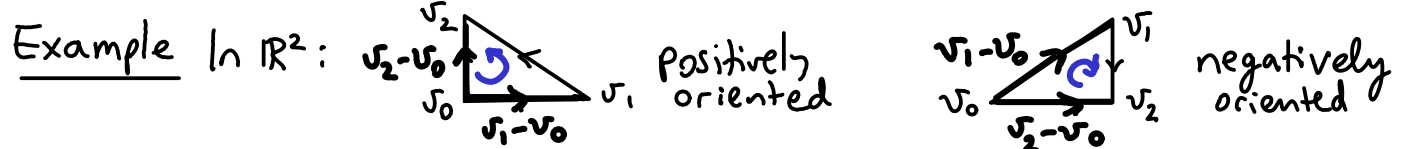
Canonical orientation on \mathbb{R}^n : e_1, \dots, e_n standard basis \leftarrow "positive orientation"

Example $[v_0, \dots, v_n]$ simplex $\Rightarrow v_1 - v_0, \dots, v_n - v_0$ is a basis of vector subspace

$$V = \{ \sum a_i v_i : \sum a_i = 0 \} \subseteq \mathbb{R}^{n+k}$$

hence a choice of orientation of V , and each transposition of vertices v_0, \dots, v_n switches the orientation class.
 (if swap v_i, v_j consider the reflection in the hyperplane $(a_i = a_j)$ in V)

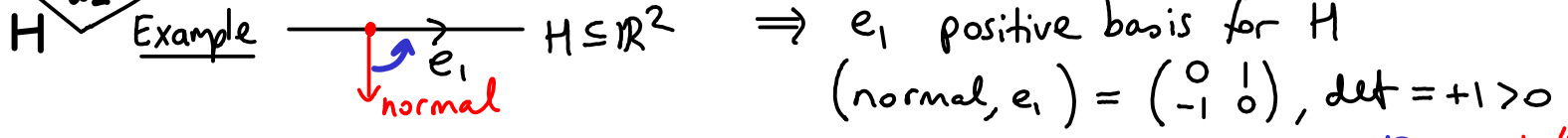
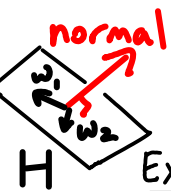
If $v_0, \dots, v_n \in \mathbb{R}^n$ then $V = \mathbb{R}^n$ so simplex's orientation can be compared with \mathbb{R}^n -orient.



- No canonical choice of orientation for an abstract vector space. Need choose basis $v_1 \rightarrow v_n$ then declare another basis positively oriented if the change of basis matrix has $\det > 0$.

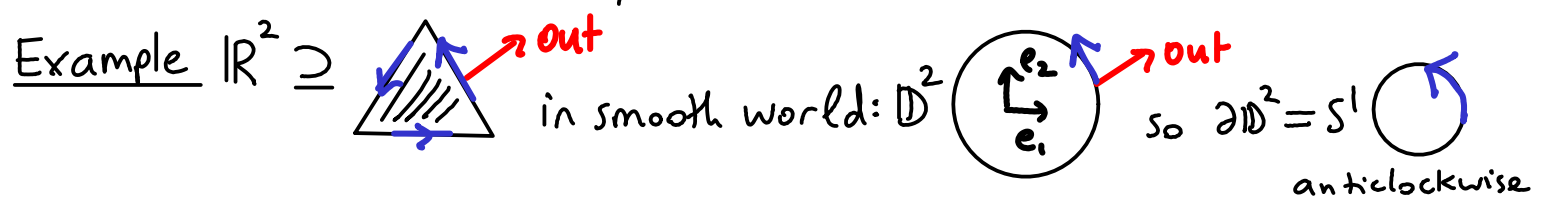
For hyperplane $H \subseteq \mathbb{R}^n$ with choice of normal can declare orientation of basis w_1, \dots, w_{n-1} of H positive if normal, w_1, \dots, w_{n-1} is positive \mathbb{R}^n -basis

convention "outward normal first"

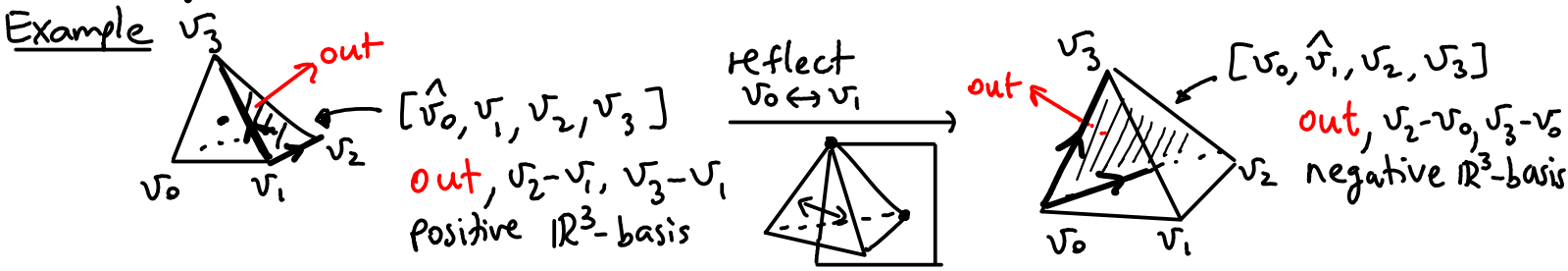


Example $\Delta^n \subseteq \mathbb{R}^{n+1}$ with normal $(1, 1, \dots, 1)$ is positively oriented, e.g. $n=2$:

UPSHOT For an n -simplex $[v_0, \dots, v_n]$ in \mathbb{R}^n , each facet lies in a hyperplane and have canonical choice of normal: outward normal. Hence facets are canonically oriented.



Any reflection of \mathbb{R}^n will swap orientation: after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get clockwise



UPSHOT $(-1)^i$ in $(-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$ in definition of simplicial ∂ secretly keeps track of whether the orientation of the simplex agrees or not with the orientation induced geometrically by the above conventions. From this point of view, the equation $\partial \circ \partial = 0$ holds because a codimension 2 face γ of a simplex σ arises as the facet of exactly two facets f_1, f_2 of σ , and the geometric orientations of f_1, f_2 induce opposite geometric orientations on γ (therefore if we keep track of orientation signs we count $+\gamma - \gamma = 0$).

(checking that they are opposite requires some thought, one approach is to say that we can deform f_1, f_2 until they make a flat angle, and then their outward normals will be opposite.

Picture: $f_1 \rightarrow f_2$ deform \rightarrow get $+\gamma$ from ∂f_1 , get $-\gamma$ from ∂f_2

Lemma $H_*^\Delta(X) \cong \bigoplus H_*^\Delta(X_i)$ where X_i are the path-components of X .

Pf $\bigoplus C_*^\Delta(X_i) \rightarrow C_*^\Delta(X)$, $\bigoplus c_i \mapsto \sum c_i$

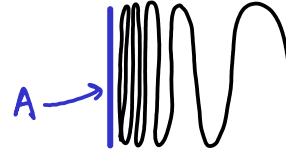
since Δ^k path-conn.

is chain isomorphism since any simplex $\sigma: \Delta^k \rightarrow X$ has path-connected image, so $\subseteq X_i$ some i . \square

Rmk X top. space \Rightarrow path conn. component \subseteq connected component since path-conn. \Rightarrow connected. For Δ -cx, these are same (since connected + locally path-conn. \Rightarrow path-connected).

However for top. space they need not be the same, e.g. Topologist's sine curve

$\{(x, \sin \frac{1}{x}) : x \in (0, 1]\} \cup \underbrace{0 \times [0, 1]}_A \subseteq \mathbb{R}^2$
 2 path-conn. components



- connected
- not path-connected
- not locally path-connected

Lemma X Δ -cx, X connected $\Rightarrow \forall$ vertices v_1, v_2, \exists path consisting of edges \leftarrow i.e. 1-simplices

Proof Pick a vertex v_0 . Let $A = \{\text{simplices } \sigma \text{ in } X \text{ such that one (hence all) vertices of } \sigma \text{ can be connected to } v_0 \text{ by a path in } X \text{ consisting of edges}\}$.

Let $B = \{\text{simplices } \sigma \text{ of } X \text{ not in } A\}$. Observe that if $\sigma \in A$ then the facets of σ are in A , similarly for B . After gluing simplices as prescribed in X , the sets A and B define subspaces of X (using the quotient topology). Exercise: A, B are closed subspaces. Note $A \cap B = \emptyset$ (otherwise they have a vertex in common and we contradict the definition of A) $\Rightarrow X = A \sqcup B$ is a disjoint union of closed sets, so X connected forces $B = \emptyset$. \square

Theorem X has Δ -cx structure $\Rightarrow H_0^\Delta(X) \cong \bigoplus \mathbb{Z}$

Pf By lemma, wlog X path-connected

path-conn. components

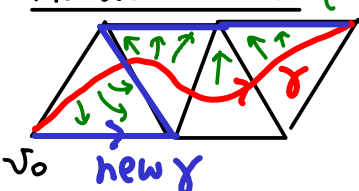
- vertex $v \Rightarrow v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) \equiv 0 \Rightarrow [v] \in H_0(X)$
- vertices $v_0, v_1 \in X \xrightarrow{\text{Lemma}} \exists$ path γ from v_0 to v_1 , consisting of edges $\Rightarrow \gamma$ is sum of 1-chains s.t. $\partial \gamma = v_1 - v_0$, so $[v_0] = [v_1] \in H_0(X)$.
- $H_0(X) = \langle [v] \rangle \leftarrow \mathbb{Z}$ is injective?
 $n v \leftarrow n$ Suppose $n v = \partial c$ some $c \in C_1(X)$

consider augmentation hom $\varepsilon: C_0(X) \rightarrow \mathbb{Z}$, $\sum n_i \sigma_i \xrightarrow{\varepsilon} \sum n_i$

\Rightarrow composition $C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$ is 0 as $\partial_1(\text{1-simplex}) = \sigma_1 - \sigma_0 \xrightarrow{\varepsilon} 1 - 1 = 0$. \leftarrow 0-simplices

$\Rightarrow n = \varepsilon(nv) = \varepsilon \partial_1 c = 0$. \square

Harder exercise (non-examinable) If $\gamma: [0, 1] \rightarrow X$ path from vertex v_0 to v_1 , then we can homotope (=continuously deform) it into a path consisting of edges.



(Hints: first show that a Δ -cx is locally path-connected. Then use the fact that the domain $[0, 1]$ of γ is compact in order to approximate γ arbitrarily well by a piecewise linear path. Finally prove the result for piecewise linear paths.)

3. SINGULAR HOMOLOGY

Motivation Not obvious that H_*^Δ is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$
 then $f \circ \sigma$ typically not a simplex: $\Delta \xrightarrow{\sigma} \Delta \xrightarrow{f} \Delta$ ↑ continuous map

Solution 1: only allow simplicial maps $f: X \rightarrow Y$ (so $f \circ \sigma$ simplex $\forall \sigma$)

Solution 2: show that any cts map $f: X \rightarrow Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on X, Y enough times. Also any two such approximations induce the same map $H_*^\Delta(X) \rightarrow H_*^\Delta(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n \rightarrow X$ WILL DO THIS. and prove $H_*^\Delta(X) \cong H_*(X)$ for Δ -complexes X .

Def Singular n -simplex is any continuous map $\sigma: \Delta^n \rightarrow X$ ↑ X is any top-space

Singular n -chains $C_n(X) =$ free abelian group generated by σ
 $= \left\{ \sum c_\sigma \cdot \sigma : c_\sigma \in \mathbb{Z} \right.$
singular n -simplices σ only finitely many $c_\sigma \neq 0$

$$\partial_n \sigma = \sum (-1)^i \cdot \sigma|_{\Delta_i^{n-1}} \quad (\text{and extend linearly})$$

← i -th facet

Rmk Here $\Delta_i^{n-1} = [e_0, \dots, \hat{e}_i, \dots, e_n]$ is identified canonically with Δ^{n-1} (send $e_k \rightarrow e_k$ $k < i$
 $e_k \rightarrow e_{k-1}$ for $k > i$)

Lemma $\partial \circ \partial = 0$

Proof $\partial_{n+1}(\partial_n \sigma) = \partial_{n+1} \left(\sum (-1)^i \sigma|_{\Delta_i^{n-1}} \right)$
 $= \sum_{j < i} (-1)^i (-1)^j \sigma|_{[e_0, \dots, \hat{e}_j, \dots, \hat{e}_i, \dots, e_n]}$
 $+ \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n]}$ ↙ antisymmetric if swap i, j
 $= 0 \quad \square$

\implies singular homology: $H_*(X) = H_*(C_*, \partial_*)$

For Δ -complex X have inclusion of subcomplex $C_*^\Delta \rightarrow C_*$

\implies induces $H_*^\Delta(X) \rightarrow H_*(X)$ Fact: isomorphism (proof later, see cellular $H_*^{CW} \cong H_*$)

Corollary $H_*^\Delta(X)$ is independent of choice of Δ -cx structure on X

Example $X = \text{point} \Rightarrow C_n(X) = \mathbb{Z} \cdot (\sigma_n: \Delta^n \rightarrow X \text{ the constant map})$

$$\partial \sigma_n = \sum (-1)^i \sigma_n|_{\Delta_i^{n-1}} = \begin{cases} \sum (-1)^i \sigma_{n-1} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} \Rightarrow \dots \rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \rightarrow 0$$

$$\Rightarrow H_*(\text{pt}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$$

Lemma $H_*(X) \cong \bigoplus H_*(X_i)$ where X_i are path-components of X

Pf Image of cts map $\Delta^n \rightarrow X$ is path conn. so lies in some X_i . \square

Cor $H_0(X) = \bigoplus_{X_i} \mathbb{Z}$ ← generators of $C_0(X)$

Pf By Lemma, wlog X path-connected. $\Delta^0 = \text{pt} \rightarrow X$ is cycle since $C_{-1}(X) = 0$

Given 2 points $x, y \in X$, a path $\Delta^1 = [0, 1] \xrightarrow{\gamma} X, \gamma(0) = x, \gamma(1) = y$ is also a 1-chain!

So $y - x = \partial \gamma$, so x, y are homologous. Finally if $n \cdot [x] = 0 \in H_0(X)$ then $nx = \partial c$ some $c \in C_1(X)$ = generated by paths. Now run the augmentation hom. trick like we did for H_0^Δ : $n = \varepsilon(nx) = \varepsilon \partial c = 0$ as $\varepsilon \partial = 0$. \square

Naturality (i.e. functoriality)

Lemma $f: X \rightarrow Y$ continuous

$\Rightarrow H_*(f) = f_*: H_*(X) \rightarrow H_*(Y)$ induced by chain map

$f_*: C_*(X) \rightarrow C_*(Y)$

$$\boxed{f_*(\sigma) = f \circ \sigma} \quad \text{and extend linearly}$$

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\sigma} & X \\ & \searrow f_* \sigma & \downarrow f \\ & & Y \end{array}$$

Pf $\partial_n(f_* \sigma) = \sum (-1)^i f \circ \sigma|_{\Delta_i^{n-1}} = f_* (\sum (-1)^i \sigma|_{\Delta_i^{n-1}}) = f_* (\partial_n \sigma)$ \square

Properties 1) $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow (g \circ f)_* = g_* \circ f_*$

2) $\text{id}_* = \text{id}$

Pf 1) $(g \circ f)_* \sigma = g \circ f \circ \sigma = g_* (f \circ \sigma) = g_* (f_* \sigma)$ \checkmark

2) $\text{id}_* \sigma = \text{id} \circ \sigma = \sigma$ \checkmark \square

Cor $H_*: \left\{ \begin{array}{l} \text{topological spaces} \\ \& \\ \text{cts maps} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{graded abelian groups} \\ \& \\ \text{graded homs} \end{array} \right\}$ is a functor

Cor $X \cong Y \Rightarrow H_*(X) \cong H_*(Y)$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

Algebra: chain homotopies

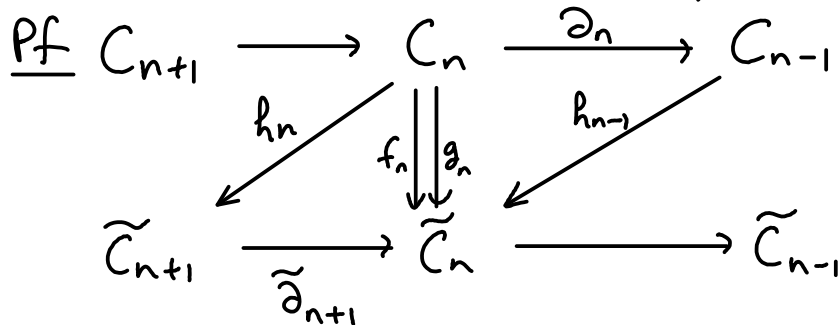
$f_*, g_* : (C_*, \partial_*) \rightarrow (\tilde{C}_*, \tilde{\partial}_*)$ chain maps

Def f_*, g_* are chain homotopic if \exists (degree +1) hom $h: C_* \rightarrow \tilde{C}_*[1]$ s.t.

$$\tilde{\partial} \circ h + h \circ \partial = f_* - g_*$$

h is called a chain homotopy

Consequence $f_* = g_* : H_+(C_*, \partial_*) \rightarrow H_+(\tilde{C}_*, \tilde{\partial}_*)$ on homology



For c cycle $\in C_n$:

$$\begin{aligned} \Rightarrow f_n(c) - g_n(c) &= \\ &= \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + \underbrace{h_{n-1} \circ \partial_n(c)}_{\text{so } 0 \in H_n(C_*)} \end{aligned}$$

Harder exercise (Non-examinable)

If chain maps $f_*, g_* : C_* \rightarrow \tilde{C}_*$ induce the same map on homology and C_n, \tilde{C}_n are free abelian groups $\forall n$, then f_*, g_* are chain homotopic.

Hints Let $B_{n-1} = \text{Im } \partial_n, K_n = \text{Ker } \partial_n$. Use fact that subgroups of free groups are free to show that B_n, K_n are free. Deduce that $C_n \cong B_{n-1} \oplus K_n$ (show $0 \rightarrow K_n \xrightarrow{\subseteq} C_n \xrightarrow{\partial_n} B_{n-1} \rightarrow 0$ splits).
More precisely deduce: the chain complex (C_*, ∂_*) is isomorphic to $D_* = \bigoplus_n (\dots \rightarrow 0 \rightarrow B_n \xrightarrow{\subseteq} K_n \rightarrow 0 \rightarrow \dots)$.
Via that iso, $f_* - g_*$ determines a chain map $\alpha: D_* \rightarrow D'_*$ such that on homology $\alpha = 0: H_*(D) \rightarrow H_*(D')$. Consider $h: D_n \rightarrow D'_{n+1}, h: K_n \rightarrow B'_{n+1}$ is α .
(hence $\alpha(K_n) \subseteq B'_{n+1} \subseteq K'_{n+1}$) $h: B_n \rightarrow 0$ is zero.

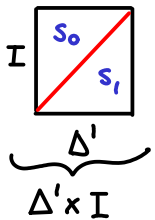
chain cx with K_n in degree n
 B_n " " $n+1$
the differential $B_n \rightarrow K_n$ is the inclusion.

Theorem

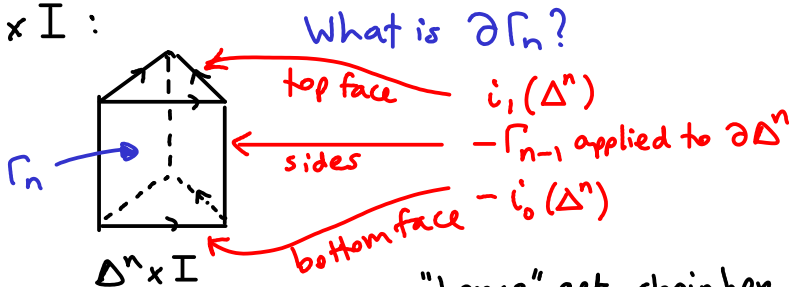
Let $i_0: X \rightarrow X \times I, i_0(x) = (x, 0)$
 $i_1: X \rightarrow X \times I, i_1(x) = (x, 1)$ where $I = [0, 1]$

$\Rightarrow i_{0*}, i_{1*}: C_*(X) \rightarrow C_*(X \times I)$ are chain hpic.

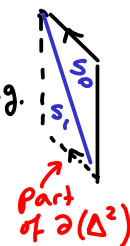
Key idea Need the "prism operator" which cuts $\Delta^n \times I$ into a sum Γ_n of $(n+1)$ -simplices in $\Delta^n \times I$:



$$\Gamma_n = s_0 - s_1$$



What is $\partial \Gamma_n$?

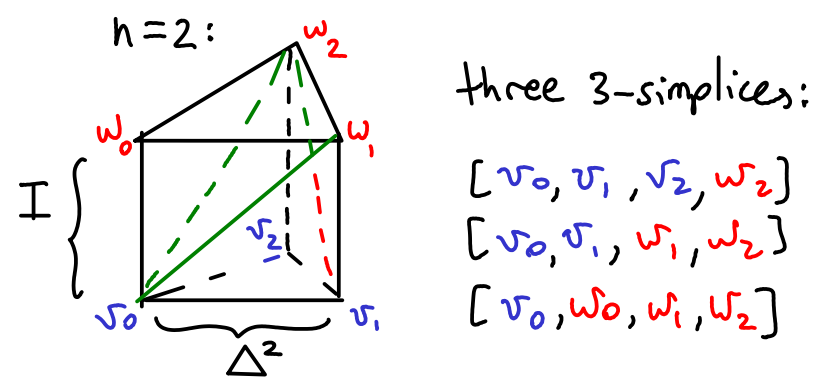
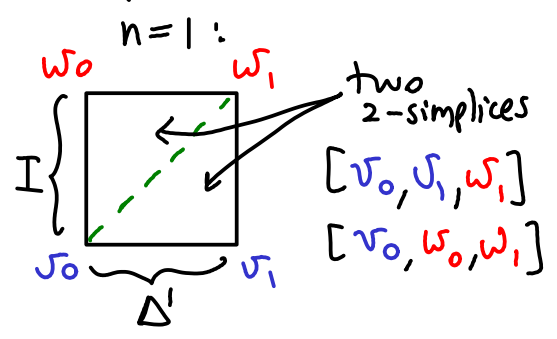


"hence" get chain hpic

Non-examinable

Pf bottom facet $\Delta^n \times 0 = [v_0, \dots, v_n]$ $\leftarrow v_i = e_i \times 0$
 top facet $\Delta^n \times 1 = [w_0, \dots, w_n]$ $\leftarrow w_i = e_i \times 1$
 $\left. \vphantom{\begin{matrix} \text{bottom facet} \\ \text{top facet} \end{matrix}} \right\} \subseteq \Delta^n \times [0,1] \subseteq \mathbb{R}^{n+1}$

Examples



Let $s_i = [v_0, \dots, v_i, w_i, \dots, w_n]$

Claim The s_i cover $\Delta \times [0,1]$ and give Δ -cx structure on $\Delta^n \times I$

Pf $\sum_{k \leq i} t_k v_k + \sum_{k > i} r_k w_k = (t_0, \dots, t_{i-1}, \underline{t_i + r_i}, r_{i+1}, \dots, r_n, \underbrace{r_i + \dots + r_n}_{\in [0,1] = I})$

So given $(x_0, \dots, x_n, a) \in \Delta^n \times I$, equate and solve:

$t_0 = x_0, \dots, t_{i-1} = x_{i-1}, r_{i+1} = x_{i+1}, \dots, r_n = x_n$, and $\begin{cases} r_i = a - x_{i+1} - \dots - x_n \\ t_i = x_i - r_i \end{cases}$

Note $x_k \geq 0, \sum x_k = 1, a \in [0,1]$ hence $\sum t_k + \sum r_k = 1 \checkmark \begin{cases} t_k \geq 0 \text{ for } k < i \\ r_k \geq 0 \text{ for } k > i \end{cases}$

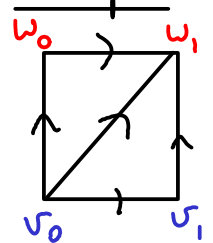
but $\begin{cases} r_i \geq 0 \\ t_i \geq 0 \end{cases} \Leftrightarrow \begin{cases} a \geq x_{i+1} + \dots + x_n \\ x_i + x_{i+1} + \dots + x_n \geq a \end{cases}$. Thus a solution exists if we pick $i = \min\{k : a \geq x_{k+1} + \dots + x_n\}$.

(There are multiple solutions if $a = x_{i+1} + \dots + x_n$ since then also "i+1" works) since $x_{i+1} + x_{i+2} + \dots + x_n \geq a \geq x_{i+2} + \dots + x_n$. This is expected since s_i, s_{i+1} meet along a facet. Example: n=1 picture above: s_0, s_1 meet along facet $[v_0, w_1]$ which arises for $i=0, r_0=0$ but also $i=1, t_1=0$.

Def
 $\Gamma_n = \sum (-1)^i s_i \in C_{n+1}(\Delta^n \times [0,1]) \leftarrow$ geometrically this "represents" $\Delta^n \times I$ as a simplicial chain

$\Rightarrow \partial \Gamma_n = \sum_i \sum_{j \leq i} (-1)^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]$
 $+ \sum_i \sum_{j > i} (-1)^i (-1)^{j+1} [v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]$
 } geometrically this "represents" $\partial(\Delta^n \times I) = (\partial \Delta^n \times I) \cup (\Delta^n \times \partial I)$

Example



$\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1]$ "is the square"

$\partial \Gamma_1 = [w_0, w_1] - [v_0, w_1] + [v_0, w_0] - [v_1, w_1] + [v_0, w_1] - [v_0, v_1]$
 "is ∂ of square" "inside facets" cancel

Prism operator

$$P : C_n(X) \longrightarrow C_{n+1}(X \times [0,1])$$

$$P(\sigma) = (\sigma \times \text{id})_* (\Gamma_n)$$

$$\sigma : \Delta^n \rightarrow X$$

$$\begin{aligned} \sigma \times \text{id} : \Delta^n \times [0,1] &\rightarrow X \times [0,1] \\ (\sigma \times \text{id})(x,t) &= (\sigma(x), t) \end{aligned}$$

$$\begin{aligned} \partial P(\sigma) &= \partial (\sigma \times \text{id})_* (\Gamma_n) \\ &= (\sigma \times \text{id})_* (\partial \Gamma_n) \end{aligned}$$

this abbreviated notation means the map
 $(t_0, \dots, t_n) \mapsto \left(\sigma \left(\begin{matrix} t_0 e_0 + \dots + t_j \hat{e}_j + t_j e_{j+1} + \dots \\ + t_{i-1} e_i + t_i e_i + \dots + t_n e_n \end{matrix} \right), t_i + \dots + t_n \right) \in X \times I$

$$\stackrel{\textcircled{\star}}{=} \sum_i \sum_{j < i} (-1)^i (-1)^j [i_0 \sigma e_0, \dots, \widehat{i_0 \sigma e_j}, \dots, i_0 \sigma e_i, i_1 \sigma e_i, \dots, i_1 \sigma e_n]$$

$$\begin{aligned} &(\sigma \times \text{id})(v_i) \\ &= (\sigma \times \text{id})(e_i, 0) \\ &= (\sigma(e_i), 0) \\ &= i_0(\sigma)(e_i) \\ &(\sigma \times \text{id})(w_i) \\ &= (\sigma(e_i), 1) \\ &= i_1(\sigma)(e_i) \end{aligned}$$

$$+ \sum_i \sum_{j > i} (-1)^i (-1)^{j+1} [i_0 \sigma e_0, \dots, i_0 \sigma e_i, i_1 \sigma e_i, \dots, \widehat{i_1 \sigma e_j}, \dots, i_1 \sigma e_n]$$

$$= i_{1*} \sigma - i_{0*} \sigma = P \partial \sigma \quad \square$$

$i=j=0$ 1st sum
 $i=j=n$ 2nd sum

the other $i=j$ terms arise in both sums with opposite signs so cancel

$$\begin{aligned} &((\partial \sigma) \times \text{id})_* \Gamma_{n-1} \\ &\sum_i (-1)^i [v_0, \dots, v_i, w_i, \dots, w_{n-1}] \end{aligned}$$

$$\partial \sigma = \sum (-1)^k \sigma | [e_0, \dots, \hat{e}_k, \dots, e_n]$$

Note for $k < i$ the P operator on $(-1)^k \sigma | [e_0, \dots, \hat{e}_k, \dots, e_n]$ gives:
 $\sum (-1)^i (-1)^k [i_0 \sigma e_0, \dots, \widehat{i_0 \sigma e_k}, \dots, i_0 \sigma e_{i+1}, i_1 \sigma e_{i+1}, \dots, i_1 \sigma e_n]$
 ← see Example.

Example to clarify
 $n=2, k=0, i=1$
 $\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1]$
 $(\sigma | [\hat{e}_0, e_1, e_2] \times \text{id})([v_0, v_1, w_1])$
 maps to $v_0 + t_1 v_1 + t_2 w_1 = (t_0, t_1 + t_2, t_2)$
 to $(\sigma(t_0, t_1 + t_2, t_2)) = (\sigma(t_0 e_1 + t_1 e_2 + t_2 e_2), t_2)$

Homotopy invariance

Def $f_0, f_1 : X \rightarrow Y$, $f_0 \simeq f_1$ homotopic if \exists continuous map $F : X \times [0,1] \rightarrow Y$ called homotopy s.t. $\begin{cases} f_0 = F \circ i_0 \\ f_1 = F \circ i_1 \end{cases}$

Idea Think of this as a continuous family of maps $f_t = F(\cdot, t) : X \rightarrow Y$ from f_0 to f_1 .

Exercise \simeq is an equivalence relation.

Homotopic relative to $A \subseteq X$ if $F(a,t) = f_0(a) = f_1(a)$ all $a \in A$, all t .
 write " $f \simeq g$ rel A "

Def $X \simeq Y$ homotopy equivalent spaces if \exists maps

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} Y \quad \text{with} \quad \begin{array}{l} g \circ f \simeq \text{id} \\ f \circ g \simeq \text{id} \end{array}$$

Rmk homeo \Rightarrow hpy equivalent

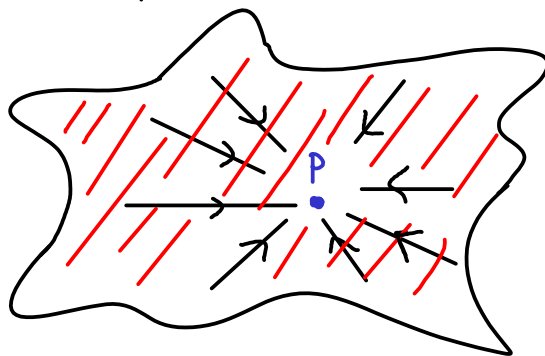
Def X contractible if $X \simeq \text{pt}$

equivalently $(X \xrightarrow{\text{id}} X) \simeq (X \xrightarrow{\text{const}} \text{point} \in X)$

Example $\mathbb{R}^n \simeq \text{pt}$

$F(x, t) = tx$ then $f_0 \equiv 0, f_1 = \text{id}$.

• (star-shaped subsets of \mathbb{R}^n) $\simeq \text{pt}$



contains line segments to a specific point p

WLOG $p=0$ & use same F
 \uparrow
 translate

(examples: \mathbb{D}^n , convex sets, ...)

Theorem $f_0 \simeq f_1 \Rightarrow f_{0*} = f_{1*} : H_* X \rightarrow H_* Y$

Pf $f_{1*} - f_{0*} = F_* i_{1*} - F_* i_{0*}$ (where $F = \text{homotopy}$,
 i_0, i_1 as in previous Thm)

$= F_* (i_{1*} - i_{0*})$

$= F_* (\partial P + P\partial)$

$= \partial \circ (F_* P) + (F_* P) \circ \partial$

previous Thm \rightarrow
 F_* chain map \rightarrow
 $\Rightarrow F_* P$ is chain hpy from f_{0*} to f_{1*} \square

Cor $X \simeq Y \Rightarrow H_* X \cong H_* Y$

Pf $f_* g_* = \text{id}_*$, $g_* f_* = \text{id}_*$ \square

Example X contractible $\Rightarrow H_* X \cong H_*(\text{pt}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Cultural Rmk (Whitehead's theorem) For nice topological spaces \leftarrow (CW complexes - see later in course)
 if X, Y are simply connected and $\exists f: X \rightarrow Y$ inducing isomorphisms on H_*
 then $X \simeq Y$ are homotopy equivalent.

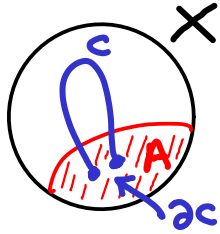
Relative homology

Def (X, A) pair of spaces if $A \subseteq X$ topological subspace
 $\Rightarrow i = \text{incl}: A \hookrightarrow X$ induces a subcx $i_*: C_* A \rightarrow C_* X$
 $\Rightarrow C_* X / C_* A$ quotient chain cx (recall $\partial[x] = [\partial x]$)

$$H_*(X, A) = H_*(C_* X / C_* A)$$

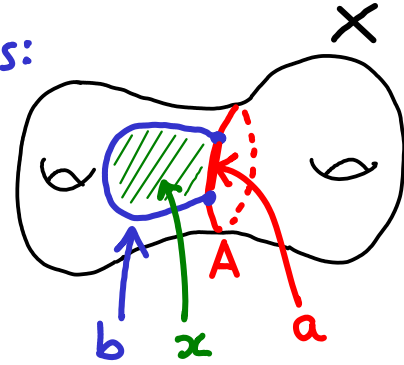
Idea: relative cycles:

$c \in C_* X$
 s.t. $\partial c \in C_* A$



relative boundaries:

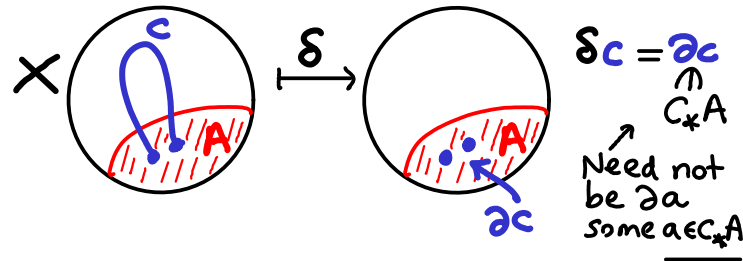
$b \in C_* X$
 s.t. $\exists x \in C_{*+1} X$
 $\partial x = b + a \in C_* A$



$$\Rightarrow 0 \rightarrow C_* A \rightarrow C_* X \rightarrow C_* X / C_* A \rightarrow 0 \text{ SES}$$

$$\text{Cor } \boxed{\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(X) \rightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \dots}$$

LES for the pair



Reduced homology

$$\tilde{H}_* X = \text{Ker} (H_* X \rightarrow H_*(pt))$$

\uparrow
induced by $X \rightarrow pt$

For $X \neq \emptyset$, $\tilde{H}_* X = H_*$ of augmented chain complex:

$$\dots \rightarrow C_n(X) \rightarrow \dots \rightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

augmentation $\epsilon(\sum n_i \cdot p_i) = \sum n_i$
 \uparrow
 $\in \mathbb{Z}$ \uparrow points $\in X$

can view $C_{-1}(X) = \mathbb{Z} \cdot (\text{map } \emptyset \rightarrow X)$ where allow the empty simplex \emptyset

Example $\tilde{H}_*(\mathbb{R}^n) \cong \tilde{H}_*(\mathbb{D}^n) \cong \tilde{H}_*(pt) = 0$

Check $H_* X = \tilde{H}_* X$ $* \neq 0$, and $H_0 X \cong \tilde{H}_0 X \oplus \mathbb{Z}$ for $X \neq \emptyset$

$f: X \rightarrow Y \Rightarrow f_*: \tilde{H}_* X \rightarrow \tilde{H}_* Y$

Lemma (X, A) pair $\Rightarrow \exists$ LES

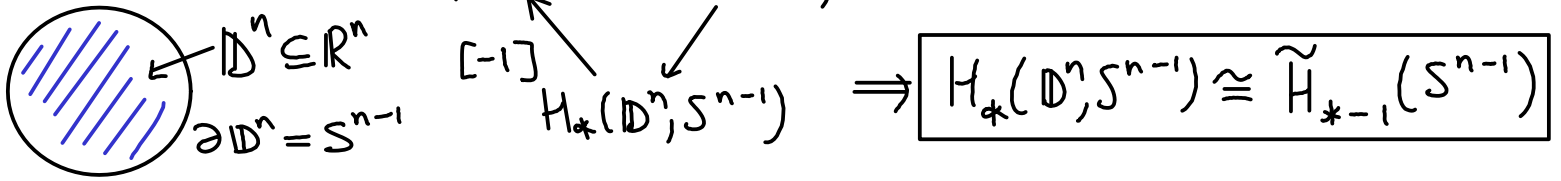
$$\boxed{\dots \rightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \rightarrow H_n(X, A) \rightarrow \tilde{H}_{n-1}(A) \xrightarrow{i_*} \dots \rightarrow H_0(X, A) \rightarrow 0}$$

Pf we augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0 \square$

Cor $H_*(X, pt) \cong \tilde{H}_*(X)$ for $X \neq \emptyset$

Pf $\tilde{H}_*(pt) = 0$. \square

Example LES: $\tilde{H}_*(S^{n-1}) \rightarrow \tilde{H}_*(D^n) = 0$



Naturality of the LES for pairs

Def A map of pairs of spaces $(X, A) \xrightarrow{f} (Y, B)$

means $f: X \rightarrow Y$ and $f(A) \subseteq B$.

Lemma
$$\begin{array}{ccccccc} \dots & \rightarrow & H_* A & \rightarrow & H_* X & \rightarrow & H_*(X, A) \rightarrow H_{*-1} A \rightarrow \dots \\ & & f_* \downarrow & & f_* \downarrow & & \downarrow & & f_* \downarrow \\ \dots & \rightarrow & H_* B & \rightarrow & H_* Y & \rightarrow & H_*(Y, B) \rightarrow H_{*-1} B \rightarrow \dots \end{array}$$

and similarly for \tilde{H}_* .

Pf
$$\begin{array}{ccccccc} 0 \rightarrow C_* A & \rightarrow & C_* X & \rightarrow & C_* X / C_* A & \rightarrow & 0 \\ & & f_* \downarrow & & f_* \downarrow & & \\ 0 \rightarrow C_* B & \rightarrow & C_* Y & \rightarrow & C_* Y / C_* B & \rightarrow & 0 \end{array} \Rightarrow \text{claim follows by naturality of LES induced by SESs of chain complexes. } \square$$

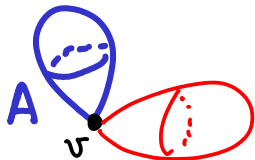
5. EXCISION THEOREM AND QUOTIENTS

(X, A) pair

Def $r: X \rightarrow X$ retraction onto A if $\begin{cases} r(X) = A \\ r|_A = id_A \end{cases}$

$\left(\begin{array}{l} \text{equivalently} \\ r^2 = r \\ \text{then define} \\ A = \text{im}(r) \end{array} \right)$

Example



$X = \underbrace{S^2}_A \vee S^2 = \text{two spheres glued at one point } v$
(wedge sum)

$r: X \rightarrow A$ map second sphere to v

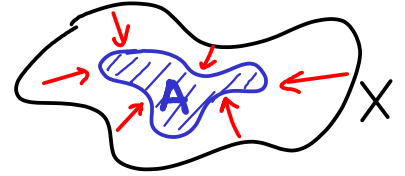
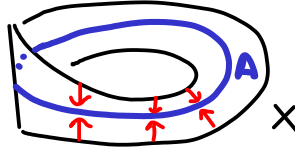
Example In Pf of Brouwer fixed pt thm we built a retraction r by contradiction

Cor r retraction $\Rightarrow r_*: H_* X \rightarrow H_* A$ surjective
 $incl_*: H_* A \rightarrow H_* X$ injective

Pf $A \xrightarrow{incl} X \xrightarrow{r} A$ now use H_* functorial \square
 $\underbrace{\quad \quad \quad}_{id_A}$

Def $r: X \rightarrow X$ deformation retraction onto A if $\begin{cases} r \text{ retraction} \\ r \simeq \text{id}_X \text{ rel } A \end{cases}$

Example $X = \text{Möbius Strip}$
 $A = \text{equator}$



Lemma r def. retr. \Rightarrow $A \xrightarrow{\text{incl}} X$ is a homotopy equivalence.

\cdot incl_* and r_* are isos on H_* , so $H_* A \cong H_* X$

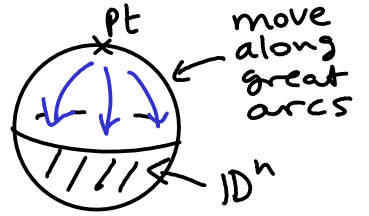
Pf $A \xrightarrow{\text{incl}} X$ $\text{incl} \circ r = r \simeq \text{id}_X$, $r \circ \text{incl} = r|_A = \text{id}_A$ \square

Example $S^n \setminus \text{pt}$ def. retracts to $D^n \cong$ lower hemisphere:

$$S^n \setminus \text{pt} \simeq D^n$$

$$S^n \setminus \{2 \text{ points}\} \simeq D^n \setminus \text{pt} \cong D^n \setminus 0 \simeq S^{n-1}$$

$$S^n \setminus \{3 \text{ points}\} \xrightarrow{\text{def. retr.}} \text{figure-eight} \xrightarrow{\text{ret.}} S^{n-1} \vee S^{n-1}$$



(Warning check these separately, not true that $X \simeq Y$ implies $X \setminus \text{point} \simeq Y \setminus \text{point}$.

(e.g. consider $X = \mathbb{R}^2, Y = \mathbb{R}, X \simeq Y \simeq \text{pt}, X \setminus \text{point} \simeq S^1, Y \setminus \text{point} \simeq 2 \text{ points}$)

Excision theorem

$E \subseteq A \subseteq X$ subspaces $\Rightarrow (X \setminus E, A \setminus E) \rightarrow (X, A)$ induces iso

with $\bar{E} \subseteq A^\circ$

$$H_*(X \setminus E, A \setminus E) \cong H_*(X, A)$$

Proof later.

Example $X = S^1 \vee S^1 =$ $\supseteq A =$ $\supseteq E =$ $\cong S^1$

$$\Rightarrow H_1(X, A) \cong \underbrace{H_1(\text{red loop}, \text{red loop})}_{\text{exc. thm.}} \cong \underbrace{H_1(D^1, \partial D^1)}_{\text{hpy invce}} \cong \underbrace{\tilde{H}_0(S^0)}_{2 \text{ points}} \cong \mathbb{Z}$$

Example Invariance of dimension from chapter 0 also holds if replace $\mathbb{R}^n, \mathbb{R}^m$ by non-empty open sets $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ because for $p \in U$:

$$H_*(U, U-p) \cong \underbrace{H_*(\mathbb{R}^n, \mathbb{R}^n-p)}_{\text{excise } \mathbb{R}^n \setminus U} \cong \underbrace{\tilde{H}_{*-1}(\mathbb{R}^n-p)}_{\text{LES of pair using } H_*(\mathbb{R}^n)=0} \cong \underbrace{\tilde{H}_{*-1}(S^{n-1})}_{\text{deformation retract } \mathbb{R}^n-p \simeq S^{n-1}}$$

(statement becomes: $U \cong V \Rightarrow n=m$)

Rephrasing of Excision Thm

$$X = A^\circ \cup B^\circ \Rightarrow H_*(X, A) \cong H_*(B, A \cap B)$$

($A, B \subseteq X$ subspaces)

induced by inclusion $(X, A) \leftarrow (B, A \cap B)$



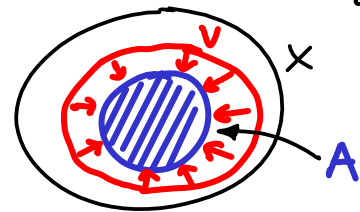
(check $\bar{E} \subseteq A^\circ$: $E = X \setminus B \subseteq X \setminus B^\circ$ closed in $X \Rightarrow \bar{E} \subseteq X \setminus B^\circ \subseteq A^\circ$)

Pf Take $E = X \setminus B$ so $X \setminus E = B$ and $A \cap B = A \setminus E$. \square

Idea why excision holds: $C_*(A) + C_*(B) \rightarrow C_*(X)$ is a homotopy equivalence and $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea \uparrow can subdivide chains in X many times, and small enough chains lie either in A or in B (or in both).

Good pairs and quotients For (X, A) pair :

- Quotient $X/A = X/\sim$ ← equivalence relation $x \sim y \Leftrightarrow \begin{cases} x, y \in A \\ \text{(or)} \\ x = y \end{cases}$
- (X, A) good pair if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \\ A \text{ deformation retract} \\ \text{of nbhd } V \text{ of } A \end{cases}$



Example $X = S^1 \vee S^1 = \bigcirc \cup \bigcirc \supseteq V = \text{red } \bigcirc \cup \bigcirc \supseteq A = \bigcirc \cong S^1$
 $X/A \cong \bigcirc$ ← (all points of A are identified with the node)

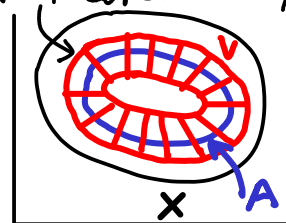
Non-example Topologist's sine curve

$\{(x, \sin \frac{1}{x}) : x \in (0, 1]\} \cup \underbrace{0 \times [0, 1]}_A \subseteq \mathbb{R}^2$ $A \rightarrow$
 not a good pair. (connected, not path-connected, not locally connected, not locally path-connected)

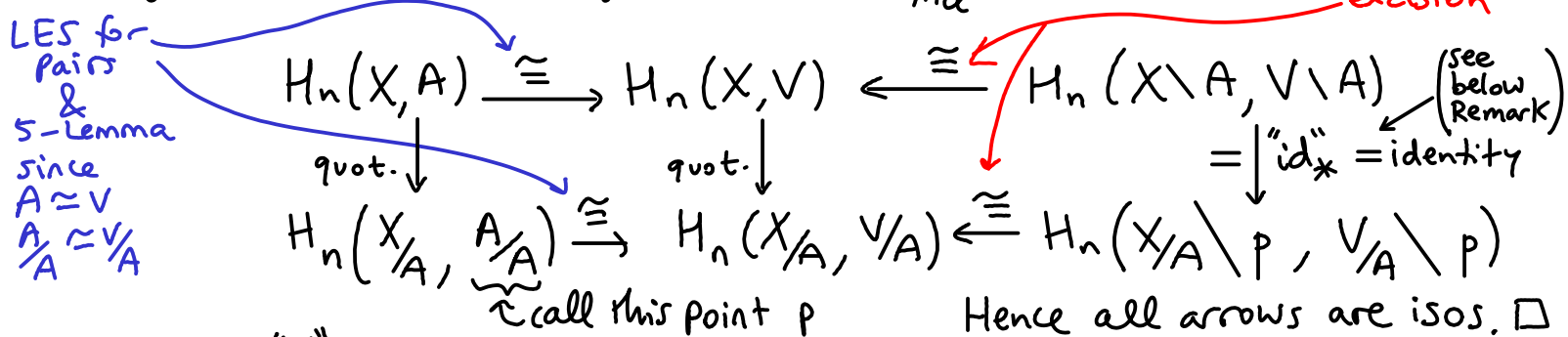
Cultural Rmk Smooth submanifold \subseteq Smooth manifold is a good pair. (by tubular neighbourhood theorem)

Cor (X, A) good $\Rightarrow (X, A) \rightarrow (X/A, \text{pt})$ induces iso

$$H_*(X, A) \rightarrow H_*(X/A, \text{pt}) = \tilde{H}_*(X/A)$$



Pf good $\Rightarrow \exists$ nbhd V of A , and $A \xrightarrow{\text{incl}} V$.



Remark $X \setminus A \xrightarrow{\text{"id"}} X/A \setminus p$, $x \mapsto [x] = \{x\}$ is essentially an identity map but we need to check it is a homeomorphism. The composite $X \setminus A \xrightarrow{\text{cts}} X \xrightarrow{\text{cts}} X/A$ is cts and lands in subspace $X/A \setminus p$, but also need $X \setminus A \rightarrow X/A \setminus p$ to send open sets to open sets. Since A is closed, open subsets $U \subseteq X \setminus A$ are open in X and yield an open set $U/A \subseteq X/A$ not containing p , hence "id"(U) = $U/A \subseteq X/A \setminus p$ open.

Example $\mathbb{D}^n \supseteq S^{n-1}$ good:
 $\Rightarrow H_*(\mathbb{D}^n, S^{n-1}) \xrightarrow{\text{Cor}} \tilde{H}_*(\mathbb{D}^n/S^{n-1}) \cong \tilde{H}_*(S^n)$ $\mathbb{D}^n/S^{n-1} \cong S^n$

Exercise Check that the iso in the Cor is natural (for a map $(X, A) \rightarrow (Y, B)$ of good pairs get comm. diagram...)

Recall we proved $H_*(\mathbb{D}^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})$ (from LES & $\tilde{H}_*(\mathbb{D}^n) = 0$)

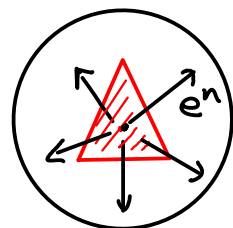
\Rightarrow inductively, using Example $\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) \cong \dots \cong \tilde{H}_{k-n}(S^0) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$

$H_0(2 \text{ pts}) = \mathbb{Z} \oplus \mathbb{Z}$

Generator of $H_n(S^n) \cong \tilde{H}_n(\mathbb{D}^n/S^{n-1}) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$

Observe \exists homeo $e^n: \Delta^n \cong \mathbb{D}^n$ (homework)

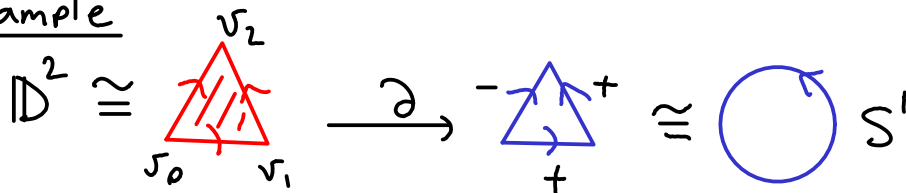
inducing Δ -cx structure on S^{n-1} :



stretch ctly outwards from barycentre (Δ^n)

$$\partial \Delta^n \cong \partial \mathbb{D}^n = S^{n-1}$$

Example



Upshot ($n \geq 2$)

$$\begin{aligned} H_n(\mathbb{D}^n, S^{n-1}) &= \mathbb{Z} \cdot e^n \\ H_{n-1}(S^{n-1}) &= \mathbb{Z} \cdot \partial e^n \\ \tilde{H}_n(\mathbb{D}^n/S^{n-1}) &= \mathbb{Z} \cdot [e^n] \end{aligned}$$

LES for $n-1 \geq 1$, so $n \geq 2$
by Cor $[e^n]$ really lives in $H_n(\mathbb{D}^n, S^{n-1}) \cong H_n(\mathbb{D}^n/S^{n-1}, S^{n-1}/S^{n-1})$

Exercise Recall another Δ -cx structure on S^n :



$$S^n = \Delta_1 \cup \Delta_2 / \text{glue along } \partial \Delta_1$$

call this Δ_1 this Δ_2

then $H_n S^n = \mathbb{Z} \cdot (\Delta_1 - \Delta_2)$ because $\Delta_1 - \Delta_2$ is a cycle and $H_n(S^n) \cong H_n(S^n, \Delta_2) \cong H_n(\Delta_1, \partial \Delta_1) \cong H_n(\mathbb{D}^n, \partial \mathbb{D}^n)$

$\Delta_1 - \Delta_2 \xrightarrow{\text{by LES for } (S^n, \Delta_2) \text{ using } n \geq 1} \Delta_1 \xrightarrow{\text{generator}} e^n$

Another remark about orientations

Fact $\{\text{homeos } \Delta^n \rightarrow \mathbb{D}^n\}$ has 2 path-components

Above we chose a path-component by constructing e^n .

If r is any reflection in \mathbb{R}^{n+1} then $e^n \circ r$ is in the other path-component

$$\begin{aligned} H_n(S^n) &\cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z} \\ e^n &\longmapsto +1 \\ e^n \circ r &\longmapsto -1 \end{aligned}$$

e.g. swap 2 coordinates in Δ^n

We will see later in the course that this corresponds to a choice of orientation of D^n and S^n .

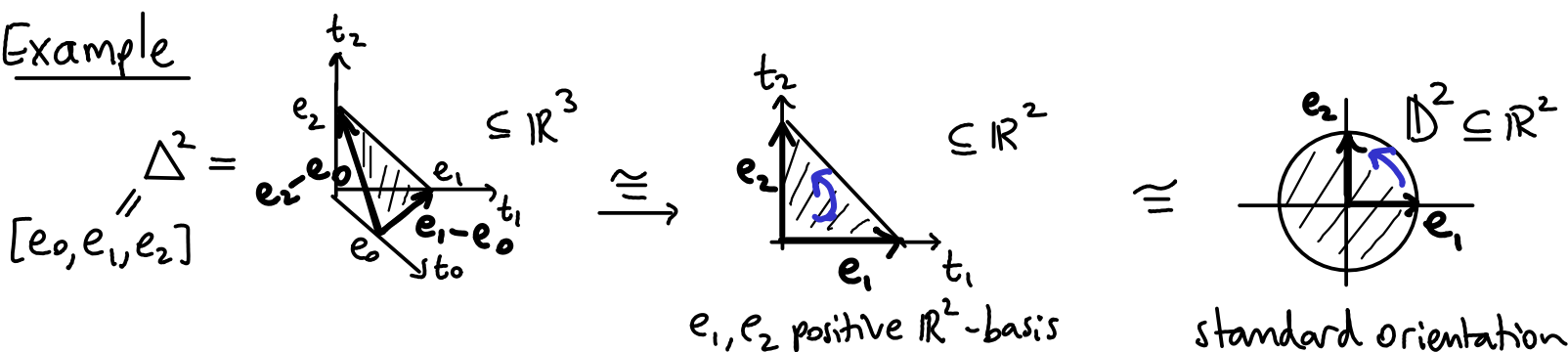
Our choice is consistent with the inclusion $D^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of \mathbb{R}^n) and the inclusion

$$(\Delta^n \subseteq \mathbb{R}^{n+1}) \xrightarrow{\cong} \{(t_1, \dots, t_n) \in \mathbb{R}^n : t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$$

$$(\underline{t_0}, \dots, \underline{t_n}) \mapsto (\underline{t_1}, \dots, \underline{t_n})$$

$t_i \geq 0, \sum t_i = 1$

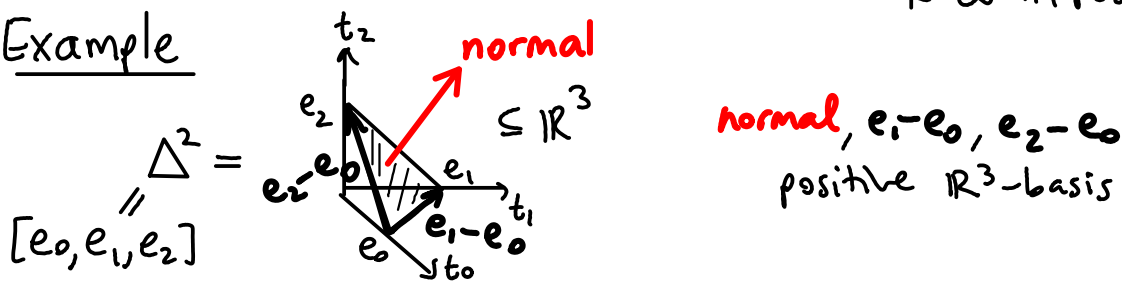
Example



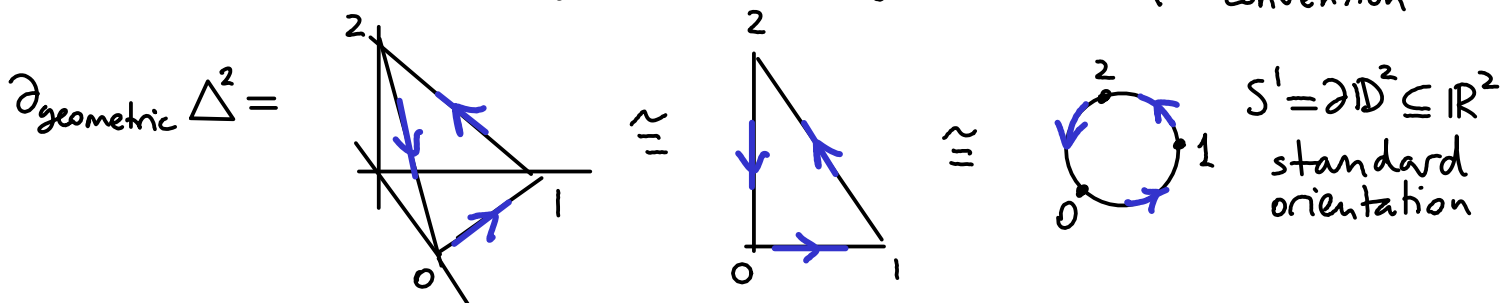
Our choice is also consistent with the "normal first" convention for orienting hyperplanes with a given choice of normal:

$\Delta^n \subseteq$ hyperplane $\{(t_0, \dots, t_n) : \sum t_j = 1\} \subseteq \mathbb{R}^{n+1}$ normal $(1, 1, \dots, 1)$ (so pointing to ∞ in positive quadrant)

Example



Consistent also with the geometric boundary orientation (outward normal first) convention



Compare $\partial \Delta = +[\hat{e}_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$

This $-[e_0, e_2]$ is not equal to singular chain $[e_2, e_0]$ since they are different maps and we take free abelian group generated by maps. But $[e_0, e_2] + [e_2, e_0]$ is homologous to 0 (Homework).

Locality (or "small simplices theorem")

$\mathcal{U} = \{ \text{subspaces } U_i \subseteq X \}$ whose interiors cover X :
 $X = \bigcup U_i^\circ$

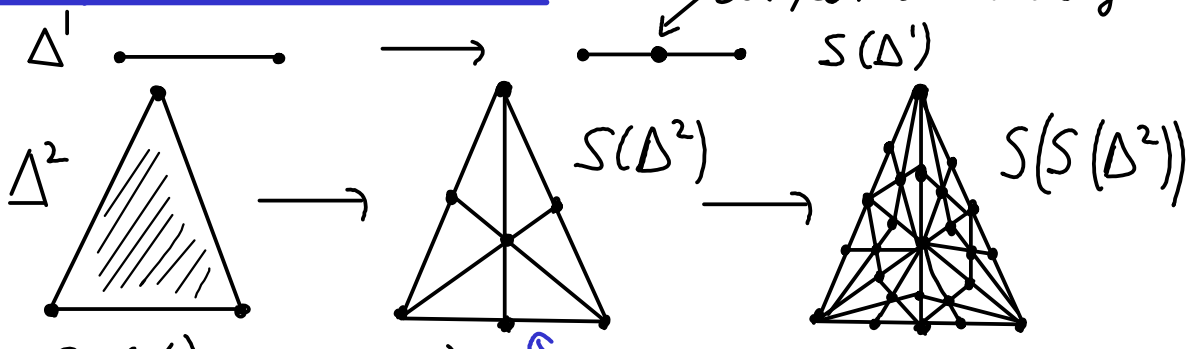
Def $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$ subcx generated by n -simplices σ with $\sigma(\Delta^n) \subseteq U_i$ some i

Theorem $H_* (C_*^{\mathcal{U}}(X)) \cong H_* (C_*(X)) = H_* X$

barycentre of $[v_0, \dots, v_n]$ is $\frac{1}{n+1}(v_0 + \dots + v_n)$
 barycentre divides edge in 2

Sketch Pf ① Barycentric subdivision

Non-examinable



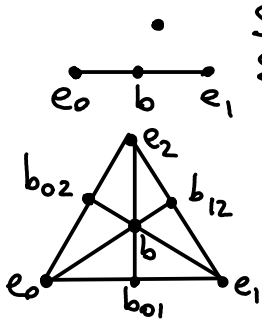
⇒ chain map $S: C_*(X) \rightarrow C_*(X)$
 $\sigma \mapsto \sigma \circ S$

and $S(C_*^{\mathcal{U}}) \subseteq C_*^{\mathcal{U}}$

subdivide the boundary (inductively by dimension) then draw the new faces obtained by convex combinations involving the new vertices and the barycentre

Construction of " $\sigma \circ S$ " is inductive:

On linear simplices (then for maps σ you restrict $\sigma|_{\dots}$)



geometrically $e_0 \xrightarrow{-} b \xrightarrow{+} e_1$
 (= " $[b, S\partial[e_0, e_1]]$ ")

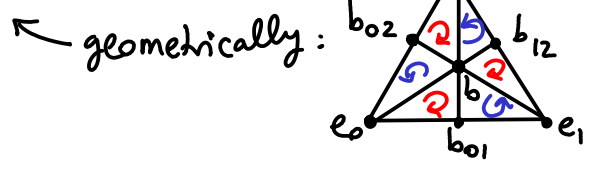
$$S[e_0] = [e_0]$$

$$S[e_0, e_1] = [b, e_1] - [b, e_0]$$

$$S[e_0, e_1, e_2] = "[b, S\partial[e_0, e_1, e_2]]"$$

$$= "[b, S[e_1, e_2]] - [b, S[e_0, e_2]] + [b, S[e_0, e_1]]"$$

$$= ([b, b_{12}, e_2] - [b, b_{12}, e_1]) - ([b, b_{02}, e_2] - [b, b_{02}, e_0]) + ([b, b_{01}, e_1] - [b, b_{01}, e_0])$$



so for $\sigma: \Delta^2 \rightarrow X$ you take $S(\sigma) = \sigma|_{[b, b_{12}, e_2]} - \sigma|_{[b, b_{12}, e_1]} - \dots$

② S chain hpic to id:

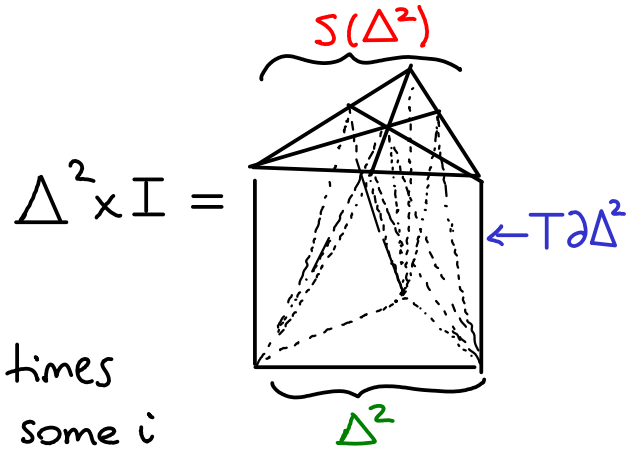
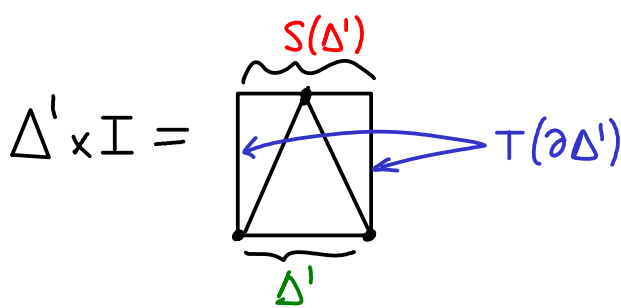
$$T: C_n(X) \rightarrow C_{n+1}(X)$$

$$T(\sigma): \Delta^n \times I \xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X$$

exercise: $\partial T + T\partial = S - id$

$$S_*: H_*(X) \xrightarrow{id} H_*(X)$$

Idea:



③ $\forall n$ -simplex $\sigma: \Delta^n \rightarrow X$, apply $S(\cdot)$ enough times until $\sigma(\text{each } n\text{-simplex of subdivision}) \subseteq U_i$ some i

\forall cycle $c, \exists n$ s.t. $S^n(c) \in C_*^U(X)$ cycle
 $\Rightarrow H_*^U(c) \rightarrow H_*(X)$ surjective

$[S^n(c)] \mapsto S_*^n[c] = [c]$ by ②

\forall bdry $c = \partial b, \exists n$ s.t. $S^n(b) \in C_*^U(X)$

claim: $H_*^U(c) \rightarrow H_*(X)$ injective

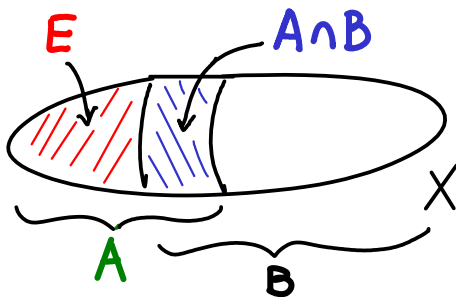
suppose $[c] \mapsto 0$ then $c = \partial b$ for $b \in C_*(X)$

now $S^n c, S^n b \in C_*^U(X)$ for large n

$\Rightarrow \partial S^n b = S^n \partial b = S^n c$ in $C_*^U(X)$

$\Rightarrow [c] \stackrel{\text{②}}{=} S_*^n[c] = [S^n c] = [\partial S^n b] = 0$ in $H_*^U(X) \checkmark \square$

Proof of excision theorem



Let $B = X \setminus E$

use $\mathcal{U} = \{A, B\}$

so $C_*^U(X) = C_*(A) + C_*(B) \subseteq C_*(X)$

$$\Rightarrow \frac{C_*(X \setminus E)}{C_*(A \setminus E)} = \frac{C_*(B)}{C_*(A \cap B)} \cong \frac{C_*(B)}{C_*(A) \cap C_*(B)} \cong \frac{C_*^U(X)}{C_*(A)}$$

\Rightarrow Compare LES's:

$H_*(X \setminus E, A \setminus E)$

$\cong \leftarrow$ by above isos

\uparrow 2nd isomorphism theorem for groups

$$H_*(A) \rightarrow H_*(C_*^U X) \rightarrow H_*(C_*^U X / C_* A) \rightarrow H_{*-1}(A) \rightarrow H_{*-1}(C_*^U X)$$

\parallel locality $\downarrow \cong$

\downarrow iso by 5-lemma \parallel

locality $\downarrow \cong$

$$H_*(A) \rightarrow H_*(X) \rightarrow H_*(C_* X / C_* A) \rightarrow H_{*-1}(A) \rightarrow H_{*-1}(X)$$

\parallel
 $H_*(X, A)$

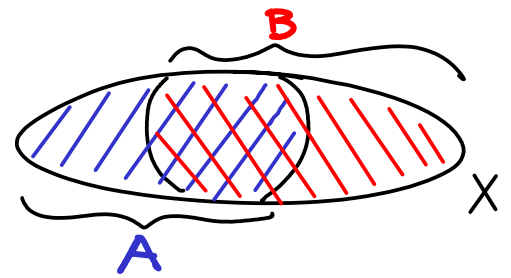
(we are using naturality of LES's induced by SES's)

\square

6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$$X = A \cup B \text{ s.t. } X = A^\circ \cup B^\circ$$

↙ ↘
any subspaces



MV Theorem \exists LES :

$$\boxed{\dots \rightarrow H_*(A \cap B) \xrightarrow{i_*} H_*(A) \oplus H_*(B) \rightarrow H_*(X) \rightarrow H_{*-1}(A \cap B) \xrightarrow{i_*[-1]} \dots}$$

& same holds for \tilde{H}_* provided $A \cap B \neq \emptyset$.

Pf SES $0 \rightarrow C_*(A \cap B) \rightarrow C_*(A) \oplus C_*(B) \rightarrow C_*^U(X) \rightarrow 0$

$\sigma \longmapsto (\sigma, -\sigma)$
 $(\alpha, \beta) \longmapsto \alpha + \beta$

\Rightarrow induces the LES (using locality $H_*^U(X) \cong H_*X$). \square

Exercise connecting map is $\delta: H_*(X) \rightarrow H_{*-1}(A \cap B)$

$$[\alpha + \beta] \longmapsto [\partial\alpha] = -[\partial\beta]$$

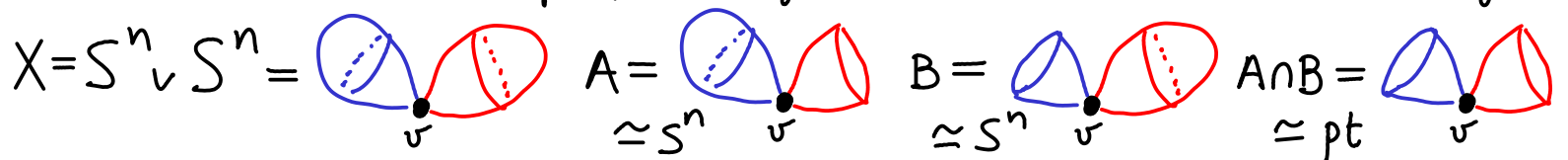


$$\dots \rightarrow H_2(pt) \oplus H_2(pt) \rightarrow H_2S^2 \rightarrow H_1(S^1) \rightarrow H_1(pt) \oplus H_1(pt) \rightarrow \dots$$

$\begin{matrix} \parallel \\ 0 \end{matrix}$ $\begin{matrix} \uparrow \\ \text{hence } \mathbb{Z} \end{matrix}$ $\begin{matrix} \parallel \\ \mathbb{Z} \end{matrix}$ $\begin{matrix} \parallel \\ 0 \end{matrix}$

Exercise Compute $H_*S^n = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ using MV

Example wedge sum of X, Y with basepoints $x \in X, y \in Y$

$$X \vee Y = \frac{X \sqcup Y}{x \sim y}$$


$$0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} H_n(X) \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{\mapsto (1, -1)} \mathbb{Z} \oplus \mathbb{Z} \rightarrow H_0(X) \xrightarrow{\cong} \mathbb{Z} \rightarrow 0$$

Similarly $\boxed{H_*(X \vee Y) \cong H_*(X) \oplus H_*(Y)}$ for $* \neq 0$ if \exists contractible nbhds of $x \in X, y \in Y$.

Cones and suspensions

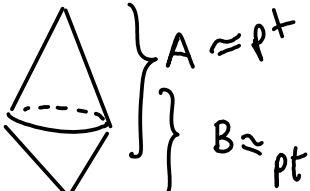
$$\text{Cone}_X = (X \times [0,1]) / (x,s) \sim (y,t) \text{ iff equal or } s=t=1$$

$$\Sigma X = (X \times [0,1]) / (x,s) \sim (y,t) \text{ iff equal}$$

Example $CS^n \cong \mathbb{D}^{n+1}$, $\Sigma S^n \cong S^{n+1}$.

or $s=t=0$
or $s=t=1$

Lemma $\tilde{H}_*(\Sigma X) \cong \tilde{H}_{*-1}(X)$

Pf  $A \cong pt$, $B \cong pt$, $A \cap B \cong X$ now apply MV. \square

Rmk $\phi \neq A \subseteq X \Rightarrow \tilde{H}_*(X \cup_A CA) \stackrel{LES}{\cong} H_*(X \cup_A CA, CA) \stackrel{exc.}{\cong} H_*(X, A)$

Connected sum

identify $a \in A \subseteq X$ with $(a,0) \in CA$

M, N connected n -manifolds $\Rightarrow M \# N = (M \setminus \text{open } n\text{-ball}) \cup (N \setminus \text{open } n\text{-ball})$

identify ∂ balls via a homeo



Fact compact connected orientable surfaces are homeo to S^2 or $T^2 \# \dots \# T^2$
and " " non-orientable ones: $\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$.

genus $g=0$
 $g = \# \text{ copies}$
called Σ_g

Exercise (Homework) For M, N compact connected n -mfd's:

By MV, $H_*(M \# N) \cong H_*(M) \oplus H_*(N)$ for $1 \leq * \leq n-2$

If M or N orientable: $* = n-1$ also works

If both non-orientable: $* = n-1$ one of $\mathbb{Z}/2$ summands becomes \mathbb{Z}

Cor 1) $\chi(M \# N) = \chi(M) + \chi(N) - \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{" " odd} \end{cases}$

2) $H_*(\Sigma_g) \leftarrow \text{genus } g \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \end{cases}$

$\chi(S^n)$

$H_0(M \# N) \cong \mathbb{Z}$
since connected
fact:
 $H_n(M \# N)$ is \mathbb{Z} or 0
 \uparrow else
if M, N both orientable
(see later in course)

7. DEGREE OF MAPS OF SPHERES

← ASSUME $n \geq 1$ IN THIS CHAPTER

$$f: S^n \rightarrow S^n \implies f_n: H_n S^n \rightarrow H_n S^n$$

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad} & \mathbb{Z} \\ \parallel & & \parallel \\ 1 & \xrightarrow{\quad} & \text{deg}(f) \end{array}$$

$$\implies f_*: \tilde{H}_* S^n \rightarrow \tilde{H}_* S^n \text{ is } \text{deg}(f) \cdot \text{id}$$

$$1 \longmapsto \text{deg}(f) \in \mathbb{Z}$$

Properties 1) $\text{deg}(\text{id}) = 1$

2) $\text{deg}(f \circ g) = \text{deg} f \cdot \text{deg} g$

3) $f \simeq g \implies \text{deg} f = \text{deg} g$

4) $f \simeq \text{const} \implies \text{deg} f = 0$

5) f homeomorphism $\implies \text{deg} f = \pm 1$

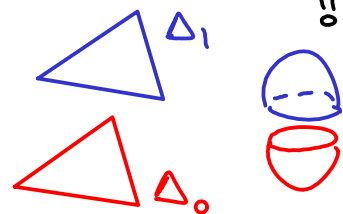
← (sign depends on whether f is orientation-preserving or reversing)

Pf

$\text{id}_* = \text{id}$, $(f \circ g)_* = f_* \circ g_*$, $f \simeq g \implies f_* = g_*$, $\text{const}_* = 0$, f homeo $\implies f_n$ iso. \square

Examples

1) $S^n = \Delta_1^n \times 1 \cup \Delta_0^n \times 0 \leftarrow \text{call this } \Delta_0$
 call this Δ_1 $(b, 1) \sim (b, 0)$ if $b \in \partial \Delta$



recall $H_n S^n = \mathbb{Z} \cdot (\Delta_1 - \Delta_0)$

reflection: $r: S^n \rightarrow S^n$, $r(x, t) = (x, 1-t)$

so $\Delta_0 \leftrightarrow \Delta_1$ swapped by r , so $r_*(\Delta_1 - \Delta_0) = -(\Delta_1 - \Delta_0)$

$\implies \text{deg}(r) = -1$

2) antipodal map $-id: S^n \rightarrow S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$

$\implies \text{deg}(-id) = (-1)^{n+1}$

Pf $-id = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} \circ \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{pmatrix} \circ \dots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ composition of $n+1$ reflections each homotopic to r . \square

3) $A \in O(n) \implies A: S^{n-1} \rightarrow S^{n-1} \implies \text{deg} A = \det A \in \{\pm 1\}$

Pf fact $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq \text{id}$ so $\text{deg} A = \det A = +1$

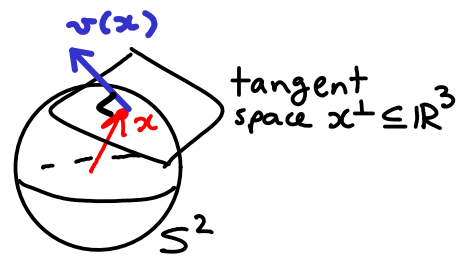
The other path-component of $O(n)$ is $r \circ SO(n)$ where r is any reflection. \square

4) f not surjective $\implies \text{deg} f = 0$

Pf If $y \notin \text{Im} f \implies H_n(S^n) \xrightarrow{f_*} H_n(S^n)$
 $f_* \searrow \quad \nearrow f_*$
 $H_n(S^n \setminus y) \cong H_n(\mathbb{R}^n) = 0$ \square

Application to vector fields on S^n

$v: S^n \rightarrow \mathbb{R}^{n+1}$ tangent vector field on S^n
 so $v(x) \perp x$



Cor Hairy ball theorem \exists nowhere zero v.f. on $S^n \iff n$ odd

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \quad \forall x$

\Rightarrow hpy $F: S^n \times [0,1] \rightarrow S^n$

$$F(x,t) = \cos(\pi t)x + \sin(\pi t) \frac{v(x)}{\|v(x)\|}$$

$\Rightarrow F_0 = id, F_1 = -id$

$\Rightarrow 1 = dy F_0 = dy F_1 = (-1)^{n+1}$

$\Rightarrow n$ odd

For n odd $\exists v(x) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}) \in \mathbb{R}^{2k}$. \square

Cultural Remark Adams in 1962 proved using alg. topology:

(max # pointwise linearly independent vector fields on S^n) = $2^b + 8a - 1$
 where $n+1 = 2^{4a+b}$. (odd number), $0 \leq b \leq 3, a, b \in \mathbb{N}, n \geq 1$.

get 0 if n even
 \Rightarrow Cor \checkmark

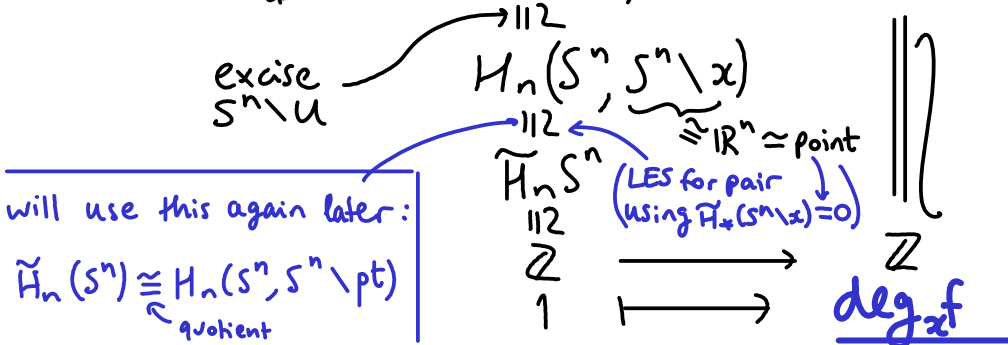
Local degree $f: S^n \rightarrow S^n$ ($n \geq 1$)
 $x \rightarrow y = f(x)$

\star Suppose points $\neq x$ near x do not map to y :

\exists nbhds $x \in U, y \in V$ s.t. $(U, U \setminus x) \xrightarrow{f} (V, V \setminus y)$

$\Rightarrow (f|_x)_* : H_n(U, U \setminus x) \xrightarrow{f_*} H_n(V, V \setminus y)$

call this $f|_x$
local map at x



will use this again later:
 $\tilde{H}_n(S^n) \cong H_n(S^n, S^n \setminus pt)$
 quotient

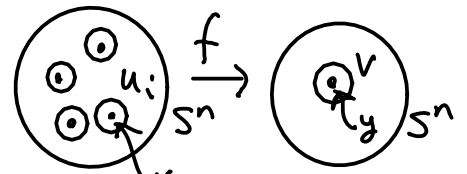
Rmk By left vertical arrows:
 $H_n(U, U \setminus x) \cong \mathbb{Z}$
 again we made a choice of iso by picking a generator "1" by combining excision, quotient and choice of $\tilde{H}_n(S^n) \cong \mathbb{Z}$.

Lemma $f: S^n \rightarrow S^n$, $f^{-1}(y) = \{x_1, \dots, x_k\}$ ($n \geq 1$)

$$\Rightarrow \boxed{\deg f = \sum \deg_{x_i} f}$$

Pf

$$\begin{array}{ccc}
 H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\
 \downarrow \text{quotient} & & \downarrow \text{quotient} \\
 H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus \{y\}) \\
 \downarrow \text{exc. } S^n \setminus \cup U_i & & \downarrow \text{exc. } S^n \setminus V \\
 \bigoplus H_n(U_i, U_i \setminus \{x_i\}) & \xrightarrow{\bigoplus (f|_{x_i})_*} & H_n(V, V \setminus \{y\})
 \end{array}$$



Rmk
 can use same V for all i by taking $\tilde{V} = \cap U_i$
 $\tilde{U}_i = f^{-1}(V) \cap U_i$

(the 2 squares commute:
 1st: quotient is natural
 2nd: excision is natural)

map to each summand is exc. of $S^n \setminus U_i$ so iso.

$$\begin{array}{ccc}
 \text{is: } 1 \in \mathbb{Z} & \xrightarrow{\deg f} & \mathbb{Z} \\
 \downarrow & & \downarrow \text{id} \\
 (1, \dots, 1) \in \bigoplus_i \mathbb{Z} & \xrightarrow{\bigoplus \deg_{x_i} f} & \mathbb{Z}
 \end{array}$$

□ (see Rmk on previous page about choice of "1")

Example $p: \mathbb{C} \rightarrow \mathbb{C}$ polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_i \in \mathbb{C}$
 $\Rightarrow f: S^2 = \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 = S^2$ (where view $\mathbb{CP}^1 = \mathbb{C} \cup \infty \cong S^2$)
 $\begin{array}{ccc} z & \mapsto & p(z) \\ \infty & \mapsto & \infty \end{array}$ stereographic projection

\Rightarrow hpy $F(z, t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$
 $F_0 = a_n z^n$ and $F_1 = f$
 hpy is continuous at ∞ since $a_n z^n$ dominates other terms: $F^{-1}(\mathbb{CP}^1 \setminus K) = \mathbb{CP}^1 \setminus (\text{some compact set}) \forall$ compact K .
 this would fail if you tried to homotope $t(a_n z^n) + a_{n-1} z^{n-1} + \dots$

$$\begin{aligned}
 \Rightarrow \deg f &= \deg(a_n z^n) \stackrel{\text{Lemma}}{=} \sum_{k=0}^{n-1} \deg_{w_k} a_n z^n \leftarrow w_k = e^{\frac{2\pi i k}{n}} \\
 &= n \\
 &= \text{degree of the poly } p.
 \end{aligned}$$

$\underbrace{w_k}_{=1}$ orientⁿ preserving homeo near w_k

Cor (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root
Pf $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0 \not\geq 1 \Rightarrow \square$

holomorphic maps are always orientation preserving: 1st order approximation is the derivative so $re^{i\theta}$ so rescale > 0 & rotate so preserve anticlockwise orientⁿ

Cultural Rmk For smooth $f: S^n \rightarrow S^n$
 counted with orientation signs
 $\deg f =$ (the number of preimages) of a generic point
 (i.e. almost any point works)

Example $S^2 \rightarrow S^2$
 $S^2 \setminus \text{North pole} \cong \mathbb{C}$
 map $\mathbb{C} \rightarrow \mathbb{C}$, $z \mapsto z^d$ and North \mapsto North
 $\Rightarrow \deg = d = \#$ preimages of a point
 except if pick North/South pole
 $\leftarrow (z = \infty) \leftarrow (z = 0)$

8. CELLULAR HOMOLOGY

Def CW complex X is sequence $\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$

s.t. X^0 is any set with discrete topology.

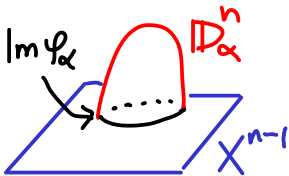
n-skeleton

$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_n} D_\alpha^n$$

$$x \sim \varphi_\alpha(x)$$

closed n -discs labelled by some index set I_n

$\varphi_\alpha: \partial D^n \rightarrow X^{n-1}$
attaching map
(any continuous map, often not injective)

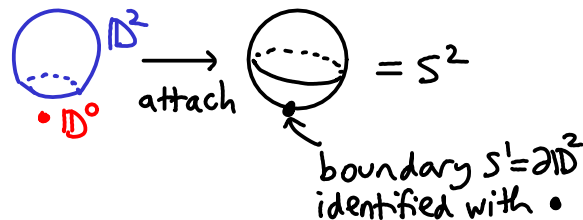


$\Rightarrow X = \bigcup_{n \geq 0} X^n$ top-space with weak topology:

$U \subseteq X$ open $\iff U \cap X^n \subseteq X^n$ open $\forall n$.
($\iff U \cap D_\alpha^n \subseteq D_\alpha^n$ open $\forall n, \alpha$)

Call X n-dimensional if $X = X^n$ and this is the least such n .

Example $S^n = (D^0 \sqcup D^n) / (D^0 \sim \partial D^n)$



Example $X = \mathbb{R}P^2 =$

$X^0 = \bullet = D^0$

$X^1 = \bullet \xrightarrow{\varphi_1} \bullet = S^1 = (D^0 \sqcup D^1) / (\bullet \sim \varphi_1(x)), \partial D^1 = S^0 = \{0, 1\} \xrightarrow{\varphi_1} X^0 = \bullet$

$X^2 = (\bullet \xrightarrow{\varphi_1} \bullet \sqcup \text{shaded disc}) / (\text{wrap } \partial \text{ of shaded disc twice around } \bullet)$

$= (X^1 \sqcup D^2) / (\partial D^2 = S^1 \xrightarrow{\varphi_2} X^1 = S^1, \varphi_2(z) = z^2, z \sim z^2)$

Fact If we homotope φ_α , we get a homotopy equivalent space

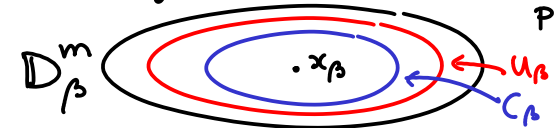
Example If we use another degree 2 map φ_2 above, get $X \simeq \mathbb{R}P^2$.

Cultural Rmk Every CW-cx X is hpy equivalent to a simplicial complex Y (so in particular a Δ -cx). [If X finite/ n -dim then can ensure Y is finite/ n -dim]

X is partitioned as a set by interiors of n-cells $e_\alpha^n = \text{Image}(D_\alpha^n \rightarrow X)$

$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_n} e_\alpha^n = \left(\bigsqcup_{\alpha \in I_0} e_\alpha^0 \right) \sqcup \left(\bigsqcup_{\alpha \in I_1} e_\alpha^1 \right) \sqcup \left(\bigsqcup_{\alpha \in I_2} e_\alpha^2 \right) \sqcup \dots$ (Rmk interior $D^0 = D^0$ so $e_\alpha^0 = e_\alpha^0$)

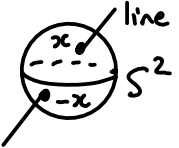
Why "CW"? "W" = weak topology, "C" = closure-finite property: each D_α^n only hits finitely many interiors of other cells. By contradiction, suppose false. Then by construction it means that the image $M := \varphi_\alpha(\partial D_\alpha^n) \subseteq X^{n-1}$ contains some point $x_\beta \in \text{int}(D_\beta^m), m < n$, for infinitely many β .



Pick a closed neighbourhood C_β of x_β inside an open nbhd U_β of x_β so $x_\beta \in C_\beta \subseteq U_\beta \subseteq \text{int}(D_\beta^m)$. Notice M is compact since φ_α is continuous and ∂D_α^n is compact. Pick any open cover $W_j \subseteq X^{n-1}$ of M . Then $W_j \cap (X^{n-1} \setminus C_\beta), U_\beta$ is also an open cover of M , and we cannot find a finite subcover since U_β is the only open set of cover containing x_β .

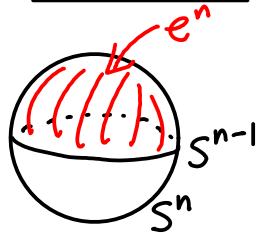
Examples

real projective space $\mathbb{R}P^n = S^n / (\mathbb{Z}/2\text{-action by } \pm \text{id})$



$X^k = \mathbb{R}P^k$ inductively

$X^n = X^{n-1} \cup e^n$ with $\varphi: S^{n-1} \rightarrow X^{n-1} = \mathbb{R}P^{n-1} = S^{n-1} / \pm \text{id}$
 $x \mapsto [x] = [-x]$



complex projective space

$\mathbb{C}P^n = (S^{2n+1} \subseteq \mathbb{C}^{n+1}) / (S^1\text{-action by } \lambda \cdot \text{Id})$ $x \sim \lambda x$ for $\lambda \in S^1 \subseteq \mathbb{C}^*$

$X^0 = X^1 = \text{pt} = \mathbb{C}P^0$

$X^2 = X^3 = X^0 \cup e^2 = \mathbb{C}P^1$, $\varphi: S^1 \rightarrow \text{pt}$ $\mathbb{C}P^1 \cong S^2$

$X^4 = X^5 = X^2 \cup e^4 = \mathbb{C}P^2$, $\varphi: S^3 \rightarrow \mathbb{C}P^1 = S^3 / (S^1\text{-action})$
 $x \mapsto [x] = [\lambda x], \forall \lambda \in S^1$

$X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{C}P^n$, $\varphi: S^{2n-1} \rightarrow X^{2n-2} = \mathbb{C}P^{n-1}$
 $x \mapsto [x]$

In coordinates: $\mathbb{C}P^n = \{ [z_0 : \dots : z_n] : \text{not all } z_i \in \mathbb{C} \text{ are } 0 \}$ and $[z] \sim [\lambda z], \forall \lambda \in \mathbb{C}^*$
 Can rescale so that $\sum |z_i|^2 = 1$ so $z \in S^{2n-1}$ and left with rescaling by $\lambda \in S^1 \subseteq \mathbb{C}^*$.

$\mathbb{C}P^{n-1} \cong X^{2n-2} = \{ [z_0 : \dots : z_{n-1} : 0] \} \subseteq \mathbb{C}P^n = X^{2n}$ and
 $e^{2n}: \mathbb{D}^{2n} = \{ (w_0, \dots, w_{n-1}) : \sum |w_j|^2 \leq 1 \} \rightarrow X^{2n}$ via $[w_0 : \dots : w_{n-1} : \sqrt{1 - \sum |w_j|^2}]$ notice this = 0 if $w \in S^{2n-1} \cong \partial \mathbb{D}^{2n}$

Observe: For X CW complex, for $n \geq 1$: (For $n=0: (X^0, X^{-1}) = (X^0, \emptyset)$
 $X^0 / X^{-1} = X^0$)

(X^n, X^{n-1}) is a good pair (since \exists nbhd of $\partial \mathbb{D}^n$ that deformation retracts to $\partial \mathbb{D}^n$)

$X^n / X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n$ $S_\alpha^n = \mathbb{D}_\alpha^n / \partial \mathbb{D}_\alpha^n$
 X^{n-1} identified to a point

Def Cellular complex for X a CW cx,

$$C_n^{CW}(X) = H_n(X^n, X^{n-1}) \cong H_n(X^n / X^{n-1})$$

= free abelian gp gen. by the n -cells e_α^n

since $\Delta^n \cong \mathbb{D}^n \rightarrow (e^n \in X^n) \rightarrow \mathbb{D}_\alpha^n / \partial \mathbb{D}_\alpha^n = S_\alpha^n$ generate

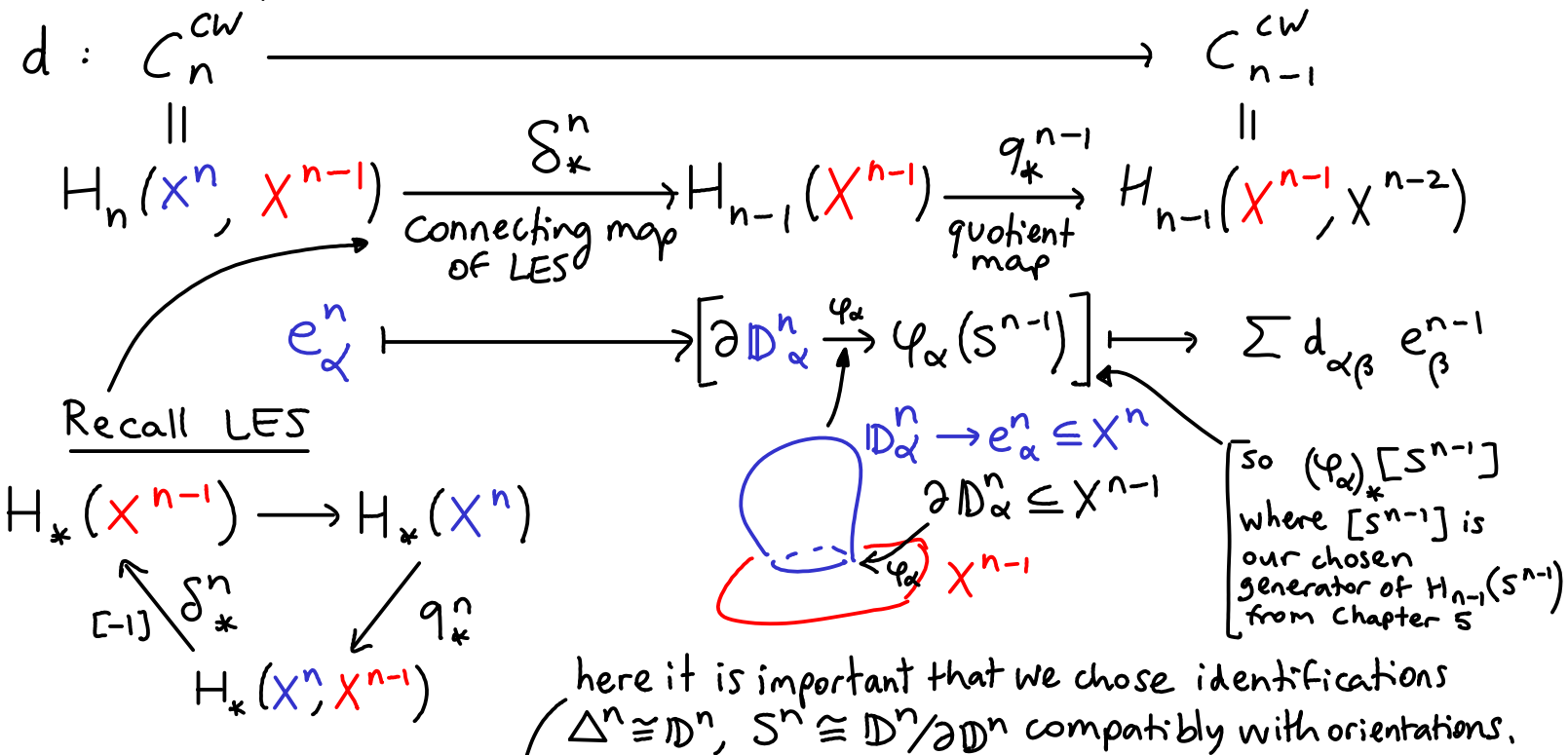
Will build cellular differential $d: C_*^{CW} \rightarrow C_{*+1}^{CW}$, prove $d \circ d = 0$ (as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.)

\Rightarrow get $H_*^{CW}(X) = H_*(C_*^{CW}(X), d)$

Example $C_k^{CW}(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} \cdot e^k & \text{for } k=0,2,4,\dots,2n \\ 0 & \text{else} \end{cases}$ hence $d=0$ so $H_*^{CW}(\mathbb{C}P^n) = C_*^{CW}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \leq * \leq 2n \text{ even} \\ 0 & \text{else} \end{cases}$

Cellular differential:

$d e_\alpha^n = \sum_{\beta \in I_{n-1}} d_{\alpha\beta}^n \cdot e_\beta^{n-1}$ now describe the coefficients $d_{\alpha\beta}^n \in \mathbb{Z}$ and why that is a finite sum.



Therefore:

$d_{\alpha\beta}^n = \deg(S^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1} / X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \rightarrow S^{n-1})$
 $\parallel \partial D_\alpha^n \quad \parallel D_\beta^{n-1} / \partial D_\beta^{n-1}$

Rmk Only finitely many $d_{\alpha\beta}^n \neq 0$ (for fixed α) because φ_α, q are continuous and S^{n-1} compact, so get a compact image in $\bigvee_{\beta} S^{n-1}$, therefore cannot surject onto ∞ many S_β^{n-1} .

Lemma $d \circ d = 0$

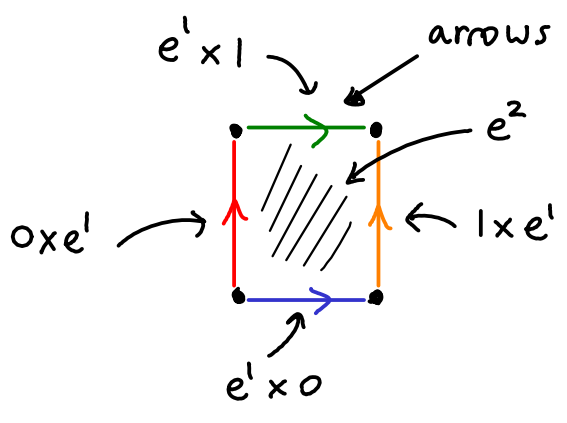
Pf $d_n = q_{n-1}^{n-1} \circ \delta_n^n$
 $d_{n-1} \circ d_n = q_{n-2}^{n-2} \circ \delta_{n-1}^{n-1} \circ q_{n-1}^{n-1} \circ \delta_n^n = 0$ by LES \square

(indeed see proof of "closure-finite": it shows φ_α only hits finitely many interiors of D_β^{n-1})

Cor $\text{rank } H_n^{CW}(X) \leq \# n\text{-cells}$

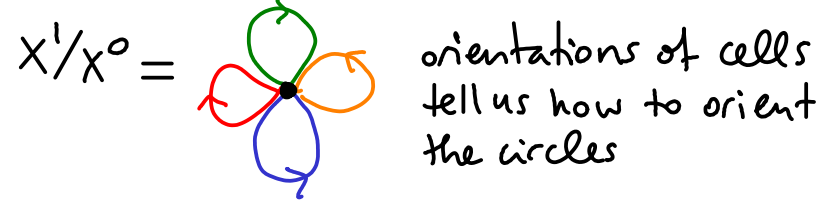
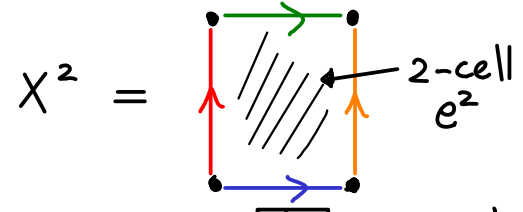
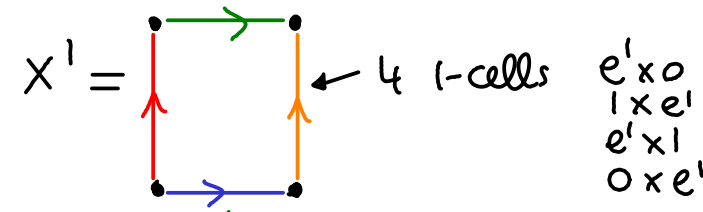
Pf $\# n\text{-cells} = \text{rank } C_n^{CW}(X) \geq \text{rank Ker } d_n^{CW} \geq \text{rank } H_n^{CW}(X) \square$

Example $I \times I$ $I = [0,1]$ $D^1 = [-1,1]$



arrows here tell us how we map $[-1,1] \rightarrow$ edge (so orientation)

$X^0 = \text{four dots} = 4 \text{ 0-cells}$

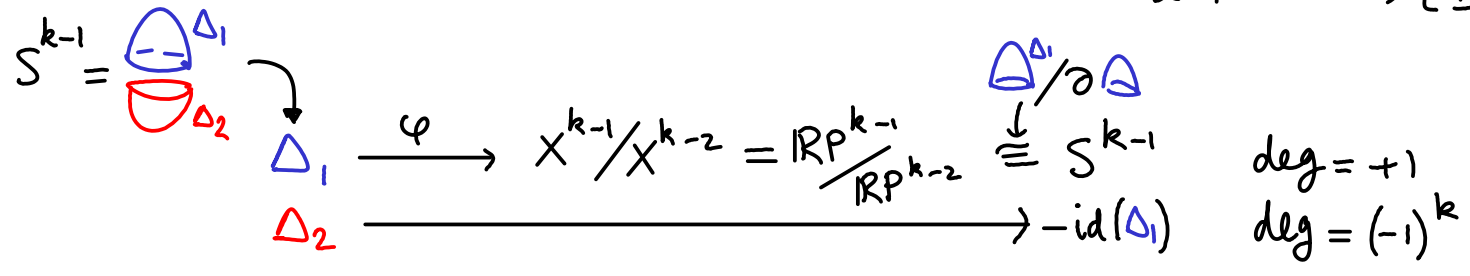


$e^2 : D^2 \cong \text{shaded square} \rightarrow X^1$

$\partial e^2 : S^1 \cong \text{square with arrows} \rightarrow X^1/X^0 =$ degree -1 because top edge of maps to by an orientation-reversing homeomorphism.

$\Rightarrow \partial e^2 = +e^1 x_0 + 1 x e^1 - e^1 x_1 - 0 x e^1$
 $(= (\partial e^1) \times e^1 - e^1 \times (\partial e^1) \leftarrow \text{we come back to this later})$

Example $\mathbb{R}P^n$ recall: 1 cell in each dim, $\varphi : S^k \rightarrow X^k = \mathbb{R}P^k$
 $x \mapsto [\pm x]$



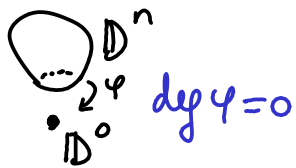
$\Rightarrow d_{\alpha\beta}^k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

$C_*^{CW}(\mathbb{R}P^n) : 0 \rightarrow \mathbb{Z} \xrightarrow{k=n} \mathbb{Z} \rightarrow \dots \rightarrow \mathbb{Z} \xrightarrow{k=1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$
 (with 0 at the bottom of the last \mathbb{Z} and -1 at the bottom of the final 0)

$H_*^{CW}(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0, \text{ also } * = n \text{ if odd} \\ \mathbb{Z}/2 & * = 1, 3, 5, \dots < n \\ 0 & \text{else} \end{cases}$

Example S^n : $C_*^{CW}(S^n): n > 2: 0 \rightarrow \mathbb{Z} \cdot D^n \xrightarrow{0} 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \cdot D^0 \rightarrow 0$

$n=1: 0 \rightarrow \mathbb{Z} \cdot D^1 \xrightarrow{0} \mathbb{Z} \cdot D^0 \rightarrow 0$

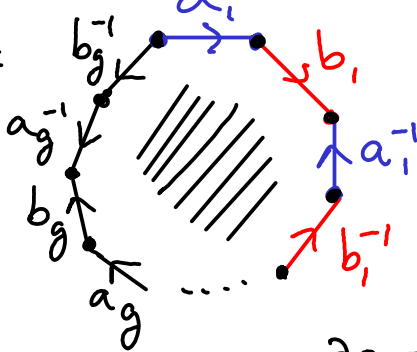


$\text{deg } \varphi = 0 \Rightarrow H_*^{CW}(S^n) \cong \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else} \end{cases}$

$H_1(S', pt) \xrightarrow{\delta} H_0(pt) \xrightarrow{\eta} H_0(pt, \emptyset)$
 $(\Delta^1 \cong [0,1] \rightarrow S^1) \xrightarrow{\sigma} \partial \sigma$
 $\sigma = \text{quotient on } \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = S^1$
 if you work with degrees, need to remember orientations:
 $\partial D^1 \cong \partial [0,1] = [1] - [0] \rightarrow \text{point}$
 so degree = $+1 - 1 = 0$

Example Σ_g

genus g surface



boundary identifications
 $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$

Notice all vertices are identified, call vertex v

$\partial a_i = v - v = 0$
 $\partial b_i = v - v = 0$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}^{2g} \xrightarrow{\partial=0} \mathbb{Z} \rightarrow 0$$

$\mathbb{Z} \cdot D \quad \mathbb{Z} \langle a_i, b_i, a_i^{-1}, b_i^{-1} \rangle \quad \mathbb{Z} \cdot v$

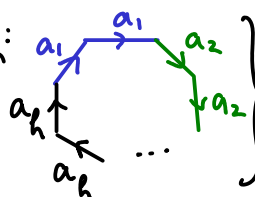
$D \mapsto -a_1 - b_1 + a_1 + b_1 + \dots - a_g - b_g + a_g + b_g = 0$

(signs: compare edge orientation with anticlockwise orientation of ∂D)

Example Non-orientable surface N_h :

$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^h \xrightarrow{0} \mathbb{Z} \rightarrow 0$
 $1 \mapsto (-2, \dots, -2)$

(since $D \mapsto -a_1 - a_1 - a_2 - a_2 - \dots - a_h - a_h$)



$\Rightarrow H_*(N_h) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2 & * = 1 \\ 0 & \text{else} \end{cases}$
 (use the standard basis for \mathbb{Z}^h except replace $(0, \dots, 0, 1)$ by $(1, 1, \dots, 1)$.)

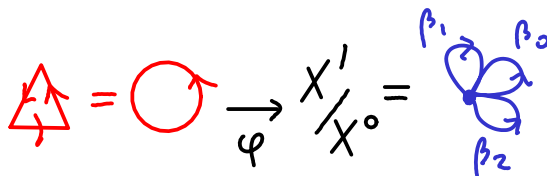
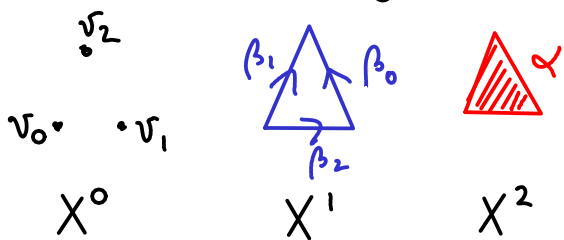
Lemma X Δ -cx structure \Rightarrow induces CW-cx structure on X and

$(C_*^{CW}(X), d^{CW}) \cong (C_*^\Delta(X), d^\Delta)$

$\Rightarrow H_*^{CW}(X) \cong H_*^\Delta(X)$

Pf $X^n = \cup_n$ -simplices of X and degrees are ± 1 depending on orientⁿ so can identify d^{CW} and d^Δ . \square

Example $X = \text{triangle} = \Delta^2$



$d_\alpha \beta_2 = d_\alpha \beta_0 = +1, d_\alpha \beta_1 = -1$

$\Rightarrow d^\Delta \alpha = \beta_0 - \beta_1 + \beta_2$

$\Rightarrow d^{CW} \alpha = d^\Delta \alpha \quad \checkmark \quad \square$

Theorem X CW cx (or Δ -cx) \implies $H_*^{CW}(X) \cong H_*(X)$

$\implies H_*^\Delta, H_*^{CW}$ independent of choice of CW-cx/ Δ -cx structure.

Pf ① $H_*(X^n, X^{n-1}) \cong \tilde{H}_*(X^n/X^{n-1}) \cong \tilde{H}_*(\bigvee_\alpha S^n) \cong \bigoplus_\alpha \tilde{H}_* S^n$
 $= 0 \iff * \neq n$ lives in degree n

LES for $(X^n, X^{n-1}) \implies H_*(X^{n-1}) \rightarrow H_*(X^n)$ iso for $* < n-1$
 $* > n$

② for $* < n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$
 by compactness each sing-chain lands in X^N , some N , by the same proof as in the "closure-finite" property (start of Chapter)

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{-1}) = 0$

④ LES: $\dots \rightarrow H_n(X^{n-1}) \rightarrow H_n(X^n) \xrightarrow{q_n} H_n(X^n, X^{n-1}) \rightarrow \dots$
 \parallel
 0 by ③
 $\implies q_n^n$ injective $\forall n$

⑤ LES: $\dots \xrightarrow{q_{n+1}^{n+1}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}^{n+1}} H_n(X^n) \xrightarrow{q_n^n} H_n(X^{n+1}) \rightarrow 0$
 hence surjective with $\text{Ker} = \text{im } \delta_{n+1}^{n+1}$

UPSHOT

$$H_n^{CW}(X) = \frac{\text{Ker } d_n^{CW}}{\text{Im } d_{n+1}^{CW}} = \frac{\text{Ker } q_{n-1}^{n-1} \circ \delta_n^n}{\text{Im } q_n^n \circ \delta_{n+1}^{n+1}}$$

④ $\text{Ker } \delta_n^n = \text{Im } q_n^n$

$$= q_n^n (H_n(X^n)) / q_n^n (\text{Im } \delta_{n+1}^{n+1})$$

$$\cong H_n(X^n) / \text{Im } \delta_{n+1}^{n+1} \stackrel{\text{⑤}}{\cong} H_n(X^{n+1}) \stackrel{\text{②}}{\cong} H_n(X). \quad \square$$

④ 1^{st} iso thm

Rmk by ① H_k not affected if attach $(k+2)$ -cells or higher

by ② Inclusion $X^n \rightarrow X$ induces iso $H_*(X^n) \rightarrow H_*(X)$ for $* < n$

Cor X n -dimensional cell cx $\implies H_*(X) = 0$ for $* > n$

Axioms for homology: Eilenberg-Steenrod axioms

It was no accident that $H_*^\Delta, H_*^{CW}, H_*^*$ all agreed.

Def A generalised homology theory (GHT)

is a functor $F: \text{TopPairs} = \left(\begin{array}{l} \text{Category of pairs} \\ \text{of spaces, and} \\ \text{maps of pairs} \end{array} \right) \rightarrow \text{Graded Abelian Gps}$

with a natural transformation $\delta: F_*(X, A) \rightarrow F_{*-1}(X, \emptyset)$ satisfying:

1) homotopy invariance: $f \simeq g \Rightarrow F(f) = F(g)$ ← abbreviated: $F_{*-1}(X)$

2) exactness: \exists LES $\boxed{\dots \rightarrow F_*(A) \xrightarrow{F(\text{incl}: A \rightarrow X)} F_*(X) \xrightarrow{F(\text{incl}: (X, \emptyset) \rightarrow (X, A))} F_*(X, A) \xrightarrow{\delta} F_{*-1}(A) \rightarrow \dots}$

3) additivity: $(X, A) = \sqcup (X_i, A_i)$, $\text{incl}_i: (X_i, A_i) \rightarrow (X, A)$

then $\boxed{\sum F(\text{incl}_i): \bigoplus F(X_i, A_i) \xrightarrow{\cong} F(X, A)}$

4) excision: $\boxed{\bar{E} \subseteq A^\circ \subseteq X \Rightarrow F(X \setminus E, A \setminus E) \xrightarrow[\uparrow F(\text{incl})]{\cong} F(X, A)}$

Remark (4) $\iff X = A^\circ \cup B^\circ$, $\text{incl}: (B, A \cap B) \rightarrow (X, A)$
then $\boxed{F(\text{incl}): F(B, A \cap B) \xrightarrow{\cong} F(X, A)}$

Pf $B = X \setminus E$, $E = X \setminus B$ noticing that $(X \setminus E)^\circ \cup A^\circ = X$

$E = A \setminus B$ noticing that $\bar{E} \subseteq \bar{A} \setminus B^\circ \subseteq A^\circ \setminus B^\circ \subseteq A^\circ$. $X = A^\circ \cup B^\circ$
so $\partial B \subseteq A^\circ$

Rmk In (3), the topology on the disjoint union $\sqcup (X_i, A_i)$ is defined by: $U \subseteq \sqcup (X_i, A_i)$ open $\iff U \cap X_i \subseteq X_i$ open $\forall i$

FACT Theorem

a) $(F, \delta_F), (G, \delta_G)$ GHTs, $\alpha: F \rightarrow G$ a natural transformation commuting with δ_F, δ_G such that $\alpha_{\text{point}}: F(\text{point}) \rightarrow G(\text{point})$ is an iso, then α is an iso.

b) If (F, δ_F) GHT satisfies (5) dimension: $\boxed{F_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}}$

Then \exists natural iso $F \cong H_*$ (such an F is called a homology theory)

Rmk In (b) if require $F_0(\text{point}) = \mathbf{G}$ an abelian group (instead of \mathbb{Z})
 $\implies F(X, A) \cong H_*(X, A; \mathbf{G}) = (\text{homology with coefficients in } \mathbf{G}) \leftarrow \text{later in course}$

9. COHOMOLOGY

(C_*, ∂_*) chain cx s.t. C_* free \mathbb{Z} -module

$$C_* \cong \bigoplus_{\alpha} \mathbb{Z}$$

Def n -cochains

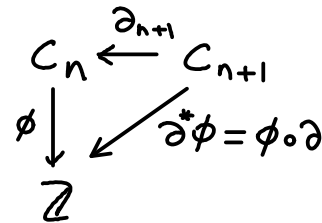
$$C^n = \text{Hom}(C_n, \mathbb{Z})$$

coboundary map

(this is the dual of ∂)

$$\partial^n : C^n \rightarrow C^{n+1}$$

$$\partial^n(\phi) = \phi \circ \partial_{n+1}$$



Notice ∂^* is degree +1 map (not -1)

$$H^m(C_*, \partial_*) = \frac{\text{Ker } \partial^m \leftarrow \text{cocycles}}{\text{Im } \partial^{m-1} \leftarrow \text{coboundaries}}$$

(Note $\partial^* \circ \partial^* = 0$:
 $\partial^* \partial^* \phi = \phi \circ \partial \circ \partial = 0$)

Rmk If use negative grading, (C^{-*}, ∂^{-*}) is a chain complex with homology so many results from H_* carry over to H^* . So abusively write

$$H^*(C_*, \partial_*) = H_*(\text{Hom}(C_*, \mathbb{Z}), \partial^*)$$

Warning A cochain $\varphi \in C^*$ takes values $\varphi(c) \in \mathbb{Z}$ on chains $c \in C_*$. However the cohomology class $\alpha = [\varphi] \in H^*$ does not have a well-defined value on c : $[\varphi] = [\varphi + \partial^*(\psi)]$ and $(\varphi + \partial^*(\psi))(c) = \varphi(c) + \psi(\partial_* c)$. If c is a cycle, so $\partial_* c = 0$ then $\alpha(c) = \varphi(c)$ is well-defined, so \exists pairing $H^* \times H_* \rightarrow \mathbb{Z}$

Remarks about dualisation

$\text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$

this is the dual of the standard basis:
 $\pi_i = e_i^* : e_i \rightarrow 1$
 $e_k \rightarrow 0, k \neq i$

$$\begin{array}{ccc} \alpha : \mathbb{Z}^n \rightarrow \mathbb{Z}^m & \Rightarrow & \text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \xleftarrow[\alpha^*]{\text{dual}} \text{Hom}(\mathbb{Z}^m, \mathbb{Z}) \\ x \mapsto A \cdot x & & \parallel \qquad \qquad \parallel \\ & & \mathbb{Z}^n \xleftarrow[\text{transpose}(A)]{\qquad} \mathbb{Z}^m \end{array} \quad \alpha^* \phi = \phi \circ \alpha$$

Def X space \Rightarrow singular cohomology

$$H^*(X) = H^*(C^*(X), \partial^*)$$

similarly define H_{Δ}^*, H_{CW}^*

dualise $C_* = C_*(X)$

Example $\mathbb{R}P^3 : C_*^{CW}(\mathbb{R}P^3) : 0 \rightarrow \mathbb{Z} \xrightarrow{\circ} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\circ} \mathbb{Z} \rightarrow 0$

dualise: $C_*^*(\mathbb{R}P^3) : 0 \leftarrow \mathbb{Z} \xleftarrow{\circ} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{\circ} \mathbb{Z} \rightarrow 0$

$$H^*(\mathbb{R}P^3) \cong H_{CW}^*(\mathbb{R}P^3) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

notice $H_1(\mathbb{R}P^3) \cong \mathbb{Z}/2$ has moved to grading 2.

Functoriality

$$f: X \rightarrow Y \Rightarrow f_*: C_* X \rightarrow C_* Y$$

← called **pull-back**

$$\Rightarrow C^* X \xleftarrow{f^*} C^* Y \text{ is dual: } \boxed{f^* \phi = \phi \circ f_*}$$

Lemma f^* is a **cochain map** (meaning $\partial^* \circ f^* = f^* \circ \partial^*$)

$$\Rightarrow \boxed{f^*: H^* Y \rightarrow H^* X}$$

Pf $(\partial^* \circ f^*)(\phi) = \partial^*(\phi \circ f_*)$

$$= (\phi \circ f_*) \circ \partial$$

$$= (\phi \circ \partial) \circ f_* \quad \text{as } f_* \text{ chain map}$$

$$= f^*(\phi \circ \partial)$$

$$= f^*(\partial^* \phi)$$

$$= (f^* \circ \partial^*)(\phi)$$

Properties • $\text{id}^* = \text{id}$

• $(f \circ g)^* = g^* \circ f^*$ notice order!

$$\Rightarrow \boxed{H^*: \text{Top} \rightarrow \text{Graded Ab Grps}} \quad \text{contravariant functor}$$

Exercise $H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ where $\pi_0 X = \{\text{path-components of } X\}$

Homotopy invariance

Lemma $f_*, g_*: C_* \xrightarrow{\text{free}} \tilde{C}_*$ chain hpic $\Rightarrow f^* = g^*: H^* \tilde{C} \rightarrow H^* C$

Pf $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$

$$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$$

some $h: C_* \rightarrow \tilde{C}_*[1]$

for dual $h^*: \tilde{C}^* \rightarrow C^*[-1]$.

(notice degree -1, not +1) \square

Def h^* called **cochain homotopy**

Cor $f \simeq g: X \rightarrow Y \Rightarrow f^* = g^*: H^* Y \rightarrow H^* X \quad \square$

Algebra: dual of SES

Lemma $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$ exact, A, B, C free

$\Rightarrow 0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$ exact

Cultural Remark

$(\bigoplus_{n \in \mathbb{N}} \mathbb{Z})^* = \prod_{n \in \mathbb{N}} \mathbb{Z}^*$
is not free.

(Baer 1937)

so A^*, B^*, C^* are not free unless A, B, C have finite ranks

Pf C free $\Rightarrow \exists$ splitting $B \xrightleftharpoons[s]{j} C$ $j \circ s = \text{id}$

pick preimages b_i for basis e_i of C , then $s(e_i) = b_i$

$\Rightarrow A \oplus C \xrightarrow[i \oplus s]{\cong} B$

dual $\Rightarrow A^* \oplus C^* \xleftarrow[i^* \oplus s^*]{\cong} B^*$ and $s^* \circ j^* = \text{id}$
 \rightarrow so i^* surj \rightarrow so j^* inj

$\Rightarrow 0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow[j^*]{s^*} C^* \leftarrow 0$

Rmk inverse is $B \cong A \oplus C$
 $b \mapsto i^{-1}(b - s(b)) \oplus j(b)$

where $0 = (j \circ i)^* = i^* \circ j^*$ so $\text{Im } j^* \subseteq \text{Ker } i^*$

prove \supseteq : $i^* b = 0 \Rightarrow b - j^* s^* b \in \text{Ker } i^* \cap \text{Ker } s^* = \{0\}$

$\Rightarrow b = j^* s^* b \in \text{Im } j^*$

$\Rightarrow \text{Ker } i^* = \text{Im } j^* \quad \square$

since $s^* j^* = \text{id}$
since $i^* \oplus s^*$ is iso.

Relative cohomology

$H^*(X, A) = H^*(\text{Hom}(C_*(X, A), \mathbb{Z}))$

recall $C_*(X, A) = C_*(X) / C_*(A)$
and homs $C_*(X) / C_*(A) \rightarrow \mathbb{Z}$
correspond precisely to homs $C_*(X) \rightarrow \mathbb{Z}$
which vanish on $C_*(A)$.
So relative cocycles are cocycles on X
which vanish on chains in A .

Excision, LES, Mayer-Vietoris

By previous Lemma get dual results:

Excision $\bar{E} \subseteq A^0 \subseteq X \Rightarrow H^*(X \setminus E, A \setminus E) \xleftarrow[\cong]{i^*} H^*(X, A)$

LES for pair $(X, A) \quad \dots \xleftarrow{q^* [+1]} H^{*+1}(X, A) \xleftarrow{\delta} H^*(A) \xleftarrow{i^*} H^*(X) \xleftarrow{q^*} H^*(X, A) \leftarrow \dots$

M.V. $X = A^0 \cup B^0 \Rightarrow \dots \leftarrow H^{*+1}(X) \leftarrow H^*(A \cap B) \leftarrow H^*(A) \oplus H^*(B) \leftarrow H^*(X) \leftarrow \dots$

where $A \cap B \begin{matrix} \xrightarrow{i_A} A \\ \xrightarrow{i_B} B \end{matrix} \begin{matrix} \xrightarrow{j_A} X \\ \xrightarrow{j_B} X \end{matrix}$ are the obvious maps

Axioms for cohomology

These are analogous to the axioms for homology except we reverse all arrows, and we change axiom (3): \prod instead of \oplus

additivity: $(X, A) = \sqcup (X_i, A_i)$, $\text{incl}_i: (X_i, A_i) \rightarrow (X, A)$

then $\prod F(\text{incl}_i): \prod F(X_i, A_i) \xleftarrow{\cong} F(X, A)$

10. CUP PRODUCT

Theorem $H^*(X)$ is unital graded-commutative ring via $\cup : H^k(X) \times H^l(X) \rightarrow H^{k+l}(X)$ determined by

$$\cup : C^k(X) \times C^l(X) \rightarrow C^{k+l}(X)$$

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_k]}) \cdot \psi(\sigma|_{[e_k, \dots, e_{k+l}]})$$

- ① $1 \in C^0(X)$ "constant function 1" $\implies 1 \cup \phi = \phi \cup 1 = \phi$
- ② $\phi \cup \psi = (-1)^{\deg \phi \cdot \deg \psi} \psi \cup \phi$ ($1(\text{point})=1$ so $1(\sum n_i \cdot p_i) = \sum n_i$)

Useful trick If X is Δ -cx, $C_*^\Delta(X) \xrightarrow[\cong]{\text{inclusion}} C_*(X)$, so $C_\Delta^*(X) \xleftarrow[\cong]{\text{restriction}} C^*(X)$
 and can define cup product on $C_\Delta^*(X)$ so that: $\phi \text{ incl} \longleftarrow \psi$

$$H_\Delta^*(X) \times H_\Delta^*(X) \xrightarrow{\cup} H_\Delta^*(X) \quad \leftarrow \text{at chain level}$$

$$\cong \uparrow \qquad \qquad \qquad \uparrow \cong$$

$$H^*(X) \times H^*(X) \xrightarrow{\cup} H^*(X)$$

$$(\phi \cup \psi)([v_0, \dots, v_n]) = \phi([v_0, \dots, v_k]) \cdot \psi([v_k, \dots, v_n])$$

$\uparrow \quad \quad \quad \uparrow$
 $n=k+l$

So you can compute cup products on $H^*(X)$ by picking simplicial cocycle representatives: so define values on the simplicial chains defining the Δ -cx structure, and use

Proof of Theorem

(cannot do this for $C_*^{CW}(X)$ because there is no meaningful analogue for D^n of the "bottom face" $[e_0, \dots, e_k]$ and "top face" $[e_k, \dots, e_n]$)

$$\begin{aligned} \partial^*(\phi \cup \psi)(\sigma) &= (\phi \cup \psi)(\partial\sigma) \\ &= (\phi \cup \psi) \sum (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_n]} \quad (n=k+l+1, \sigma \in C_n, \phi \in C^k, \psi \in C^l) \\ &= \sum_{i \leq k} (-1)^i \phi(\sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_{k+1}]}) \cdot \psi(\sigma|_{[e_{k+1}, \dots, e_n]}) \\ &\quad + \sum_{i > k} (-1)^i \phi(\sigma|_{[e_0, \dots, e_k]}) \cdot \psi(\sigma|_{[e_k, \dots, \hat{e}_i, \dots, e_n]}) \cdot \underbrace{(-1)^{i-k} (-1)^{k-i}}_1 \\ &= ((\partial^*\phi) \cup \psi)(\sigma) + (-1)^k \phi \cup \partial^*\psi \end{aligned}$$

induces $[\phi] \cup [\psi] = [\phi \cup \psi] : \quad \quad \quad \overset{=0}{\widetilde{(\partial\phi) \cup \psi}} \quad \quad \quad \overset{=0}{\widetilde{\phi \cup (\partial\psi)}}$
 well-defined: • cycles \rightarrow cycle: $\partial(\phi \cup \psi) = (\partial\phi) \cup \psi \pm \phi \cup (\partial\psi) = 0$
 • $[\phi] = [\phi + \partial\alpha]$ so need $[(\partial\alpha) \cup \psi] = 0$
 $(\partial\alpha) \cup \psi = \partial(\alpha \cup \psi)$ ✓ (using $\partial\psi=0$)
 • Similarly $[\phi] \cup [\partial\beta] = 0$

bilinear, associative, distributive: true at chain level

unital: $(\partial 1)(\sigma) = 1(\sigma|_{[e_1]}) - 1(\sigma|_{[e_0]}) = 1 - 1 = 0$

$(1 \cup \psi)(\sigma) = 1(\sigma|_{[e_0]}) \cdot \psi(\sigma|_{[e_0, \dots, e_n]}) = \psi(\sigma)$ ($\phi \cup 1 = \phi$ similar)

graded-comm. sketch proof: ← **non-examinable**

Let $r : C_n(X) \rightarrow C_n(X)$, $r(\sigma) = \epsilon_n \bar{\sigma}$ where: $\epsilon_n = (-1)^{\frac{n(n+1)}{2}}$

and $\bar{\sigma}|_{[v_0, \dots, v_n]} = \sigma|_{[v_n, \dots, v_0]}$ ← reverse order of vertices:
is product of $n + (n-1) + \dots + 1$ transpositions
 $\frac{n(n+1)}{2}$

(idea: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert ϵ_n to compensate)

one checks: • r chain map

• $\underline{r^* \psi \cup r^* \psi} = \underline{r^*(\psi \cup \psi)}$

$\epsilon_k \epsilon_l$ ← differ by $(-1)^{kl}$ → ϵ_{k+l}

• $r \simeq id$ so can drop $r^* = id$ on cohomology

$(r - id = P\partial + \partial P$ with v_i, w_i like for prism operator)
 $P\sigma = \sum (-1)^i \epsilon_{n-i} (\sigma \circ \pi)|_{[v_0, \dots, v_i, \underline{w_n, \dots, w_i}]}$ \square
projection $\Delta^n \times I \xrightarrow{\pi} \Delta^n$

Naturality of cup product

Lemma $f : X \rightarrow Y \implies f^* : H^* Y \rightarrow H^* X$ hom of unital rings

Pf $f^*(\psi \cup \psi)(\sigma) = (\psi \cup \psi)(f_* \sigma)$
 $= \psi(f_* \sigma|_{[e_0, \dots, e_k]}) \cdot \psi(f_* \sigma|_{[e_k, \dots, e_n]})$
 $= ((\psi \circ f_*) \cup (\psi \circ f_*))(\sigma)$
 $= (f^* \psi \cup f^* \psi)(\sigma)$

unital: $f^*(1) = 1 \circ f_* = 1$ \square

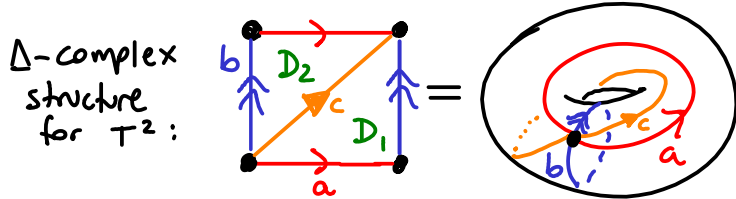
UPSHOT $H^* : \text{Top} \rightarrow \left\{ \begin{array}{l} \text{Graded-commutative unital rings} \\ \text{with graded unital ring homs} \end{array} \right\}$
contravariant functor.

Warning An (iso)morphism $H^*(Y) \rightarrow H^*(X)$ of groups will also preserve the ring structure if f^* is induced by a map of spaces $X \rightarrow Y$ (by above Lemma).

\implies Cor The excision theorem iso on cohomology is an iso of rings.

However the connecting hom in M.V. or LES cannot possibly be a ring hom since it raises gradings by 1 ($\implies \delta(a \cup b)$ and $\delta(a) \cup \delta(b)$ have different grading!)

Example $H^1(T^2) \times H^1(T^2) \xrightarrow{\cup} H^2(T^2)$ is bilinear form $\mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{Z}$ with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Pf By the Useful Trick, it is enough to work with H_Δ^* instead of H^* .



$$C_*^\Delta: 0 \rightarrow \mathbb{Z}^2 \xrightarrow{\text{gens: } D_1, D_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

$$C_\Delta^*: 0 \leftarrow \mathbb{Z}^2 \xleftarrow{\text{gens: } D_1^*, D_2^*} \mathbb{Z}^3 \xleftarrow{0} \mathbb{Z} \rightarrow 0$$

*	$H_*^\Delta(T^2)$	$H_\Delta^*(T^2)$
0	\mathbb{Z} pt	$\mathbb{Z} \cdot 1$
1	$\mathbb{Z}a \oplus \mathbb{Z}b$	$\mathbb{Z}(a^* + c^*) + \mathbb{Z}(b^* + c^*)$
2	$\mathbb{Z}(D_1 - D_2)$	$\mathbb{Z} \cdot D_1^*$

← abbreviate $\begin{cases} A = a^* + c^* \\ B = b^* + c^* \end{cases}$
 ← (Remark $[D_1^*] = -[D_2^*]$ in $H_\Delta^2(T^2)$)
 ← dual basis for basis a, b, c (e.g. $a^*(a) = 1, a^*(b) = 0, a^*(c) = 0$)

claim $A \cup B = D_1^*$

Pf

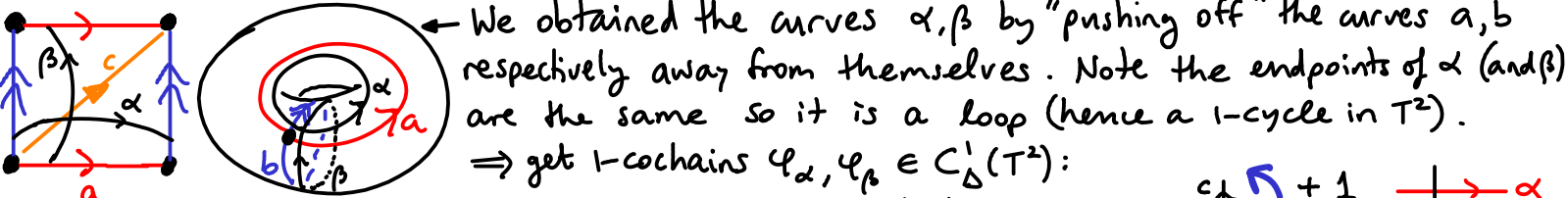
$$(A \cup B)(D_1) = A(D_1 | [e_0, e_1]) \cdot B(D_1 | [e_1, e_2]) = A(a)B(b) = 1 \cdot 1 = 1$$

$$(A \cup B)(D_2) = A(b)B(a) = 0 \cdot 0 = 0 \quad \square$$

Graded-comm. $\Rightarrow B \cup A = -D_1^*, A \cup A = (-1)^{1 \cdot 1} A \cup A = 0$, similarly $B \cup B = 0 \quad \square$

Rmk Recall that to specify a cochain in $C_\Delta^k(X)$ one needs to specify values on all generators of $C_k^\Delta(X)$ so not just on generators of $H_k^\Delta(X)$ (e.g. above A and a^* agree on gens a, b of $H_1^\Delta(T^2)$ but disagree on $c \in C_1^\Delta(T^2)$, note a^* is 1-cochain $\in C_\Delta^1(T^2)$ but is not a 1-cocycle)
 Some (but not all) k -cochains φ can be specified by drawing a "nice" $(n-k)$ -dimensional subspace $\Sigma \subseteq X$ and defining $\varphi(c) = \#(\text{times } \Sigma \text{ intersects } c)$ for all $c \in C_k^\Delta(X)$

where one must explain with what sign \pm an intersection point is counted and one has ensured that Σ intersects the generators of $C_*^\Delta(X)$ in a finite $\#$ points.



$$\varphi_\alpha^*(c) = \# \alpha \text{ intersects } c \text{ counted with orientation signs: } \begin{matrix} c \uparrow +1 \\ \downarrow \alpha \\ c \downarrow -1 \end{matrix}$$

← Written $\alpha \cdot c$, called intersection pairing

Notice $\varphi_\alpha(a) = 0, \varphi_\alpha(b) = 1, \varphi_\alpha(c) = 1$ so $\varphi_\alpha = B$. Similarly, $\varphi_\beta = -A$.

Fact Since T^2 is an orientable manifold, $\varphi_\alpha \cup \varphi_\beta = (\alpha \cdot \beta) \text{ vol}$ where vol is a generator of $H^2(T^2)$. Later in the course: vol is the "Poincaré dual" of the point class, and corresponds to the dual of the oriented sum of the top simplices. Above: $\text{vol} = D_1^*$ and

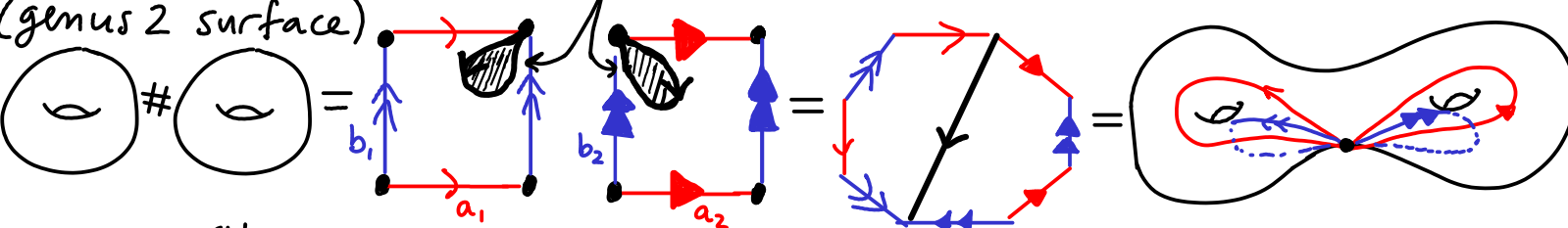
$$\varphi_\alpha \cup \varphi_\beta = B \cup (-A) = A \cup B = (\alpha \cdot \beta) \text{ vol} = \text{vol} = D_1^*$$

Defining intersection numbers rigorously is tricky, even when using smooth chains. one can calculate $\varphi_\Sigma(c)$ on a cycle c by first deforming c to a smooth homologous cycle \tilde{c} which is "transverse" to Σ , and then we count intersection points $\Sigma \cap \tilde{c}$ (with "orientation signs").

← ("tangencies" are the key issue)

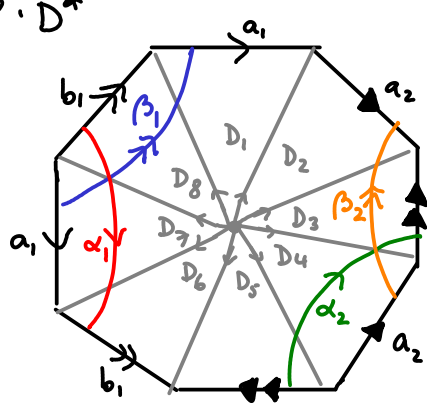
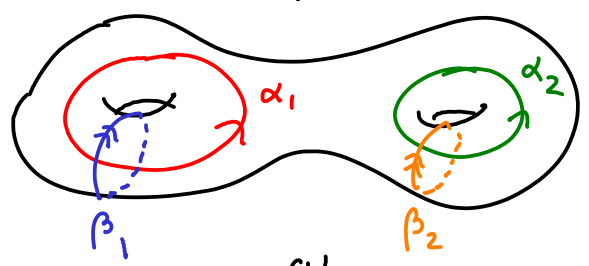
The fact that we consider the intersection number $a \cdot a = 0$ is because we can push a off itself:

Exercise Σ_2 remove balls & glue bdries



		$H_*^{CW}(\Sigma_2)$	$H_{CW}^*(\Sigma_2)$
0	\mathbb{Z}	$\mathbb{Z} \cdot pt$	$\mathbb{Z} \cdot 1$
1	\mathbb{Z}^4	$\mathbb{Z}a_1 + \mathbb{Z}b_1 + \mathbb{Z}a_2 + \mathbb{Z}b_2$	$\mathbb{Z} \langle \alpha_1^*, b_1^*, a_2^*, b_2^* \rangle$
2	\mathbb{Z}	$\mathbb{Z} \cdot D$	$\mathbb{Z} \cdot D^*$

Deform curves a_1, b_1, a_2, b_2 to get $\alpha_1, \beta_1, \alpha_2, \beta_2$:



← dual basis in C_{CW}^1 (easy to define on $C_{CW}^1(\Sigma_2)$ but not so obvious on $C_{\Delta}^1(\Sigma_2)$)

← Δ -cx structure on Σ_2

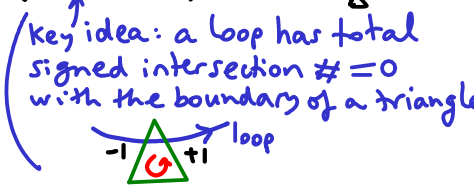
then notice for $c \in C_{CW}^1(\Sigma_2)$ signed count

$$\begin{cases} a_i^*(c) = -\#(\beta_i \text{ intersects } c) \\ b_i^*(c) = \#(\alpha_i \text{ intersects } c) \end{cases}$$

so can extend this to a definition of $a_i^*, b_i^* \in C_{\Delta}^1(\Sigma_2)$ by allowing $c \in C_{\Delta}^1(\Sigma_2)$. Check that a_i^*, b_i^* are 1-cocycles in $C_{\Delta}^1(\Sigma_2)$.

Exercise: $a_i^* \cup b_j^* = \delta_{ij} \cdot D^* = -b_j^* \cup a_i^*$

Hint: represent D as a sum of \pm triangles in last picture. orientation signs: $D = -D_1 - D_2 + D_3 + D_4 - D_5 + D_6 + D_7 - D_8$ using + if outer edge is oriented anticlockwise



$$a_i^* \cup a_i^* = b_i^* \cup b_i^* = 0$$

so same as geometric intersection numbers of corresponding curves.

Cultural Rmk on general theory (Intersection Theory/Differential Topology)

M^m oriented m -manifold $\Rightarrow H_n(N) \xrightarrow{incl_*} H_n(M)$ ← see later in course
 $N^n \subseteq M^m$ oriented compact n -dim submfd $\Rightarrow \exists$ generator $[N] \mapsto [N]$

N, M also smooth (see Differential Geometry course) $\Rightarrow \omega_N \in H^{m-n}(M)$ counts $\#$ intersections with N ← with signs

$N_1, N_2 \subseteq M$ compact oriented smooth submfds and transverse (= at every $p \in N_1 \cap N_2$ the tangent spaces to N_1, N_2 at p span the tangent space to M at p)
 (tang. space means the "best" vector space approximation at p determined by the local smooth coordinates.)

$$\omega_{N_1} \cup \omega_{N_2} = \omega_{N_1 \cap N_2} \in H^{2m-n_1-n_2}(M)$$

In particular if $n_1+n_2=m$, and M connected, then $H^m(M) \cong \mathbb{Z}$ s.t. $\omega_{N_1} \cup \omega_{N_2} \mapsto \#(N_1 \cap N_2) \in \mathbb{Z}$. In non-orientable case, this all holds if work over $\mathbb{Z}/2$

Fact (Thom 1954) $(N_1 \cap N_2$ has an orientation induced by N_1, N_2 . If $n_1+n_2=m$, get $N_1 \cap N_2$ is sum of \pm points. Sign compares oriented bases of tang. spaces: $T_p M = T_p N_1 \oplus T_p N_2$.)
 Not all $a \in H^j(M)$ arise as ω_N for connected compact oriented codim= j smooth submfd N
 But $\exists N \in \mathcal{N}$ s.t. $N \cdot a$ does arise. They do arise for $H^*(M; \mathbb{Q}), H^*(M, \mathbb{R}), H^*(M; \mathbb{Z}/2)$

11. KÜNNETH FORMULA AND PRODUCT SPACES

Algebra : tensor products

R ring (comm. with 1)

e.g. abelian groups = \mathbb{Z} -mods
vector spaces/ \mathbb{F} = \mathbb{F} -mods

Def A, B R -modules \Rightarrow Tensor product is R -module

$$A \otimes_R B = \langle (a, b) : a \in A, b \in B \rangle / \text{relations of bilinearity \& rescaling}$$

(or $A \otimes B$) R -mod generated write $a \otimes b$ for its class

bilinearity: $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$
 $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$

rescaling: $r \cdot (a \otimes b) = (ra) \otimes b = a \otimes (rb) \quad \forall r \in R$

"can move $r \in R$ across the \otimes symbol"

- So general element looks like $\sum a_k \otimes b_k$ (finite sum) ← NOT UNIQUELY!
- Don't confuse with $A \times B$: e.g. $0 \otimes b = 0 \quad \forall b$

Rmk Can define $A \otimes_R B$ also by a universal property: for all R -mods C ,

$$\text{Hom}_R(A \otimes_R B, C) \xrightarrow[\text{natural}]{\cong} \{R\text{-bilinear maps } A \times B \rightarrow C\}$$

Using above description of $A \otimes B$: $\varphi \mapsto (A \times B \rightarrow C, (a, b) \mapsto \varphi(a \otimes b))$

Example ($R = \mathbb{F}$) V, W v.s. / $\mathbb{F} \Rightarrow V \otimes W$ v.s. / \mathbb{F} basis $v_i \otimes w_j$
 $\dim_{\mathbb{F}} V \otimes W = \dim V \cdot \dim W$

Exercise V, W finite dim / $\mathbb{F} \Rightarrow V^* \otimes W \cong \text{Hom}_{\mathbb{F}}(V, W)$

Hint $f: V \rightarrow \mathbb{F}, w \in W, f \otimes w \mapsto (V \rightarrow W, v \mapsto f(v) \cdot w)$

Examples

($R = \mathbb{Z}$)

- $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{n \cdot m}$
- $\mathbb{Z}/n \otimes \mathbb{Z} \cong \mathbb{Z}/n \leftarrow \sum a_i \otimes b_i = (\sum a_i b_i) \otimes 1$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0 \leftarrow 1 \otimes x = 3 \otimes x = 1 \otimes 3x = 0$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2 \leftarrow \begin{cases} 1 \otimes 1 = 3 \otimes 1 = 1 \otimes 3 \\ 1 \otimes 2 = 2 \otimes 1 = 0 \end{cases}$

e.g. $(\mathbb{R}^n)^* \otimes \mathbb{R}^m \cong \text{Mat}_{m \times n}(\mathbb{R}) \cong \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$
 $e_i^* \otimes e_j \longleftrightarrow \text{matrix } A \text{ with } A_{ji} = 1, 0 \text{ else.}$

Examples

- $A \otimes B \cong B \otimes A$
- $(\bigoplus_i A_i) \otimes (\bigoplus_j B_j) \cong \bigoplus_i \bigoplus_j (A_i \otimes B_j)$
- $A \otimes R \cong A$ (so " $\cdot \otimes_R$ does nothing")
- $A \otimes R/d \cong A/d \cdot A$

hence now know $A \otimes B$ for any f.g. R -mods A, B .

for example $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong (\mathbb{Z}/4)/2 \cdot \mathbb{Z}/4 \cong \mathbb{Z}/2 \leftarrow \left(\begin{array}{l} \text{Rmk } (\mathbb{Z}/n)/m \cdot \mathbb{Z}/n \\ \cong \mathbb{Z}/\text{gcd}(m, n) \end{array} \right)$

More generally: $\begin{cases} R/I \otimes_R R/J \cong R/(I+J) \text{ for ideals } I, J \subseteq R. \\ A \otimes_R R/J \cong A/J \cdot A \end{cases}$

Warning \otimes often not an exact functor, i.e. does not preserve exact sequences
 indeed it can ruin injectivity: $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ now take $\cdot \otimes \mathbb{Z}/2$ get $0 \rightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$.

Fact $\cdot \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\cdot \otimes_{\mathbb{Z}} \mathbb{R}$ are exact functors on \mathbb{Z} -mods

More generally $\cdot \otimes_{\mathbb{Z}} \text{Frac}(R)$ is exact on R -mods where $\text{Frac } R$ is fraction field, and R is an integral domain "localisation is an exact functor"

example A f.g. \mathbb{Z} -mod $\Rightarrow A \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_k$ some $d_i \neq 0$
 $\Rightarrow A \otimes \mathbb{Q} \cong \mathbb{Q}^r \Rightarrow r = \text{rank } A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$

Corollary Rank-nullity thm holds for \mathbb{Z} -modules if use rank instead of dim.

Pf $0 \rightarrow A \xrightarrow{\varphi} B \rightarrow C \rightarrow 0$ exact $\Rightarrow 0 \rightarrow A \otimes \mathbb{Q} \rightarrow B \otimes \mathbb{Q} \rightarrow C \otimes \mathbb{Q} \rightarrow 0$ exact
 $\Rightarrow \dim(C \otimes \mathbb{Q}) + \dim(A \otimes \mathbb{Q}) = \dim(B \otimes \mathbb{Q})$. \square
rank-nullity for \mathbb{Q} -vector spaces.

Tensor product of chain cxes

C_* , \tilde{C}_* chain cxes of R -mods $\Rightarrow (C_* \otimes \tilde{C}_*)_n = \bigoplus_{i+j=n} C_i \otimes \tilde{C}_j$

$\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{\text{deg } x} x \otimes \tilde{\partial} y$ "Leibniz rule"

Think of ∂ as an operator of $\text{deg} = -1$ acting from left

since ∂ "jumps over x " get $(-1)^{\text{deg } \partial} \cdot \text{deg } x$

Exercise $\partial \circ \partial = 0$ \leftarrow would fail without sign

recall $Z_* = \ker \partial = \text{cycles}$
 $B_* = \text{im } \partial = \text{boundaries}$

$Z_i \otimes \tilde{Z}_j \subseteq Z_{i+j}(C_* \otimes \tilde{C}_*)$ and $\left. \begin{matrix} Z_i \otimes \tilde{B}_j \\ B_i \otimes \tilde{Z}_j \end{matrix} \right\} \subseteq B_{i+j}(C_* \otimes \tilde{C}_*)$

Cor \exists natural maps

$H_i(C_*) \otimes H_j(\tilde{C}_*) \rightarrow H_{i+j}(C_* \otimes \tilde{C}_*)$
 $\sum [c_k] \otimes [\tilde{c}_k] \mapsto \sum [c_k \otimes \tilde{c}_k]$

FACT: Algebraic Künneth Thm

C_* , $H_*(C_*)$ f.g. free R -mods \leftarrow PID (principal ideal domain) (no assumption on \tilde{C}_*)

$\Rightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(\tilde{C}_*) \cong H_n(C_* \otimes \tilde{C}_*)$ via

Algebra: Euler characteristic

C finitely generated graded abelian gp (so \mathbb{Z} -mod)

more generally: R -mod for PID R

Def Euler characteristic $\chi(C) = \sum (-1)^i \text{rank } C_i$

Example/Motivation X finite CW-cx then take $C = C_*^{\text{CW}}(X)$ to get

$\chi(X) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) - \dots$

Lemma If C_* f.g. chain cx $\Rightarrow \chi(C_*) = \chi(H_*(C_*)) (= \sum (-1)^i \text{rank } H_i(C))$

Pf Abbreviate $|C_i| = \text{rank } C_i (= \dim_{\mathbb{Q}}(C_i \otimes_{\mathbb{Z}} \mathbb{Q}))$

for R -mods, do $\dim_F(C_i \otimes_R F)$ with $F = \text{Frac}(R)$ (R integral domain) [Corollary still holds, same proof]

By previous corollary about rank-nullity:

$$\begin{aligned} 0 \rightarrow Z_i \rightarrow C_i \xrightarrow{\partial} B_{i-1} \rightarrow 0 &\Rightarrow |C_i| = |Z_i| + |B_{i-1}| \\ 0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0 &\Rightarrow |H_i| = |Z_i| - |B_i| \end{aligned}$$

$$\Rightarrow \chi(C_*) - \chi(H_*) = \sum (-1)^i |B_{i-1}| + \sum (-1)^i |B_i| = \sum (-1)^i (-|B_i| + |B_i|) = 0. \quad \square$$

Cor X space $\Rightarrow \chi(X) := \sum (-1)^i \text{rank } H_i(X) = \sum (-1)^i \text{rank } C_i(X)$
 ← if finite rank $H_*(X)$
 ← if finite rank $C_*(X)$

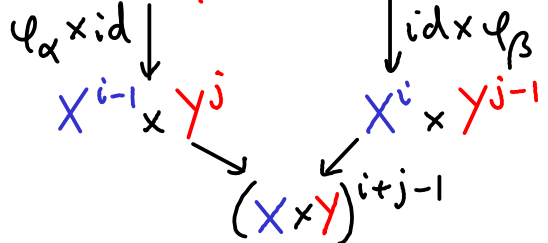
So $\chi(X)$ is invariant up to hpy equivalence! Example $\chi(\text{platonic solid}) = \chi(S^2) = 2$

Product spaces

Technical Remark The CW topology agrees with the product topology if X, Y have countably many cells, or X, Y are locally compact (see Appendix in Hatcher)

X, Y CW-cxes $\Rightarrow X \times Y$ CW-cx with cells $e_\alpha \times e_\beta$ attaching maps $\partial(D_\alpha^i \times D_\beta^j) = (\partial D_\alpha^i) \times D_\beta^j \cup D_\alpha^i \times (\partial D_\beta^j)$

Cor $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$
 \forall finite CW-cxes X, Y



Pf $\sum (-1)^k \text{rank } H_k^{CW}(X \times Y) = \sum (-1)^k \text{rank } C_k^{CW}(X \times Y) = \sum_k \sum_{i+j=k} (-1)^{i+j} \text{rank } C_i^{CW}(X) \cdot \text{rank } C_j^{CW}(Y)$ \square

Lemma $d(e_\alpha^i \times e_\beta^j) = (d e_\alpha^i) \times e_\beta^j + (-1)^i e_\alpha^i \times (d e_\beta^j)$

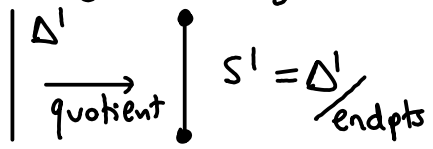
(proof later) hence $C_*^{CW}(X \times Y) \cong C_*^{CW}(X) \otimes C_*^{CW}(Y)$

Hence if $H_*(Y)$ free then by Kunneth $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.

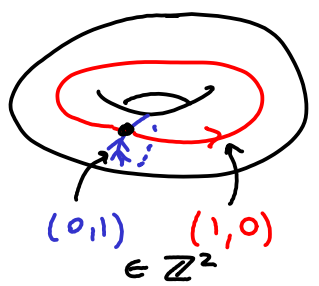
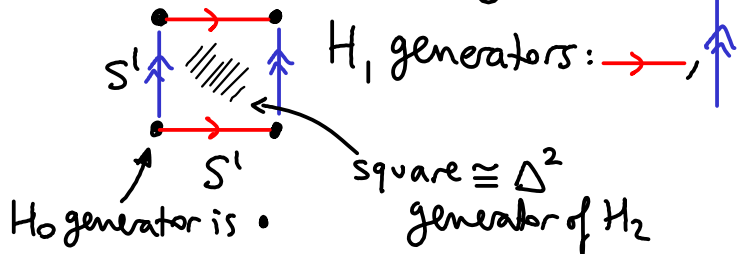
Example

$*$	$H_*(S^1)$	$*$	$H_*(S^1) \otimes H_*(S^1) \cong H_*(S^1 \times S^1)$ ← torus
0	$A \cong \mathbb{Z}$	0	$A \otimes A \cong \mathbb{Z}$
1	$B \cong \mathbb{Z}$	1	$(A \otimes B) \oplus (B \otimes A) \cong \mathbb{Z}^2$
2	0	2	$B \otimes B \cong \mathbb{Z}$
		3	0

B generated by



A generated by $\Delta^0 \rightarrow \bullet$



Sketch Pf. $(\partial D_\alpha^i) \times D_\gamma^j \xrightarrow{\varphi_\alpha \times \text{id}} X^{i-1} \times Y^j \rightarrow X^{i-1} \times Y^j$

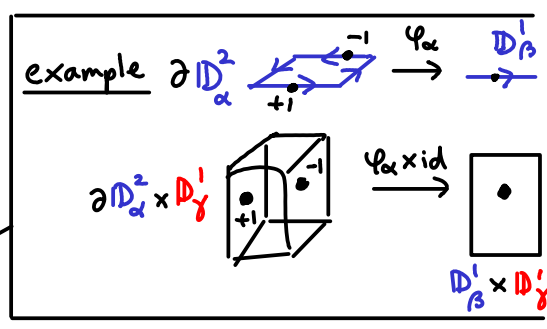
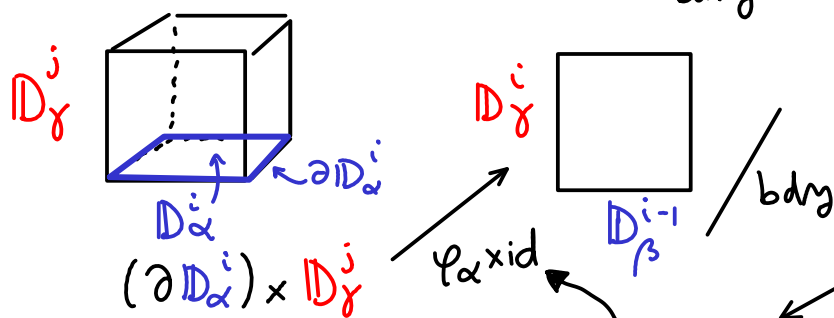
↑
This proof is Non-examinable

$(X \times Y)^{i+j-2} \cap (X^{i-1} \times Y^j)$
|| ← easy check
 $X^{i-2} \times Y^j \cup X^{i-1} \times Y^{j-1}$

$X^{i-1} = X^{i-2} \cup (D_\beta^{i-1} \cup \dots)$
 $Y^j = Y^{j-1} \cup (D_\gamma^j \cup \dots)$
} get ~ from attaching maps

$X^{i-1} \times Y^j = X^{i-2} \times Y^j \cup X^{i-1} \times Y^{j-1} \cup (D_\beta^{i-1} \times D_\gamma^j \cup \dots)$
 $\Rightarrow \textcircled{\star} = (D_\beta^{i-1} \times D_\gamma^j \cup \dots) / \text{boundaries}$
 $= D_\beta^{i-1} \times D_\gamma^j / \partial(D_\beta^{i-1} \times D_\gamma^j) \vee \dots$

$(\partial D_\alpha^i) \times D_\gamma^j \xrightarrow{\varphi_\alpha \times \text{id}} D_\beta^{i-1} \times D_\gamma^j \vee \dots$
bdry



By considering local degrees of "we get" degree = $d_\alpha d_\beta$ so contribution $(d_\alpha e_\alpha^i) \times e_\beta^j$

Rmk This step is actually quite tricky, see Hatcher Chapter 3.B (it requires homotopying maps to make them cellular). The cleanest argument may be to use smooth maps: one shows that continuous maps between discs can be homotoped a little to make them smooth, and that for smooth maps almost any point in the target has finitely many preimages s.t. the local maps are local homeomorphisms (so the local degrees are ± 1). If φ_α has been homotoped in this way, then $(\varphi_\alpha \times \text{id})^{-1}(y, t) = \varphi_\alpha^{-1}(y) \times \{t\}$ has the same # preimages as φ_α and local degrees ± 1 like φ_α (i.e. problem reduces to discussing orientation signs).

Similarly $D_\alpha^i \times \partial D_\gamma^j \xrightarrow{\text{id} \times \varphi_\gamma} D_\alpha^i \times D_\delta^{j-1} \Rightarrow \text{degree } (-1)^i d_\delta$
so get $(-1)^i e_\alpha^i \times d_\beta^j$

The $(-1)^i$ is caused by orientations:

could reorder factors: $D_\alpha^i \times D_\gamma^j \cong D_\gamma^j \times D_\alpha^i$ by $\begin{pmatrix} 0 & \text{Id}_j \\ \text{Id}_i & 0 \end{pmatrix}$
whose $\det = (-1)^{ij}$. Then $\partial D_\gamma^j \times D_\alpha^i \rightarrow D_\delta^{j-1} \times D_\alpha^i / \text{bdry}$ gives degree d_δ .
Swap factors $D_\delta^{j-1} \times D_\alpha^i / \text{bdry}$ by $\begin{pmatrix} 0 & \text{Id}_i \\ \text{Id}_{j-1} & 0 \end{pmatrix}$, $\det = (-1)^{i(j-1)}$. Total sign = $(-1)^i$.

Example Recall after definition of H_*^{CW} we had example $I \times I$:

arrows here tell us how we map $[-1,1] \rightarrow$ edge (so orientation)

$$\partial e^2 = +e^1 \times 0 + 1 \times e^1 - e^1 \times 1 - 0 \times e^1$$

$$= (\partial e^1) \times e^1 - e^1 \times (\partial e^1)$$

$(-1)^{\dim e^1}$ ✓

A further comment on orientation sign $(-1)^i$

$$\mathbb{D}^i \times \mathbb{D}^j \cong \underbrace{\Delta^i}_{[v_0, \dots, v_i]} \times \underbrace{\Delta^j}_{[w_0, \dots, w_j]}$$

← viewed in $\mathbb{R}^i, \mathbb{R}^j$
Project $\mathbb{R}^{i+j} \rightarrow \mathbb{R}^i$
(t_0, \dots, t_i) \mapsto ($\underline{t_1, \dots, t_i}$)

$$\partial(\mathbb{D}^i \times \mathbb{D}^j) \cong \underbrace{\partial \Delta^i}_{\parallel} \times \Delta^j \cup \Delta^i \times \underbrace{\partial \Delta^j}_{\parallel}$$

$$\sum_k (-1)^k [v_0, \dots, \hat{v}_k, \dots, v_i] \quad \sum_k (-1)^k [w_0, \dots, \hat{w}_k, \dots, w_j]$$

would be correct orientation sign for basis $w_1 - w_0, \dots, w_k - w_0, \dots, w_j - w_0$ but actually we have $[v_0, \dots, v_i] \times [w_0, \dots, w_j] \subseteq \mathbb{R}^i \times \mathbb{R}^j$

and $(-1)^{i+k}$ is the orientation sign for the basis

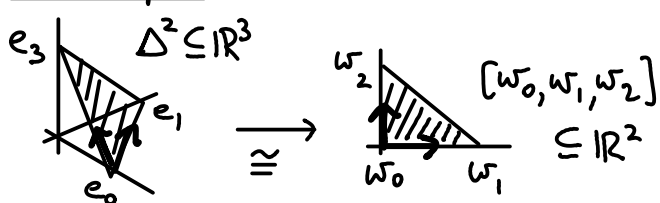
$$v_1 - v_0, \dots, v_i - v_0, w_1 - w_0, \dots, w_k - w_0, \dots, w_j - w_0$$

for the hyperplane in \mathbb{R}^{i+j+1} containing

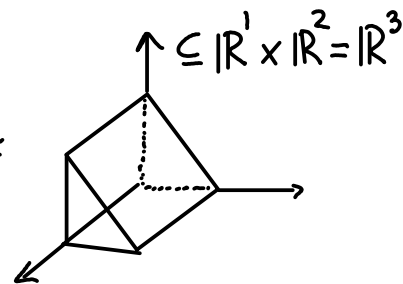
\Rightarrow need $(-1)^i$ to fix orientation sign.

Example

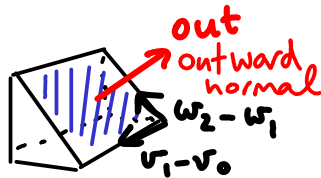
$$\Delta^1 \times \Delta^2$$



$$\Delta^1 \times \Delta^2 \cong$$



$$[v_0, v_1] \times [\hat{w}_0, w_1, w_2]$$



out, $v_1 - v_0, w_2 - w_1$ is negative \mathbb{R}^3 -basis



← differ due to $(-1)^i, i=1$.



FACT:

Künneth Theorem If $H_n(Y)$ finitely generated, free $\forall n$ ← no conditions on X ← automatic if use field coefficients

$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$	$H^n(X \times Y) \cong \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)$
	$p_X^* a \cup p_Y^* b \longleftarrow a \otimes b \quad (\star)$ <p style="text-align: center;">↑ and extend linearly</p>

Recall for cellular homology this on generators is: (chain level)

$$e_\alpha^i \times e_\beta^j \longleftarrow e_\alpha^i \otimes e_\beta^j$$

This is hom of rings if use following product $(a \otimes b) \cdot (\tilde{a} \otimes \tilde{b}) = (-1)^{|b||\tilde{a}|} (a \cup \tilde{a}) \otimes (b \cup \tilde{b})$
 think of it as "exchanging order of b, \tilde{a} "
 gradings

Rmk

An indirect proof the Thm is to write down two generalised cohomology theories $F(X,A) = H^*(X,A) \otimes H^*(Y)$ and $G(X,A) = H^*(X \times Y, A \times Y)$, and consider the natural transformation $\alpha: F \rightarrow G$ given by (\star) , notice for $X = \text{pt}$ both F, G give $H^*(Y)$
 $A = \emptyset$

Example $X = S^n, Y = S^m, n \neq m$

$$H_* (S^n \times S^m) \cong \begin{cases} \mathbb{Z} & * = 0, n, m, n+m \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^m) \quad \text{where } a_n \cup a_m = a_{n+m}$$

$$H_* (S^n \times S^n) \cong \begin{cases} \mathbb{Z} & * = 0, 2n \\ \mathbb{Z}^2 & * = n \leftarrow \text{gens: } a_n^{(1)}, a_n^{(2)} \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^n) \quad \begin{matrix} a_n^{(1)} \cup a_n^{(2)} = a_{2n} \\ \text{(but } a_n^{(i)} \cup a_n^{(i)} = 0) \end{matrix}$$

Cor $H^k(T^n) \cong \wedge^k [x_1, \dots, x_n]$ ← n-torus $S^1 \times \dots \times S^1$

where $x_i = p_i^*(\text{gen. of } H^1(S^1))$ ← $\text{deg } x_i = 1$
 $p_i: T^n \rightarrow S^1$ projections to factors.

Pf Künneth & induction $(T^n = T^{n-1} \times S^1)$ \square

exterior algebra

= free abelian gp. on gens. $\{x_i \wedge \dots \wedge x_{i_k} : i_1 < \dots < i_k\}$
 so rank = $\binom{n}{k}$
 product is " \wedge " using the rule $x_i \wedge x_j = -x_j \wedge x_i$
 (compare graded-commutativity of cup product)

FACT ← Non-examinable cup product equals composition

$$\cup : H^i(X) \otimes H^j(X) \xrightarrow{\text{Künneth map}} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

Idea: \exists natural chain maps $C_*(X) \otimes C_*(Y) \rightarrow C_*(X \times Y)$ whose composites are chain homotopic to identity maps (Eilenberg-Zilber Thm)
 Then get $C^*(X) \otimes C^*(Y) \rightarrow \text{dual}(C_*(X) \otimes C_*(Y)) \simeq \text{dual}(C_*(X \times Y)) = C^*(X \times Y)$
 $\alpha \otimes \beta \longmapsto (\sigma \otimes \tau \longmapsto \alpha(\sigma) \cdot \beta(\tau))$ For $X=Y$ this is the Künneth map.
 $\Delta = \text{diagonal map } X \rightarrow X \times X, x \mapsto (x, x)$

12. UNIVERSAL COEFFICIENTS THEOREM

MOTIVATION: What is difference between $H^*(\text{Hom}(C_*, \mathbb{Z}))$ and $\text{Hom}(H_*(C_*), \mathbb{Z})$?
Similarly: $H_*(C_* \otimes G)$ vs. $H_*(C_*) \otimes G$.

Proof is non-examinable. For (C_*, ∂_*) chain C_* :

$$\implies 0 \rightarrow Z_* = \text{Ker } \partial_* \xrightarrow{\text{incl}} C_* \xrightarrow{\partial_*} B_{*+1} = \text{Im } \partial_{*+1} \rightarrow 0 \text{ is SES}$$

$\uparrow \partial=0$ $\partial=0 \uparrow$

FACT: Submodules of a free \mathbb{Z} -module are free
Rmk The same holds for R -mods if R is PID

\mathbb{Z} -module \equiv abelian gp
free means: $\bigoplus_{\text{indexing set}} \mathbb{Z}$
(PID) = principal ideal domain = integral domain R s.t. every ideal = $R \cdot a$ some a

Assume C_* free \mathbb{Z} -mod

FACT $\implies Z_*, B_*$ free (as $\text{Ker } \partial^*, \text{Im } \partial^*$ are submods of C_*)

\implies SES splits, choose splitting $C_* \xrightleftharpoons[S]{\partial^*} B_{*+1}$ so $\partial_* \circ S = \text{id}$ recall just pick preimages under ∂_* of a basis for B_*

dual SES \implies

$$\begin{array}{ccccccc} 0 & \leftarrow & Z^* & \xleftarrow{\text{incl}^*} & C^* & \xleftarrow{\partial^*} & B^{*+1} & \leftarrow & 0 \\ & & \uparrow \partial=0 & & \uparrow \partial & & \uparrow \partial=0 & & \\ 0 & \leftarrow & Z^n & \leftarrow & C^n & \xleftarrow{\partial^*} & B^{n-1} & \leftarrow & 0 \\ & & \uparrow \partial=0 & & \uparrow \partial & & \uparrow \partial=0 & & \\ 0 & \leftarrow & Z^{n-1} & \leftarrow & C^{n-1} & \xleftarrow{\partial^*} & B^{n-2} & \leftarrow & 0 \end{array}$$

note: $\text{incl}^* = \text{restrict to } Z_*$ since $\text{incl}^* \circ \phi: Z_* \xrightarrow{\text{incl}} B_* \xrightarrow{\phi} \mathbb{Z}$
(Rmk) Although $\partial^* = 0: B^* \rightarrow B^{*+1}$ the map $\partial^*: B^{n-1} \rightarrow C^n$ need not = 0:
 $\psi: B_{n-1} \rightarrow \mathbb{Z}$
 $\implies \partial^* \psi = \psi \circ \partial: C_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\psi} \mathbb{Z}$

Connecting map

$$\begin{array}{ccc} 0 & \leftarrow & \partial^* \psi & \xleftarrow{\partial^*} & \psi|_{B_*} = \phi|_{B_*} \\ \uparrow & & \uparrow & & \uparrow \\ \delta: Z^{n-1} \rightarrow B^{n-1} & & \exists \psi & & \phi|_{B_*} \\ \text{of LES:} & & & & B_* \subseteq Z_* \end{array} \implies \boxed{\delta(\phi) = \phi|_{B_*}}$$

LES \implies

$$\dots \leftarrow \delta^n Z^n \leftarrow H^n C \xleftarrow{\partial^*} B^{n-1} \xleftarrow{\delta^{n-1}} Z^{n-1}$$

$\psi \longleftarrow [\psi] \qquad \phi|_{B_{n-1}} \longleftarrow \phi$

$(H^* B = B^*, H^* Z = Z^*$ since $\partial^* = 0)$

$$\implies 0 \leftarrow \text{Ker } \delta^n \leftarrow H^n C \leftarrow B^{n-1} / \text{Im } \delta^{n-1} \leftarrow 0$$

$$\text{Ker } \delta^n = \{ \phi \in Z^n : \phi(B_n) = 0 \} \implies \text{so: } \phi: Z_n \rightarrow \mathbb{Z}$$

$Z_n / B_n = H_n(C_*)$

(★) Universal Coefficients Thm: $0 \rightarrow B^{n-1} / \text{Im } \delta^{n-1} \rightarrow H^n(C) \rightarrow \text{Hom}(H_n(C_*), \mathbb{Z}) \rightarrow 0$ is SES
 $\text{Ext}^1(H_{n-1}(C), \mathbb{Z}) \xrightarrow{[\psi]} (\psi: H_n(C_*) \rightarrow \mathbb{Z})$ and natural

(evaluation of a cohomology class on cycles)

and SES splits (but not naturally): $B^{n-1} / \text{Im } \delta^{n-1} \xrightleftharpoons[S^*]{\partial^*} H^n(C)$ $S^* \circ \partial^* = \text{id}$
 $\implies H^n(C) \cong \text{Hom}(H_n(C), \mathbb{Z}) \oplus \text{Ext}^1(H_{n-1}(C), \mathbb{Z})$ (well-defined: $S^*(\partial^* \psi) = \psi \circ \partial_* \circ S$ since $\partial_* \circ S = \text{id} \implies \text{id} = (\partial_* S)^* = S^* \circ \partial^*$)

Lemma $\text{Ext}^1(H_{n-1}(C); \mathbb{Z}) \cong B^{n-1} / \text{Im } \delta^{n-1}$ canonically

Algebra background on Extension groups $\text{Ext}^i(M; R)$ ($= \text{Ext}_R^i(M, R)$)

general case

M R -module, R ring (comm. with 1)

$\Rightarrow \exists$ free resolution:

$$\dots \rightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \rightarrow 0 \text{ exact, } P_i \text{ free } R\text{-mods}$$

(pick gens x_α for $M \Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow{\varphi_0} M, e_\alpha \mapsto x_\alpha$

" " y_β for $\text{Ker } \varphi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow{\varphi_1} \text{Ker } \varphi_0, e_\beta \mapsto y_\beta$

continue inductively)

our case

$H_{n-1}(C_*)$ \mathbb{Z} -mod ($R = \mathbb{Z}$)

$$\begin{array}{ccccccc} 0 & \rightarrow & B_{n-1} & \hookrightarrow & \mathbb{Z}_{n-1} & \rightarrow & H_{n-1}(C) \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & P_1 & & P_0 & & M \end{array}$$

Take $\text{Hom}(\cdot; R)$ and drop $\text{Hom}(M; R)$

$$0 \rightarrow \text{Hom}(P_0; R) \xrightarrow{\varphi_1^*} \text{Hom}(P_1; R) \xrightarrow{\varphi_2^*} \dots$$

Is cochain complex but not exact

\Rightarrow take cohomology groups:

Def $\text{Ext}^0(M; R) = \text{Ker } \varphi_1^*$

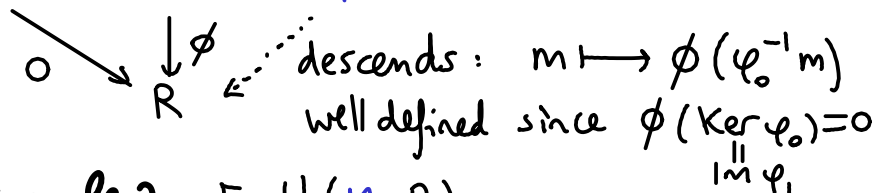
$\text{Ext}^1(M; R) = \text{Ker } \varphi_2^* / \text{Im } \varphi_1^*$

...

Fact
independent
of choices P_i, φ_i

Example 1 $\text{Ext}^0(M; R) \cong \text{Hom}(M, R)$

$$P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M$$



Example 2 $\text{Ext}^1(M; R) =$

$$\left\{ \phi : P_2 \xrightarrow{\varphi_2} P_1 \rightarrow P_0 \right\} / \left\{ \phi = \varphi_0 \varphi_1 : P_1 \xrightarrow{\varphi_1} P_0 \right\}$$

$$0 \rightarrow \mathbb{Z}^{n-1} \rightarrow B^{n-1} \rightarrow 0$$

Proof of Lemma

By Example 2,

$$\text{Ext}^1(H_{n-1}(C_*); \mathbb{Z}) =$$

$$= \left\{ \begin{array}{c} 0 \rightarrow B_{n-1} \rightarrow \mathbb{Z}_{n-1} \\ \phi \downarrow \\ \mathbb{Z} \end{array} \right\} \text{ modulo}$$

those arising from restriction

$$\left\{ \begin{array}{c} 0 \rightarrow B_{n-1} \rightarrow \mathbb{Z}_{n-1} \\ \phi \downarrow \searrow \downarrow \phi \\ \mathbb{Z} \end{array} \right\}$$

Thus $B^{n-1} / \text{Im } \delta^{n-1}$. \square

Rmk If R PID, then $\text{Ker}(P_0 \rightarrow M)$ is free (since submod of free mod P_0)

\Rightarrow can pick $P_1 = \text{Ker}(P_0 \rightarrow M)$, $P_k = 0$ for $k \geq 2 \Rightarrow \text{Ext}^k(M; R) = 0$ $k \geq 2$

(Co)homology with coefficients in a ring/field/module

Motivation

So far we had (C_*, ∂_*) chain cx of abelian groups } in graded sense
 $\Rightarrow H_*(C_*) = \text{Ker } \partial_* / \text{Im } \partial_*$ abelian group (since $\text{Ker } \partial, \text{Im } \partial$ are)

We cannot use a chain cx of (non-abelian) groups, because $\text{Im } \partial_*$ need not be a normal subgroup of $\text{Ker } \partial_*$.

However, abelian groups can be thought of as \mathbb{Z} -modules, then given any **abelian group G** , define **homology with coeffs in G**

$$H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G) \leftarrow \begin{array}{l} \text{with differential} \\ \partial_* \otimes \text{id} \end{array}$$

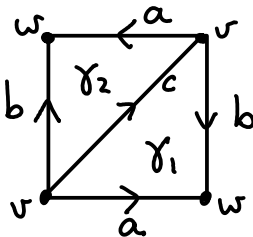
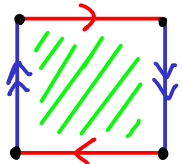
Def X space $\Rightarrow H_*(X; G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)$

Explanation:

$C_k(X)$ free \mathbb{Z} -mod $\cong \bigoplus_{\mathbb{I}_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{\mathbb{I}_k} G$: just replace \mathbb{Z} by G (as $\mathbb{Z} \otimes \cdot \cong \cdot$)

Why care? We hope to get more/new invariants of spaces

Example $X = \mathbb{RP}^2 =$



$*$	$C_*^\Delta(\mathbb{RP}^2; G)$
0	$G \vee \oplus G \vee w$
1	$G a \oplus G b \oplus G c$
2	$G \gamma_1 \oplus G \gamma_2$

for $G = \mathbb{Z}/2$: $0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\begin{pmatrix} \partial_2 \\ \vdots \\ \vdots \end{pmatrix}} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\begin{pmatrix} \partial_1 \\ \vdots \\ \vdots \end{pmatrix}} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow 0$

$\Rightarrow H_*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & *=0 \\ \mathbb{Z}/2 & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases}$ compare: $H_*(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & *=1 \\ 0 & \text{else} \end{cases}$ ($G = \mathbb{Z}$ case)

Form cochain complex using $\text{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs to G) in place of $\text{Hom}(\cdot, \mathbb{Z})$

$$\begin{array}{l} H^*(C_*; G) = H_*(\text{Hom}_{\mathbb{Z}}(C_*, G)) \leftarrow \begin{array}{l} \text{with differential } \partial^*: \\ \partial^* \phi = \phi \circ \partial_* \end{array} \\ H^*(X; G) = H_*(\text{Hom}_{\mathbb{Z}}(C_*(X), G)) \leftarrow \begin{array}{l} \text{so:} \\ H^*(C_*(X); G) \end{array} \end{array}$$

Universal coefficients thm (same proof using $\text{Hom}_{\mathbb{Z}}(\cdot, G)$)

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_*); G) \rightarrow H^n(C_*; G) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n(C_*), G) \rightarrow 0$$

$$[\varphi] \mapsto (\varphi: H_n(C_*) \rightarrow G)$$

Example $X = \mathbb{R}P^2$, $G = \mathbb{Z}/2$, apply $\text{Hom}_{\mathbb{Z}}(\cdot, G)$ to the complex in previous example

$$0 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow{\partial^1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow{\partial^0} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow 0$$

$$\begin{pmatrix} | & | & | \\ | & | & | \\ | & | & | \end{pmatrix} \quad \begin{pmatrix} | & | \\ | & | \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H^*(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$
 ($G = \mathbb{Z}$ case)

(compare $\text{Hom}(H_*(\mathbb{R}P^2), \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & * \geq 2 \end{cases}$
 caused by $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$
 $\leftarrow H_1(\mathbb{R}P^2) \leftarrow G = \mathbb{Z}$)

Can generalise further:

C_* = chain cx of ...	coefficients in:	
abelian gps (\mathbb{Z} -mods)	abelian gp G (\mathbb{Z} -mod)	$H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$
R -modules \leftarrow ring (comm. with 1)	R -module M	$H_*(C_*; M) = H_*(C_* \otimes_R M)$

Rmk $H_*(C_*; M)$ will be an R -module since $\ker \partial, \text{Im } \partial$ are (∂_* is R -linear hom by assumption)

X space $\Rightarrow C_{\mathbb{R}}(X; R) = C_{\mathbb{R}}(X) \otimes_{\mathbb{Z}} R \cong \bigoplus_{\mathbb{I}_k} R$: just replace \mathbb{Z} by R (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

$$\Rightarrow H_*(X; M) = H_*(C_*(X; R); M) = H_*(C_*(X; R) \otimes_R M)$$

so just replace each \mathbb{Z} by M in $C_*(X)$

Form cochain complex using $\text{Hom}_R(\cdot, M)$ ($= R$ -linear homs to M) in place of $\text{Hom}(\cdot, \mathbb{Z})$

$$H^*(C_*; M) = H_*(\text{Hom}_R(C_*, M))$$

with differential ∂^* : $\partial^* \phi = \phi \circ \partial_*$

$$H^*(X; M) = H_*(\text{Hom}_R(C_*(X; R), M))$$

so: $H^*(C_*(X; R); M)$

Rmk These are R -mods. If we use $M = R$, then they are also rings via cup product

Universal Coefficients Thm For R any PID, C_* chain cx of R -mods,

$$0 \rightarrow \text{Ext}_R^1(H_{n-1}(C_*); M) \rightarrow H^n(C_*; M) \rightarrow \text{Hom}_R(H_n(C_*), M) \rightarrow 0$$

is SES and natural.

$B^{n-1} / \text{im } \delta^{n-1}$ working over R using homs to M $[\varphi] \mapsto (\varphi: H_n(C_*) \rightarrow M)$

and the SES splits but the splitting is not natural. Same proof using $\text{Hom}_R(\cdot, M)$

Example $R = \mathbb{F}$ field $\Rightarrow C_*, H_*, H^*$ are vector spaces / \mathbb{F} .

Rmk all \mathbb{F} -mods (i.e. vector spaces / \mathbb{F}) are free \mathbb{F} -mods $\cong \bigoplus \mathbb{F} b_i$
 up to iso they are determined by $\dim_{\mathbb{F}} =$ cardinality of basis. basis b_i

Cor $C_* =$ chain cx of \mathbb{F} -vector spaces $\Rightarrow H^n(C_*; \mathbb{F}) \cong (H_n(C_*))^*$ ← dual v.s.: $\text{Hom}_{\mathbb{F}}(H_n(C_*); \mathbb{F})$

Pf Pick any basis v_i for \mathbb{F} -v.s. B_{n-1} , extend it to a basis v_i, w_j of Z_{n-1} (also works in ∞ dim case).

\Rightarrow can extend any \mathbb{F} -linear map $\psi: B_{n-1} \rightarrow \mathbb{F}$ to $\phi: Z_{n-1} \rightarrow \mathbb{F}$ just pick any values $\phi(w_j) \in \mathbb{F}$ e.g. $\phi(w_j) = 0$.

$\Rightarrow B^{n-1}/\text{img } \delta^{n-1} = 0$ so $H^n(C_*; \mathbb{F}) \rightarrow \text{Hom}(H_n(C_*); \mathbb{F})$ iso \square

Cor $H^n(X; \mathbb{F}) \cong H_n(X; \mathbb{F})^*$ ← dual v.s. for any field \mathbb{F} .

Cor $H^n(X; \mathbb{M}) \cong H_{CW}^n(X; \mathbb{M}) \cong H_{\Delta}^n(X; \mathbb{M})$
 if X is CW-cx if X is Δ -cx

Pf Cor holds for homology and the isos are natural. ← i.e. functorial w.r.t. maps

The universal coeff. thm SES is natural. So result holds by 5-Lemma. \square

Algebra: structure thm for f.g. abelian groups and R-mods

Fact 1 A f.g. abelian gp $\Rightarrow A \cong \underbrace{\mathbb{Z}^r}_{\text{free part } F} \oplus \underbrace{\mathbb{Z}/p_1^{n_1} \oplus \dots \oplus \mathbb{Z}/p_a^{n_a}}_{\text{torsion part } T}$

where $p_i \in \mathbb{Z}$ prime (need not be distinct)
 Also r, a, p_i, n_i are unique (up to reordering)

Example $\mathbb{Z}/4 = \mathbb{Z}/2^2 \not\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$
 $\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$ (Chinese Remainder Thm)

Fact 2 $T \cong \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_k$ with $d_1 | d_2 | \dots | d_k$ ($d_i \in \mathbb{N}$ unique)

Example $\mathbb{Z}/2 \oplus \mathbb{Z}/2^2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cdot 2^2$ $d_1=2, d_2=12$

Fact 3 M f.g. R -mod, R PID, then:

M	\cong	$F \oplus T$	← $r \in \mathbb{N}$ unique, called <u>rank</u> of M ← $p_i \in R$ primes, $p_i^{n_i}$ unique up to ordering & mult ⁿ by ← $d_1 \dots d_k$ non-zero, not invertible d_i called <u>invariant factors</u> unique up to mult ⁿ by invertible elements e.g. ± 1 if $R = \mathbb{Z}$
F	\cong	R^r	
T	\cong	$R/p_1^{n_1} \oplus \dots \oplus R/p_a^{n_a}$	
	\cong	$R/d_1 \oplus R/d_2 \oplus \dots \oplus R/d_k$	

Rmk $T = \{m \in M : rm = 0 \text{ some } r \neq 0 \in R\} =$ torsion elements
 $F \cong M/T$

Torsion shift

(as you would expect for $\text{Ext}^0 = \text{Hom}$ since a hom from $\bigoplus M_i \rightarrow \dots$ means you have to specify it on each M_i)

Easy Exercise

$$\text{Ext}_R^*(\bigoplus_i M_i; \prod_j N_j) \cong \prod_i \prod_j \text{Ext}_R^*(M_i; N_j)$$

← any R-mods M_i, N_j

↓
Upshot

To compute $\text{Ext}_R^1(M; R)$ for $M = R^r \oplus R/d \oplus \dots$ just need:

$$\begin{aligned} \text{Ext}_R^1(R; R) &= 0 \\ \text{Ext}_R^1(R/d; R) &\cong R/d \end{aligned}$$

← since $0 \rightarrow R \xrightarrow{\text{id}} R \rightarrow 0$
 $\begin{matrix} \parallel & \parallel \\ P_1 & P_0 \end{matrix}$

← since $0 \rightarrow R \xrightarrow{d} R \xrightarrow{1} R/d \rightarrow 0$
 $\begin{matrix} \downarrow \phi \\ R \end{matrix}$ so choice of $\phi(1) \in R$
 modulo ϕ coming from
 $\begin{matrix} R & \xrightarrow{d} & R \\ \downarrow \phi & \swarrow \varphi & \\ R & & \end{matrix}$ so $\phi(1) = d \cdot \varphi(1) \in d \cdot R$

$$\Rightarrow \text{Ext}_R^1(M; R) \cong \text{Torsion}(M)$$

Exercises

- $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n; \mathbb{Z}/m) \cong \mathbb{Z}/\text{gcd}(m, n)$
- Abelian gp $\Rightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}; G) = 0$ and $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/d; G) \cong G/d \cdot G$ (if $d \neq 0$)
- R any ring (comm. with 1)
 $x \in R$ not zero divisor $\Rightarrow \text{Ext}_R^*(R/(x); N) \cong \begin{cases} \{n \in \mathbb{N} : x \cdot n = 0\} & * = 0 \\ N/x \cdot N & * = 1 \\ 0 & \text{else} \end{cases}$

Cor If $H_n(X; R)$ f.g. R -mod $\forall n$, R PID,
 $\Rightarrow H_n(X; R) = R^{r_n} \oplus T_n$ (free & torsion parts)

$$\Rightarrow H^n(X; R) \cong R^{r_n} \oplus T_{n-1}$$

← torsion moves up!

not natural

Pf $0 \rightarrow T_{n-1} \rightarrow H^n(X; R) \rightarrow \text{Hom}(R^{r_n} \oplus T_{n-1}; R) \rightarrow 0$
 $\text{Hom}(R^{r_n} \oplus T_{n-1}; R) \cong (\text{Hom}(R; R))^{r_n} \times \text{Hom}(T_{n-1}; R)$

$R \rightarrow R \xrightarrow{1 \mapsto x} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \cong \mathbb{Z}^{r_n}$
 x determines the hom

(R -linear hom of R -modules)
 proof: $T_{n-1} \xrightarrow{\varphi} R$ then $r \cdot t = 0$ some $r \neq 0 \in R$ since t torsion.
 Then $0 = \varphi(r \cdot t) = r \cdot \varphi(t)$ but R is an integral domain, so $\varphi(t) = 0$. \square

$$\Rightarrow 0 \rightarrow T_{n-1} \rightarrow H^n(X; R) \rightarrow R^{r_n} \rightarrow 0$$

free, so can split the SES (pick lifts of basis). \square

← so not canonical

Example

$*$	$H_*(\mathbb{R}P^3)$	$H^*(\mathbb{R}P^3)$
0	\mathbb{Z}	\mathbb{Z}
1	$\mathbb{Z}/2$	0
2	0	$\mathbb{Z}/2$
3	\mathbb{Z}	\mathbb{Z}

torsion moves up

Universal coefficients Theorem in homology (recall $H_*(C_* \otimes_R M) = H_*(C_*; M)$)

FACT Theorem C_* chain cx of free R -mods, M R -module

$$\Rightarrow \begin{matrix} \text{natural} \\ \text{SES} \end{matrix} 0 \rightarrow H_*(C_*) \otimes_R M \rightarrow H_*(C_* \otimes_R M) \rightarrow \text{Tor}_1^R(H_{*-1}(C_*); M) \rightarrow 0$$

$[C] \otimes m \mapsto [C \otimes m]$

defined below.

The SES splits, but the splitting is not natural.

Torsion groups: A, B R -mods (R comm. ring with 1) exact sequence, P_i free R -mods

pick $\dots \rightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \rightarrow 0$ free resolution

$\Rightarrow \dots \rightarrow P_2 \otimes B \xrightarrow{\varphi_2 \otimes \text{id}} P_1 \otimes B \xrightarrow{\varphi_1 \otimes \text{id}} P_0 \otimes B \rightarrow 0$ not exact but is chain cx

$\text{Tor}_k^R(A, B) = H_k(\text{this complex}) \leftarrow \text{fact independent of choices of } P_i, \varphi_i$

Rmk R PID $\Rightarrow \ker \varphi_0$ free \Rightarrow pick $\begin{cases} P_1 = \ker \varphi_0 \\ P_k = 0 \text{ for } k \geq 2 \end{cases} \Rightarrow$ only $\text{Tor}_0^R, \text{Tor}_1^R$ can be non-zero

Example $\text{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = ?$

take $\cdot \otimes \mathbb{Z}/b$ drop $\mathbb{Z}/a \otimes \mathbb{Z}/b$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \xrightarrow[\text{quotient}]{\varphi_0} \mathbb{Z}/a \rightarrow 0 \text{ free resolution}$$

$$\Rightarrow 0 \rightarrow \mathbb{Z}/b \xrightarrow{\cdot a} \mathbb{Z}/b \rightarrow 0 \text{ (since } \mathbb{Z} \otimes_{\mathbb{Z}} G \cong G \text{ any } G)$$

$$\text{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = (\mathbb{Z}/b) / a \cdot \mathbb{Z}/b \cong \mathbb{Z}/\langle a, b \rangle \cong \mathbb{Z}/\text{gcd}(a, b) \cong \mathbb{Z}/a \otimes \mathbb{Z}/b$$

$$\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = \{x \in \mathbb{Z}/b : a \cdot x = 0\} \cong \mathbb{Z}/\text{gcd}(a, b)$$

Facts $\text{Tor}_0^R(A, B) = P_0 \otimes B / \text{Im}(\varphi_0 \otimes \text{id}) \cong A \otimes B$ via: $\frac{b}{\text{gcd}(a, b)} \leftarrow +1$

$\text{Tor}_*^R(A, B) \cong \text{Tor}_*^R(B, A)$

Exercise $\text{Tor}_*^R(\oplus A_i, \oplus B_j) \cong \oplus \oplus \text{Tor}_*^R(A_i, B_j)$

$\text{Tor}_*^R(A, B) = 0$ for $* \geq 1$ if A or B is free (use $M \otimes_R R \cong M$)

$$\text{Tor}_*^R(R/u, M) \cong \begin{cases} M/u \cdot M & *=0 \\ \text{u-torsion}(M) = \{x \in M : u \cdot x = 0\} & *=1 \\ 0 & \text{else} \end{cases}$$

$u \in R$ not zero divisor, R any ring (comm. with 1)

deduce $\text{Tor}_i^R(A, M)$ for f.g. R -mods A R PID

Example $H_*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & *=0 \\ \mathbb{Z}/2 & *=1 \\ \mathbb{Z}/2 & *=2 \end{cases} \quad H_*(\mathbb{R}P^2) \otimes \mathbb{Z}/2 \cong \begin{cases} \mathbb{Z} \otimes \mathbb{Z}/2 \\ \mathbb{Z}/2 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 \\ 0 \end{cases}$

Künneth Thm

caused by $\text{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{R}P^2); \mathbb{Z}/2) = \text{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

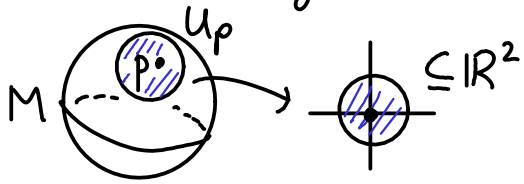
R PID \Rightarrow natural SES: $0 \rightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \rightarrow H_n(C_* \otimes D_*) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}_1^R(H_i(C_*), H_j(D_*)) \rightarrow 0$

and the SES splits but the splitting is not natural. Example $R = \text{field} \Rightarrow$ that $\text{Tor}_1 = 0$ degree increases by 1

Example (take $C_* := C^{-*}(X), D_* := C^{-*}(Y)$): $0 \rightarrow H^*(X) \otimes H^*(Y) \rightarrow H^*(X \times Y) \rightarrow \text{Tor}_1(H^*(X), H^*(Y)) \rightarrow 0$ exact (if $C_*(X)$ has ∞ rank then $C^{-*}(X)$ may not be free but it will be "flat" and Thm holds if C_* is flat R -mod)

13. MANIFOLDS : POINCARÉ-LEFSCHETZ DUALITY

- M n -mfd is Hausdorff topological space s.t. $\forall p \in M$
 \exists open neighbourhood $U_p \subseteq M$ homeomorphic to \mathbb{R}^n



(equivalently: to an open ball, or any open set in \mathbb{R}^n)

One also requires M second countable i.e. \exists countable basis of open sets
 $\iff M$ is covered by countably many such U_p :
 ← exercise

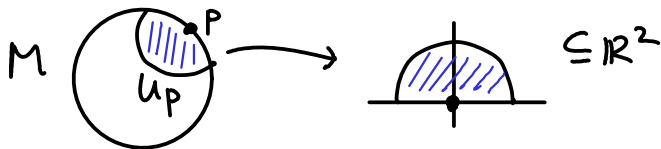
A submanifold $N \subseteq M$ is a mfd s.t. inclusion $N \rightarrow M$ is an embedding (i.e. a homeomorphism onto its image)

$$\{x \in \mathbb{R}^n : x_n \geq 0\}$$

$$\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$$

- M n -mfd with boundary if also allow $U_p \cong$ upper half space \mathbb{H}^n
 such p are called boundary points they form the boundary ∂M which is an $(n-1)$ -mfd without boundary.

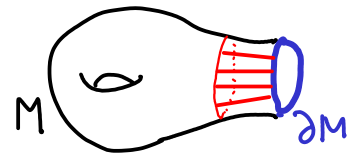
$p \mapsto 0$
 equivalently: any open nbhd of $0 \in \mathbb{H}^n$



FACT (collar nbhd thm) $\partial M \subseteq M$ has an open neighbourhood $\cong \partial M \times (0,1]$
 $\partial M \rightarrow \partial M \times 1$

M is closed if compact without boundary.

Rmk For manifolds, connected components = path components.
 (since locally \cong disc, so locally path-connected, so conn. \iff path-conn.)



Examples

n -torus

closed mfd's: $S^n, \mathbb{R}P^n, T^n = S^1 \times \dots \times S^1, \mathbb{C}P^n, O(n), SU(n)$

non-compact mfd's: $\mathbb{R}^n, \text{Mat}_{m \times n} \cong \mathbb{R}^{mn}, GL(n, \mathbb{R})$

mfd's with bdry: $\mathbb{D}^n, \mathbb{D}^1 \times S^1 = \text{rectangle}, \text{Möbius band} = \text{torus with hole}, T^2 \setminus \text{open disc} = \text{torus with hole}$

FACT (Milnor 1959) Any mfd is homotopy equivalent to a CW-cx

fact If M is a compact manifold then $H_*(M)$ are finitely generated

Rmk M triangulable if $M \cong$ simplicial cx.

Not all mfd's are triangulable, but most of those we encounter are.

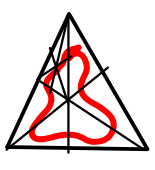
Compact manifolds have f.g. homology ← Non-examinable proof

- ① X space is a Euclidean neighbourhood retract if
 \exists embedding $j: X \rightarrow \mathbb{R}^N$ some N , s.t. $i(X)$ is a retract of a nbhd $V \subseteq \mathbb{R}^N$
 ↑ (homeo onto image)
- ② X is weakly locally contractible if \forall nbhd $x \in U \subseteq X$, \exists nbhd $x \in V \subseteq U$
 s.t. V is contractible inside U .

FACT compact $X \subseteq \mathbb{R}^n$ is ① \iff X is ②

Rmk If we find nbhd V as in ① with retraction $V \xrightarrow{f} X$ then any smaller nbhd V' also retracts using $f|_{V'}: V' \rightarrow X$. Similarly in ② $V' \subseteq V$ is contractible: restrict the hom.

Lemma A X compact & ① $\implies X$ is the retract of a finite simplicial cx
Pf $i(X) \subseteq \mathbb{R}^n$ compact \implies lies inside some large n -simplex $\Delta^n \rightarrow \mathbb{R}^n$



Apply barycentric subdivision until simplices have diameter $< \text{dist}(X, \partial V)$.
 Simpl. cx. = $\cup \{ \text{subsimplices which intersect } X \}$ using the restriction of retraction $V \rightarrow X$. \square

Rmk Also deduce X has f.g. homology since retractions are surjective on H_* .
 $(\oplus \mathbb{Z} \rightarrow H_*(\text{finite simpl. cx.}) \xrightarrow{\text{retract}} H_*(X)$ so get surjection from free \mathbb{Z} -mod, so f.g.)

Lemma B M compact mfd $\implies M$ embeds into \mathbb{R}^N , some N .

Pf "Just do it proof":
 $\forall p \in M, \exists$ homeo $\mathbb{D}^n \xrightarrow{\psi_p} \text{nbhd}(p \in M)$
 Pick finite subcover of ψ_p : of $M = \cup_{p \in M} \psi_p(\mathbb{D}^n)$. Say $i=1, \dots, k$
 $\psi_{p_i}: M \xrightarrow{\psi_{p_i}^{-1}} \mathbb{D}^n / \partial \mathbb{D}^n \cong S^n \subseteq \mathbb{R}^{n+1}$ define embedding $(\psi_{p_1}, \dots, \psi_{p_k}): M \rightarrow \mathbb{R}^{k \cdot (n+1)}$
 ↑ (send $M \setminus \text{Im}(\psi_{p_i})$ to the point corresponding to $\partial \mathbb{D}^n \in \mathbb{D}^n / \partial \mathbb{D}^n$).

Finally use: a continuous bijection from a compact space to a Hausdorff space is $\cong \square$

Rmk Same works if M has boundary, just consider its double $M \cup M$
 and apply the Lemma to the double.
 identify along ∂M

Cor M compact mfd (possibly with bdry) $\implies M$ has f.g. homology

Pf Mfds satisfy ② since locally ball \simeq pt. M embeds in \mathbb{R}^N by Lemma B.
 ① holds by FACT. Done by Lemma A. \square

Local orientations and orientability

Def A local orientation of M at $x \in M$ is a choice of generator excise complement of nbhd $V_x \cong \mathbb{D}^n$

$$\mu_x \in H_n(M, M \setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n \setminus \{0\}) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$$

choice of homeo is not canonical!

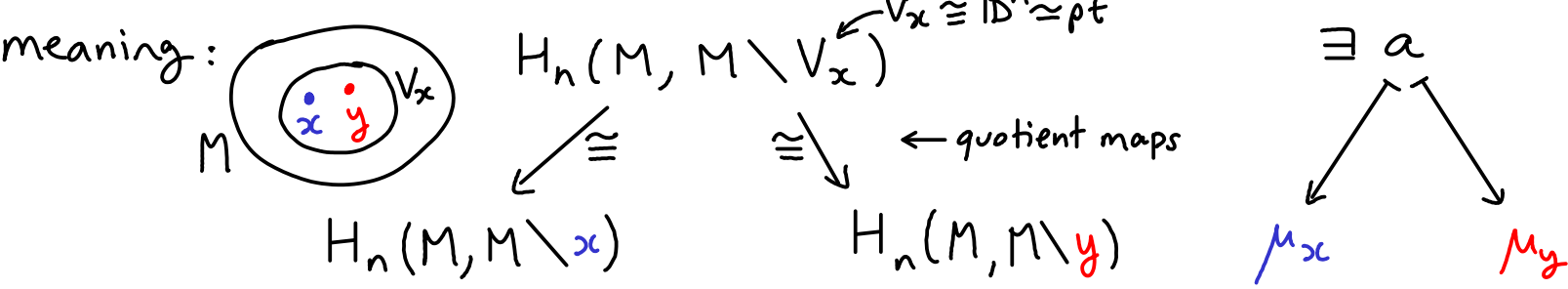
$\partial \mathbb{D}^n = S^{n-1}$

(see section 5 of these notes)

Rmk Even though we can pick the last 2 isos in a canonical way, we cannot make the first one canonical. The intuitive notion that an orientation of \mathbb{R}^n is a choice of basis for \mathbb{R}^n can be made rigorous now, since a generator for $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ can be specified by an \mathbb{R} -linear map $\Delta^n \xrightarrow{\varphi} \mathbb{R}^n$ sending $e_i \mapsto i$ -th basis vector, provided that $0 \in \text{interior}(\text{image } \varphi)$. By Gram-Schmidt, one can homotope any basis to an orthonormal one, and by successive rotations can homotope any o.n. one to another o.n. basis one basis vector at a time except possibly the last one when a reflection may be necessary.

This argument shows that $GL(n)$ has 2 connected components depending on whether $\det > 0$ or < 0 (once we homotoped the matrix to an orthogonal matrix $O(n)$ we have $\det = +1$ for $SO(n)$ and $\det = -1$ for $SO(n)$ o reflection, which are the 2 connected components of $O(n)$).

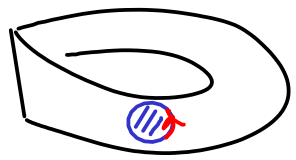
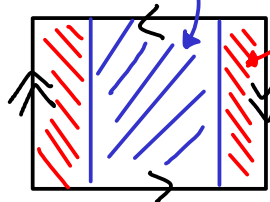
Def An orientation of M is a locally consistent choice $x \mapsto \mu_x$



Def M orientable if \exists orientation on M
oriented if we chose an orientation

Examples $S^n, \mathbb{R}^n, \mathbb{C}P^n$, orientable surfaces Σ_g , $\mathbb{R}P^n \leftarrow$ odd n

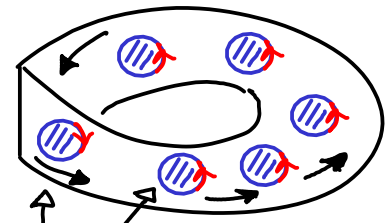
Non-example $\mathbb{R}P^2 = \text{Möbius band} \cup \mathbb{D}^2$



choice of μ_x is choice of orientation of boundary circle of small disc containing x

$\Rightarrow \mathbb{R}P^2$ not orientable

by local consistency can move disc continuously and preserves orientation



discs are differently oriented \Rightarrow contradicts local consistency.

The fundamental class [M]

FACT
Theorem

For M closed n -mfd :

1) M orientable connected $\Rightarrow H_n(M) \cong H_n(M, M \setminus x) = \mathbb{Z} \cdot \mu_x$

natural map from the LES

\uparrow
once we choose an orientation

$\Rightarrow \exists [M] \longleftarrow \mu_x$

\uparrow
once we choose an orientation $(\mu_x)_{x \in M}$

\uparrow
called fundamental class

(if swap orientation: for $-\mu_x$ get $-[M]$)

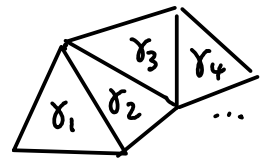
2) M not orientable connected $\Rightarrow H_n(M) = 0$

$H_n(M; \mathbb{F}_2) = \mathbb{F}_2 \cdot [M] \cong \mathbb{F}_2$

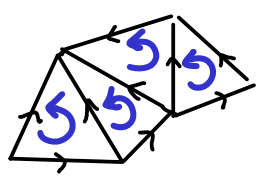
\leftarrow (or any field of characteristic 2)

Construction of [M] if M has Δ -complex structure

M compact \Rightarrow finite # n -simplices $\gamma_1, \dots, \gamma_N$



M oriented \Rightarrow pick orientations of $\gamma_1, \dots, \gamma_N$ to agree with given orientation of M : \leftarrow for $x \in \text{Int}(\gamma_i)$



$$\mathbb{Z} \cdot \mu_x = H_n(M, M \setminus x) \xrightarrow[\text{exc.}]{\cong} H_n(\gamma_i, \gamma_i \setminus x) = \mathbb{Z} \cdot \gamma_i$$

$\mu_x \longmapsto \gamma_i$

$\Rightarrow [M] := \sum \gamma_i$ satisfies $\partial [M] = 0$ ✓
(each facet arises twice with opposite signs)

$$H_n(M) \rightarrow H_n(M, M \setminus x) \xrightarrow{\cong} H_n(\gamma_i, \gamma_i \setminus x)$$

$[M] \longleftarrow \mu_x \longmapsto \gamma_i$

More generally:
 $[M] := \sum \pm \gamma_i$
 where signs come from
 $H_n(M, M \setminus x) \xrightarrow{\cong} H_n(\gamma_i, \gamma_i \setminus x)$
 $\mu_x \longmapsto \pm \gamma_i$
 (so compare orientation of μ_x with orientation of γ_i)

Not difficult to see that $H_n^\Delta(M) = \mathbb{Z} \cdot [M]$, so $\Rightarrow H_n(M) \cong H_n(M, M \setminus x)$

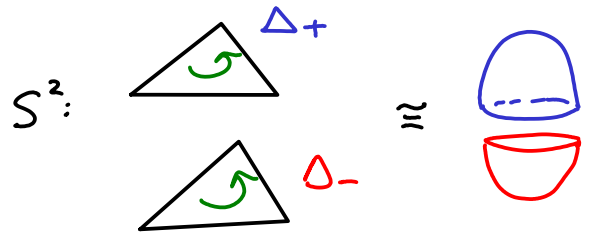
$[M] \longmapsto \mu_x$

Also $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \cdot [M]$ since $C_{n+1}(M) = 0$ (\nexists $(n+1)$ -simplices since $\dim M = n$)

M non-orientable \Rightarrow each facet of γ_i appears twice in $\partial \sum \gamma_i$
 $\Rightarrow \partial \sum \gamma_i = 0$ over \mathbb{F}_2 independently of choices of orientations of γ_i . ✓

Examples

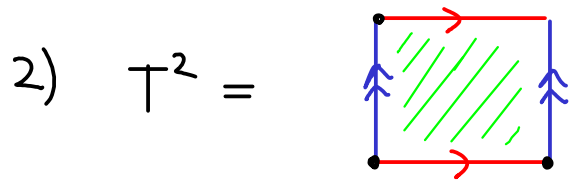
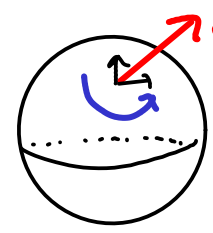
1) $S^n = \frac{\Delta_+^n \cup \Delta_-^n}{\text{glue bdris}}$



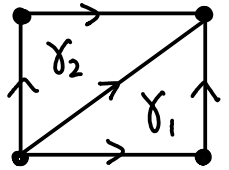
$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed

hence $\partial[S^n] = \partial\Delta_+ - \partial\Delta_- = 0$

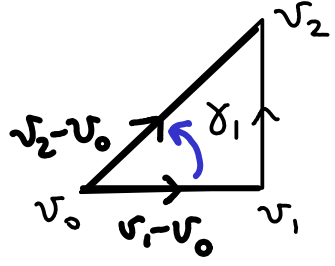
$\mathbb{D}^n \subseteq \mathbb{R}^n$ canonical orientation $\Rightarrow S^{n-1} = \partial\mathbb{D}^n$ " using outward normal first rule



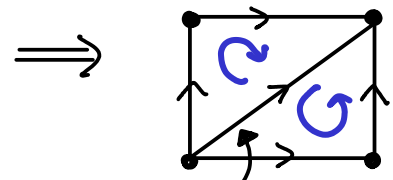
Δ -complex structure (compatibly with side identifications!)



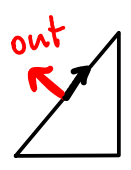
Want orientation induced by square $\in \mathbb{R}^2$



$v_1 - v_0, v_2 - v_0$ positive \mathbb{R}^2 -basis $\Rightarrow \gamma_1$ agrees with orientation

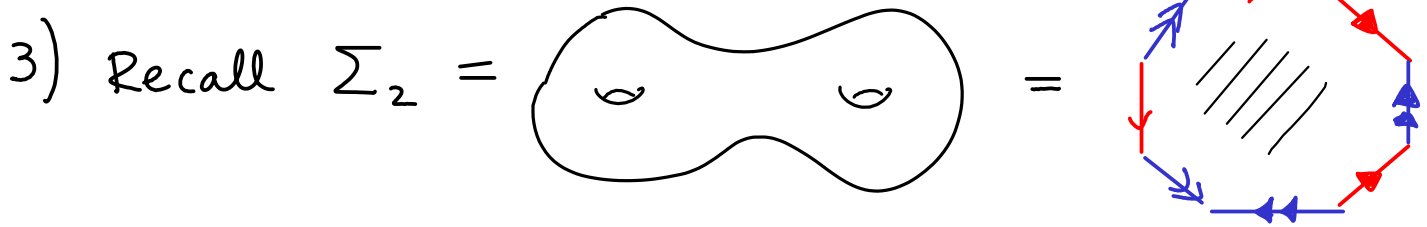


$[T^2] = +\gamma_1 - \gamma_2$
 \uparrow γ_2 orientation disagrees

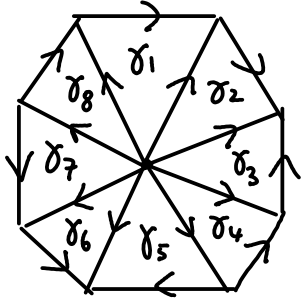


Rmk general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

So consistency \Rightarrow $\left\{ \begin{array}{l} \text{either simplices are compatibly oriented and the two} \\ \text{induced orientations on facet are } \underline{\text{opposite}} \\ \text{or not compatibly oriented but facet orient}^n \text{ is } \underline{\text{same}}, \\ \text{then } \underline{\text{need sign}} \text{ like in example when build } [T^2] \end{array} \right.$

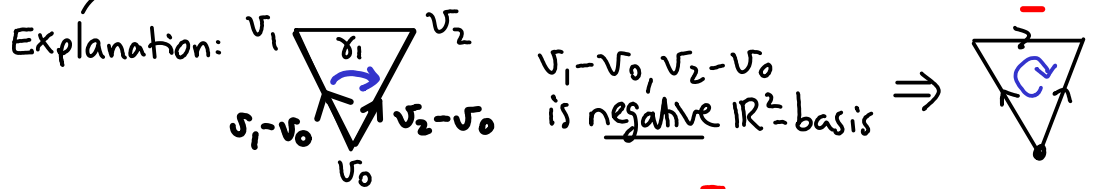


Δ -cx structure (compatible with side identifications!):



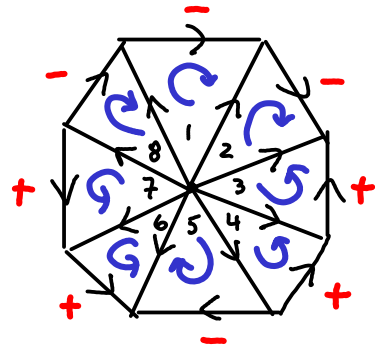
Use the orientation induced by polygon $\subseteq \mathbb{R}^2$

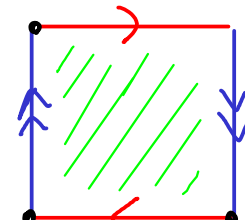
$$\Rightarrow [\Sigma_2] = -\delta_1 - \delta_2 + \delta_3 + \delta_4 - \delta_5 + \delta_6 + \delta_7 - \delta_8$$

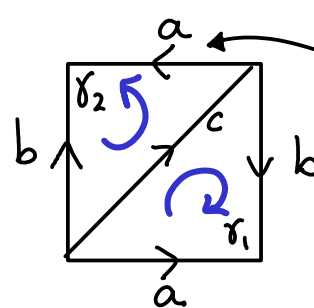



All simplices δ_i have $v_0 =$ centre of polygon

\Rightarrow sign $\begin{cases} - & \text{if outer edge clockwise} \\ + & \text{anti} \end{cases}$



3) $\mathbb{RP}^2 =$ 
(non-orientable) example



won't get Δ -cx structure if you try  since get issue here

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$$\Rightarrow [\mathbb{RP}^2] = -\delta_1 + \delta_2$$

$$\partial [\mathbb{RP}^2] = -(b - a + c) + (a - b + c)$$

$$= -2b + 2a$$

$$\neq 0 \quad \text{so not cycle in } C_*^{CW}(\mathbb{RP}^2)$$

However, working modulo 2:

$$\partial [\mathbb{RP}^2] = 0 \in C_*^{CW}(\mathbb{RP}^2; \mathbb{F}_2) \text{ since } 2=0 \text{ in } \mathbb{F}_2$$

$$\Rightarrow [\mathbb{RP}^2] \in H^2(\mathbb{RP}^2; \mathbb{F}_2)$$

Degree

Def M, N oriented closed connected n -mfds, $f: M \rightarrow N$
 $f_*: H_n(M) \rightarrow H_n(N)$
 $[M] \mapsto \underline{\deg(f)} \cdot [N]$

\mathbb{Z}^n

Lemma If $f^{-1}(y)$ finite,

then $\deg(f) = \sum_{x \in f^{-1}(y)} \deg(f_x)_*$

local degree

local map like in chapter 7:

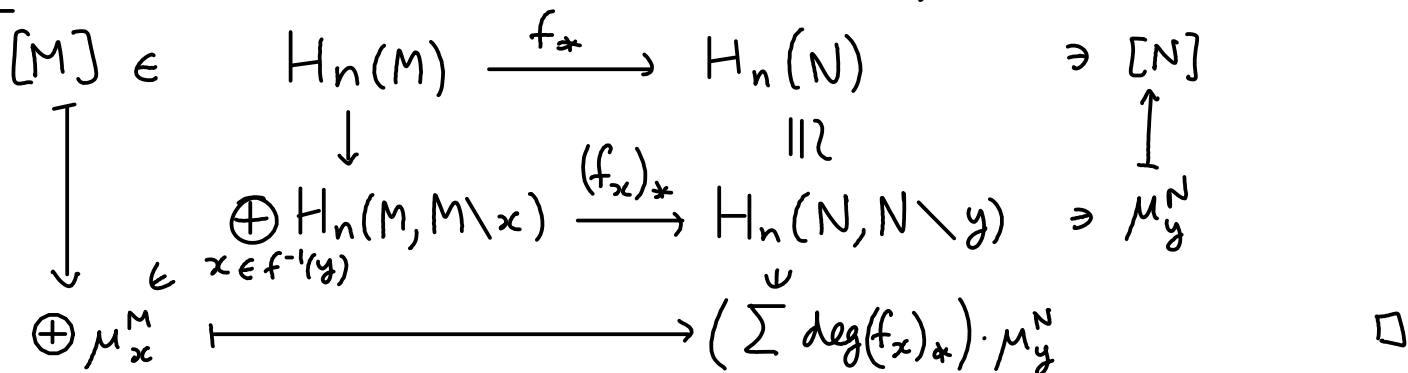
$$f_x: (M, M \setminus x) \rightarrow (N, N \setminus f(x))$$

$$(f_x)_*: H_n(M, M \setminus x) \rightarrow H_n(N, N \setminus f(x))$$

$$\cong H_n(\mathbb{D}^n, \mathbb{D}^n \setminus \{0\}) \cong \mathbb{Z}$$

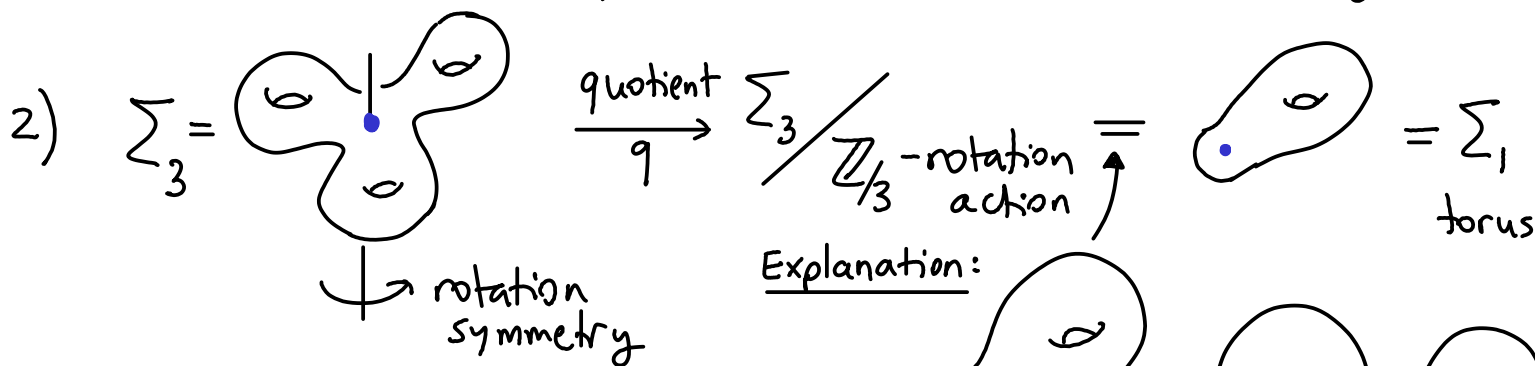
$$\mu_x \mapsto (\deg f_x) \cdot \mu_{f(x)}$$

pf

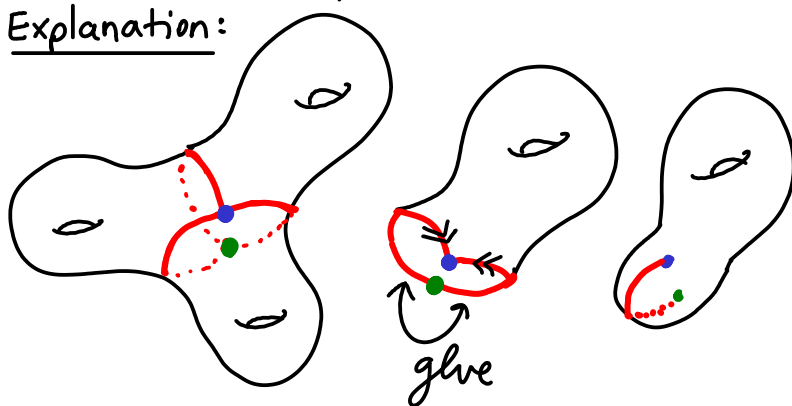


Examples

1) $S^1 \rightarrow S^1, z \mapsto z^n, [S^1] \mapsto n \cdot [S^1]$ so $\deg = n$



Easy check: $\deg(q) = 3$
 (e.g. use local degrees)



Cultural Rmk

For M, N, f smooth, the $\deg f = \#$ (preimages of a generic point of N)
 Idea: $\deg f$ tells you how many times you cover N . (almost all points work)

Poincaré duality

FACT Theorem For M closed n -mfd

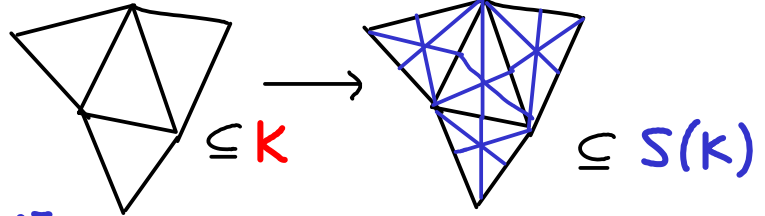
$$M \text{ oriented} \Rightarrow H^k(M) \cong H_{n-k}(M)$$

s.t. $1 \leftrightarrow [M]$
 $\hat{H}^0(M) \cong \hat{H}_n(M)$

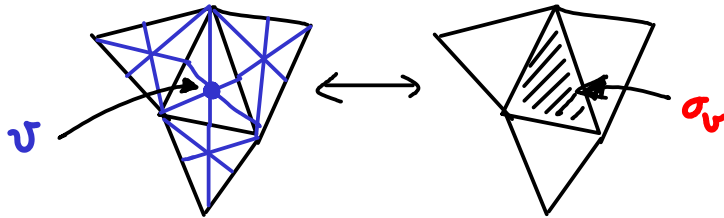
M non-oriented \Rightarrow same holds with \mathbb{F}_2 coefficients

Sketch proof when M is a simplicial complex K (Non-examinable)

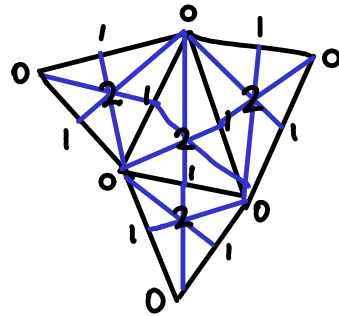
$S(K)$ = barycentric subdivision



1) simplex $\sigma = \sigma_v$ of K with barycentre $v \leftrightarrow v = v_\sigma$ vertex of $S(K)$



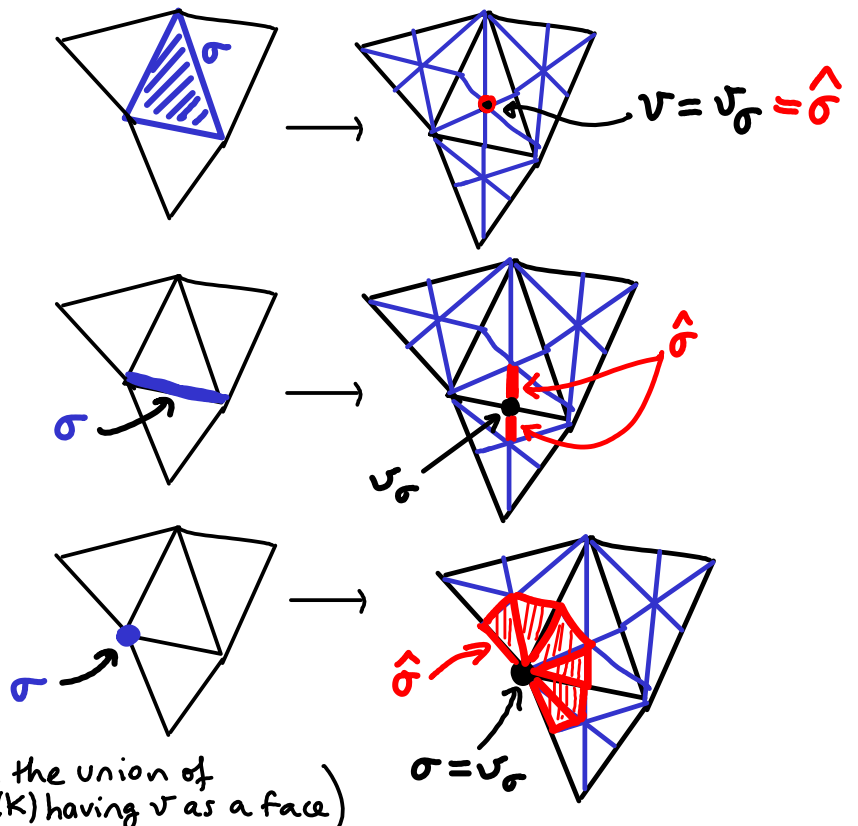
2) $ht(v) = (\text{height of } v) = \dim \sigma_v$



3) σ k -simplex of K
 \downarrow
dual simplex

$$\hat{\sigma} = \bigcup_{\tau \in S(K), v_\sigma \in \tau} \tau$$

$ht(v_\sigma)$ is min of heights of vertices of τ



Rmk: $\bigcup \tau$ with $ht(v_\sigma)$ max will give back σ .
 Thus $\hat{\sigma}, \sigma$ intersect transversely at v_σ .
 One can also describe $\hat{\sigma}$ as

$$\hat{\sigma} = \bigcap_{\text{vertices } v \in \sigma} \text{Star}_{S(K)}(v)$$

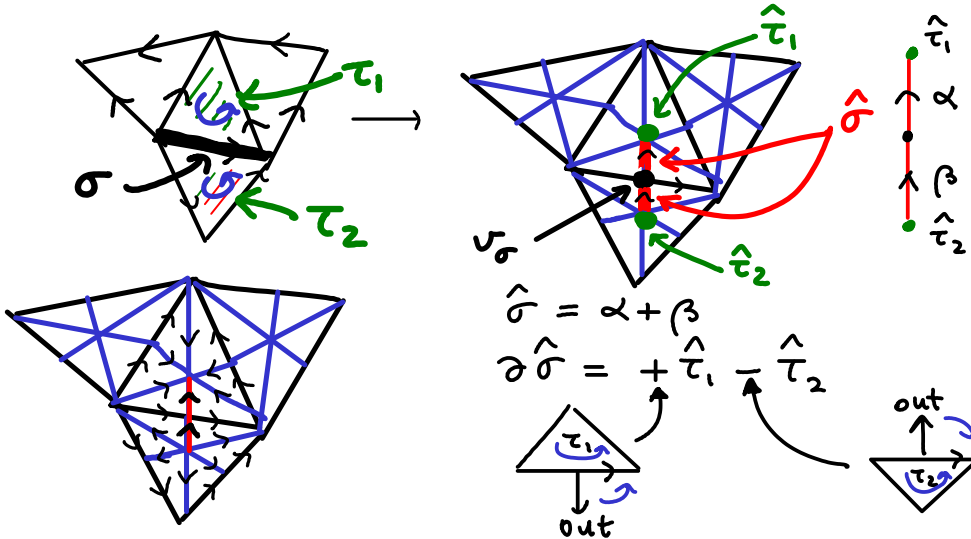
(closed star is the union of simplices of $S(K)$ having v as a face)

FACTS • $\dim \hat{\sigma} = n - \dim \sigma$

• dual cells $\hat{\sigma}$ give a cell decomposition of M

(*) • $\partial \hat{\sigma} = \sum_{\substack{\text{facet} \\ \sigma \subsetneq \tau \\ \tau \in K}} \pm \hat{\tau}$

("polygonal" complex rather than Δ -cx)
 need compare orientations of σ, τ
 (+ if σ as a facet of τ has boundary orientation)



4) dual chain complex $D_{n-k} =$ free abelian group on dual chains $\hat{\sigma}$
 $H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat{\sigma}$ give a cell decomp. of M)

5) $\varphi: D_{n-k} \rightarrow C^k(M)$ where $\sigma^*(\alpha) = \begin{cases} 0 & \text{for } k\text{-cells } \alpha \neq \sigma \\ 1 & \text{for } \alpha = \sigma \end{cases}$
 $\hat{\sigma} \mapsto \sigma^*$

Rmk notice that $\sigma^*(\alpha) = \# \alpha$ intersects $\hat{\sigma}$ counted with orientation signs.

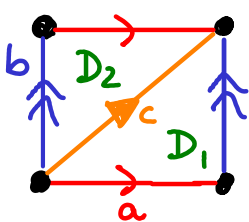
- φ linear bijection \checkmark
- chain map:

$\varphi(\partial \hat{\sigma}) \stackrel{(*)}{=} \varphi \sum \pm \hat{\tau} = \sum \pm \tau^*$

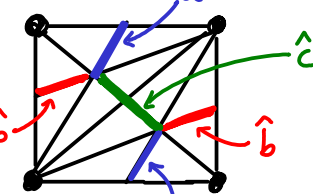
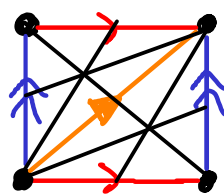
$\partial^* \varphi(\hat{\sigma}) = \partial^* \sigma^* = \left(\sigma^* \circ \partial: \tau \mapsto \sum_{\text{facets } \sigma_i \text{ of } \tau} \pm \sigma_i \right) \xrightarrow{\sigma^*} \begin{cases} \pm 1 & \text{if one of } \sigma_i = \sigma \\ 0 & \text{else} \end{cases} = \sum \pm \tau^* = \varphi(\partial \hat{\sigma})$

UPSHOT φ chain iso so get iso: $H_*(M) \stackrel{(*)}{\cong} H_*(D_*, \partial_*) \xrightarrow{\varphi} H^{n-*}(M)$

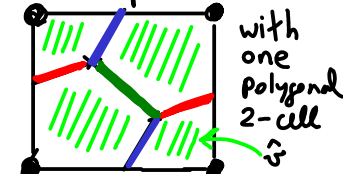
Example T^2 :



barycentric subdivision:



polygonal cell decomposition:



for example $\hat{a} + \hat{c}$ (ignoring orientation signs for simplicity)

is a 1-cycle in the polygonal cell decomposition so $\in H_1(D_*)$ corresponding to $a^* + c^*$ indeed $\hat{a} + \hat{c}$ intersects a once, b zero times and c once. Compare with Chapter 10 the computation of cup product for T^2 .

Cor χ (odd dimensional closed orientable mfd) = 0

Pf Betti numbers $b_i = \text{rank } H_i(M) \stackrel{\text{universal coeff. thm.}}{=} \text{rank } H^i(M) \stackrel{\text{Poincaré duality}}{=} \text{rank } H_{n-i}(M)$

$$\chi(M) = b_0 - b_1 + \dots + b_{\dim M - 1} - b_{\dim M}$$

equal. \square

(Poincaré-)Lefschetz duality

Theorem

M compact oriented n -mfd with boundary

$$H^k(M) \cong H_{n-k}(M, \partial M)$$

$$1 \in H^0(M) \leftarrow [M, \partial M] \in H_n(M, \partial M)$$

relative fundamental class

$$H_k(M) \cong H^{n-k}(M, \partial M)$$

Non-oriented \Rightarrow same holds with \mathbb{F}_2 coefficients.


"Pf" basically same as Poincaré duality. \square

(In simplicial case you extend a simplicial decomposition of ∂M to a simplicial decomposition of M , using a collar neighbourhood of ∂M)

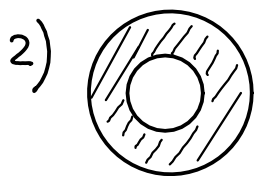
either by universal coefficient thm since $H_0(M, \partial M) = 0$ or by hand since given $p \in M, \varphi \in C^0(M, \partial M)$ consider $(\partial\varphi)(\gamma)$ for γ path from p to any $q \in \partial M$.

Cor M compact, connected, $\partial M \neq \emptyset \Rightarrow \begin{cases} H^n(M) \cong H_0(M, \partial M) = 0 \\ H_n(M) \cong H^0(M, \partial M) = 0 \end{cases}$

Examples

1) D^n  $\partial D^n = S^{n-1}$

$$\mathbb{Z} \cong H^0 D^n \cong H_n(D^n, S^{n-1})$$



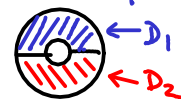
2) $A = \text{annulus} \subseteq \mathbb{R}^2 \cong S^1$

$$\mathbb{Z} \cong H^0 A \cong H_2(A, \partial A)$$

$$\mathbb{Z} \cong H^1 A \cong H_1(A, \partial A)$$

$$0 \cong H^2 A \cong H_0(A, \partial A)$$

generator $D_1 - D_2$

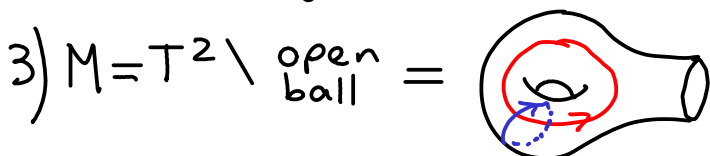


generator

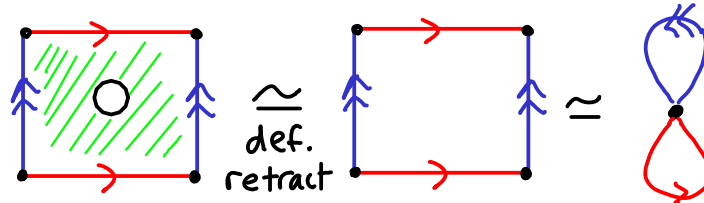


(notice $\partial D^1 \rightarrow \partial A$)

Rmk notice gen. of $H_1(A)$ is \odot which intersects gen. of $H_1(A, \partial A)$ once transversely.



$$\cong S^1 \vee S^1$$



$$\Rightarrow H_*(M, \partial M) \cong H^{2-*}(S^1 \vee S^1) = \begin{cases} \mathbb{Z}^2 & * = 1 \leftarrow \text{gen. by 2 loops} \\ \mathbb{Z} & * = 2 \leftarrow \text{gen. by } [M, \partial M] \\ 0 & \text{else} \end{cases}$$



What happens in the non-compact case?

Locally finite homology (Borel-Moore)

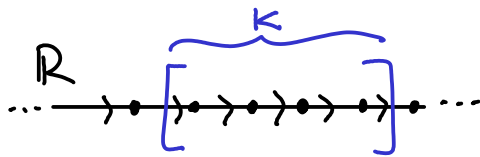
$C_*^{lf}(X)$ allow infinite sums $\sum n_i \sigma_i$ ← generators of $C_*(X)$

s.t. given any compact subset $K \subseteq X$,

$$\#\{n_i \neq 0 : K \cap \text{Im } \sigma_i \neq \emptyset\} < \infty.$$

Examples

• $C_1^{lf}(\mathbb{R}) \ni \sum_{m \in \mathbb{Z}} \sigma_m, \sigma_m: I \cong [m, m+1] \subseteq \mathbb{R}$



⇒ get cycle $[\mathbb{R}] \in H_1^{lf}(\mathbb{R})$

• $C_0^{lf}(\mathbb{R}) \ni [\text{point}] \ni \partial(\sigma_1 + \sigma_2 + \dots)$ is a boundary:

exercise $H_*^{lf}(\mathbb{R}) = \begin{cases} \mathbb{Z} & * = 1 \\ 0 & \text{else} \end{cases} (\cong H^{1-*}(\mathbb{R}))$

e.g. in simplicial case $[M] = \sum \pm \sigma_i \in C_n^{lf}$

FACT Theorem M orientable n -mfd \Rightarrow (possibly not compact)

$$H^*(M) \cong H_{n-*}^{lf}(M)$$

cohomology with compact supports $H_c^*(X)$

$C_c^*(X)$: only allow cochains $\phi: C_* X \rightarrow \mathbb{Z}$ s.t. \exists compact $K \subseteq X$ with $\phi(C_*(X \setminus K)) = 0$ (vanish on chains in $X \setminus K$)

Example $c \in C_*(X) \Rightarrow \phi(\alpha) =$ signed # intersections of c with α (geometric intersection #)

FACT $\Rightarrow \phi \in C_c^*(X)$ since $\phi(\alpha) = 0$ if $\alpha \subseteq X \setminus \text{Im}(c)$

Thm M orientable n -mfd (possibly not compact)

$$H_*(M) \cong H_c^{n-*}(M)$$

Warning H_*^{lf}, H_c^* are not homotopy invariant (indeed non-trivial for \mathbb{R}^n)

Caused because they are not functorial. They are however functorial for proper maps

Mayer-Vietoris holds for H_c^* but not for H_*^{lf} .

(preimages of compact sets are compact)

Fact $H_c^*(X) \cong \varinjlim H^*(X, X \setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M \setminus K_1) \rightarrow H^*(M, M \setminus K_2)$

Direct limit $\varinjlim G_i$ via maps $G_i \rightarrow G_j$ means $\sqcup G_i / \text{identify } g \in G_i \text{ with its images under those maps}$

(The indices are partially ordered & directed: $\forall i, j, \exists k \succ i, j$ so can compare G_i, G_j inside G_k (via $G_i \rightarrow G_k, G_j \rightarrow G_k$)

Fact \varinjlim is an exact functor.

Cap product and Poincaré duality revisited

X space, $k \geq l$

(sometimes write) $\phi \cap \sigma$

$$\cap: C_k(X) \times C^l(X) \rightarrow C_{k-l}(X)$$

cap product

$$(\sigma: \Delta^k \rightarrow X) \cap (\phi: C^l(X) \rightarrow \mathbb{Z}) = \underbrace{\phi(\sigma|_{[e_0, \dots, e_l]})}_{\substack{\text{"bottom face"} \\ \in \mathbb{Z}}} \cdot \underbrace{\sigma|_{[e_l, \dots, e_k]}}_{\substack{\text{"top face"} \cong \Delta^{k-l} \\ \in C_{k-l}(X)}}$$

(easy) Properties

- \cap bilinear
- $\partial(\sigma \cap \phi) = (-1)^l (\partial\sigma \cap \phi - \sigma \cap \partial^* \phi)$
- cycle \cap cocycle is cycle
- boundary \cap cocycle are boundaries
- cycle \cap coboundary

Rmk if $\varphi \in C^{k-l}(X)$ then $\varphi(\sigma \cap \phi) = (\phi \cup \varphi)(\sigma)$

$$\Rightarrow \boxed{\cap: H_k(X) \times H^l(X) \rightarrow H_{k-l}(X)} \quad \text{bilinear}$$

Theorem (Poincaré duality) The map $\boxed{\phi \mapsto [M] \cap \phi}$ gives following isos

① For M closed oriented n -mfd

$$[M] \cap \cdot: H^*(M) \xrightarrow{\cong} H_{n-*}(M)$$

② For M non-compact oriented n -mfd,

$$[M] \cap \cdot: H_c^*(M) \xrightarrow{\cong} H_{n-*}(M) \quad \star$$

$$[M] \cap \cdot: H^*(M) \xrightarrow{\cong} H_{n-*}^{lf}(M)$$

Sketch Pf of ① for smooth mfd (Non-examinable)

If M smooth $\Rightarrow \exists$ "good cover" U_i of M meaning open cover s.t.

FACT from Riemannian geometry ("convex neighbourhoods")

$$U_i \cong \mathbb{R}^n$$

$$U_i \cap \dots \cap U_{i_k} \cong \mathbb{R}^n \text{ or } \emptyset$$

Then compute $H_c^*(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ and \star holds for \mathbb{R}^n .

$\Rightarrow \star$ holds $\forall U_i$

\Rightarrow by naturality of \star and of Mayer-Vietoris get \star for $\cup U_i$ finite

$\Rightarrow \star$ for M , which is ①. \square

\nwarrow use 5-lemma

General Pf of Poincaré duality ← Non-examinable

Step 1: holds for \mathbb{R}^n

Pf $H_c^k(\mathbb{R}^n) \cong H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

(recall fact: $H_c^*(X) \cong \varinjlim H^*(X, X \setminus K)$ then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$ can make K larger by picking $K = \text{large ball}$)

Pick Δ -cx structure for \mathbb{R}^n . So $[\mathbb{R}^n] = \sum \pm \sigma_i \leftarrow \text{sum over } n\text{-simplices.}$

Say \exists simplex $\sigma_0: \Delta^n \rightarrow \mathbb{R}^n$. Define $\phi: C_c^{CW}(\mathbb{R}^n) \rightarrow \mathbb{Z}$, $\phi(\sigma_0) = \pm 1$ (*)

$\Rightarrow \delta\phi = 0$ for dim reasons

$\phi(\text{other simplices}) = 0$

$[\mathbb{R}^n] \cap \phi = \sum \phi(\pm \sigma_i) = \phi(\pm \sigma_0) = 1$ (pick sign in *)

Step 2 holds for $A, B, A \cap B \Rightarrow$ holds for $A \cup B$

Pf Mayer-Vietoris for H_c^* , naturality, 5-lemma ✓

Step 3 holds for A_i , and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \Rightarrow$ holds for $\cup A_i$

Pf By applying \varinjlim : both sides of P.D. iso commute with limits ✓

Step 4 holds for open subsets in \mathbb{R}^n

Pf Every such set is a union of convex open sets (e.g. balls)

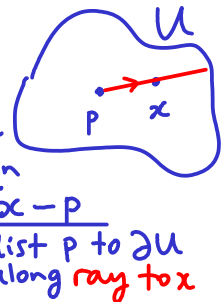
By Step 3 enough to consider case of finite union.

By induction on # convex open sets:

1 convex set $U \cong \mathbb{R}^n$ via a proper homeomorphism, now use Step 1 ✓

2 convex sets: KEY TRICK convex set \cap convex set is convex in $\mathbb{R}^n!$
 \Rightarrow use Step 2 & previous case

$k+1$ convex sets: $A = \cup \{\text{first } k \text{ convex sets}\}$, $B = \text{last convex set}$
 $\Rightarrow A \cap B \subseteq B$ is a union of k convex sets \Rightarrow use Step 2 & Inductive hypothesis ✓



fix $p \in U$
 $U \rightarrow \mathbb{R}^n$
 $x \mapsto \frac{x-p}{\text{dist } p \text{ to } \partial U \text{ along ray to } x}$

Step 5 holds for mfd M

Consider open sets in M for which it holds.

By a Zorn's Lemma argument we get a maximal open subset U where holds.

If $U \neq M$ pick $p \in M \setminus U$ and nbhd $V \cong \mathbb{R}^n$ of p . Then holds for $U, V, U \cup V$

(note $U \cup V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by Step 2 holds for $U \cup V$

Contradicts maximality. ✓ □

Recall there is a well-defined evaluation of H^* -classes on H_* :

$$\langle \cdot, \cdot \rangle : H_k(M; \mathbb{R}) \otimes H^k(M; \mathbb{R}) \rightarrow \mathbb{R}$$

$$c \otimes \alpha \longmapsto \langle c, \alpha \rangle = \varphi(c)$$

any representative cocycle φ for α

Easy exercise

$$\langle c, \alpha \cup \beta \rangle = \langle c \cap \alpha, \beta \rangle$$

any $\alpha, \beta \in H^*$, $c \in H_*$

Corollary of Poincaré duality

M compact oriented n -mfd, \mathbb{F} field.

$$\Rightarrow H^k(M; \mathbb{F}) \otimes H^{n-k}(M; \mathbb{F}) \xrightarrow{\circledast} \mathbb{F}$$

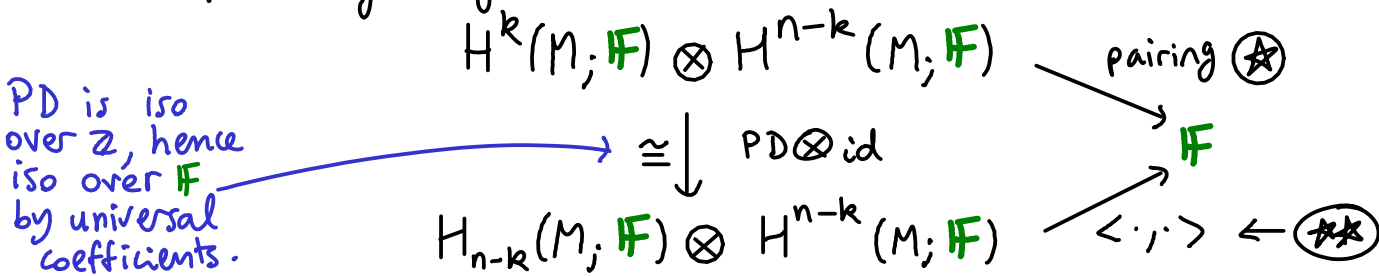
$$\alpha \otimes \beta \longmapsto \langle [M], \alpha \cup \beta \rangle$$

is a non-singular bilinear form.

hence: $H^*(M; \mathbb{F}) \cong (H^{n-*}(M; \mathbb{F}))^*$ dual

Pf. By exercise, $\langle [M], \alpha \cup \beta \rangle = \langle [M] \cap \alpha, \beta \rangle = \langle PD(\alpha), \beta \rangle$

So the following diagram commutes:



By universal coefficients, $H^*(M; \mathbb{F}) \cong \text{Hom}(H_*(M; \mathbb{F}), \mathbb{F})$ via $\beta \mapsto \langle \beta, \cdot \rangle$

Hence \circledast is a non-degenerate bilinear pairing

Hence so is the pairing \circledast in the diagram. \square

using that for any \mathbb{F} -vector space V
 $V \otimes V^* \rightarrow \mathbb{F}, v \otimes \varphi \mapsto \varphi(v)$
 $\cong \text{Hom}(V, \mathbb{F})$ is non-deg. pairing.

Remark For M non-orientable, the same holds for \mathbb{F} of characteristic 2, e.g. $\mathbb{Z}/2$

For \mathbb{Z} coefficients it can fail if $H^*(M) \not\cong \text{Hom}(H_*M, \mathbb{Z})$. So we define:

Betti group $B^k(M) = H^k(M) / \text{torsion}(H^k(M))$ has no torsion

$B_k(M) = H_k(M) / \text{torsion}(H_k(M))$

By what we proved in the section on universal coefficients, $B^q(M) \cong \text{Hom}(B_q(M), \mathbb{Z})$ whenever $H_{q-1}(M)$ is finitely generated (which we know holds for compact mfd)

The iso is given by $\langle \cdot, \cdot \rangle$ again: this descends to quotients since $\langle c, \alpha \rangle = 0 \in \mathbb{Z}$ if c or α has finite order (i.e. torsion). The same proof as above yields:

$$M \text{ compact oriented } n\text{-mfd} \Rightarrow B^k(M) \otimes B^{n-k}(M) \rightarrow \mathbb{Z}, \alpha \otimes \beta \mapsto \langle [M], \alpha \cup \beta \rangle$$

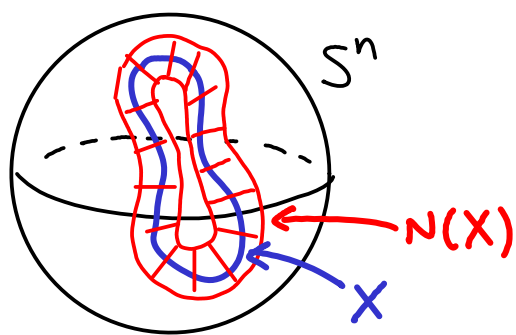
is non-degenerate bilinear form.

Also the Remark holds.

Example Use this to prove ex. 4(c) sheet 3. (Hint: $H^{2k}(\mathbb{C}P^n) \cup H^{2n-2k}(\mathbb{C}P^n) = H^{2n}(\mathbb{C}P^n)$)

Alexander duality

(in fact, enough to assume)
 X is locally contractible



$\emptyset \neq X \subsetneq S^n$ compact subset s.t.

\exists open neighbourhood $N(X)$ which deformation retracts to X

such that $\overline{N(X)} \subseteq S^n$ is an n -mfd with boundary.

Fact assumption about $N(X)$ holds if X is a smooth submanifold by tubular neighbourhood thm see Chapter 5

Theorem

$$\tilde{H}_*(X) \cong \tilde{H}^{n-* - 1}(S^n \setminus X)$$

Pf later

Example $X \subseteq S^3$ knot (i.e. $X = \text{image}(S^1 \xrightarrow{\text{homeomorphism}} S^3)$)

$\Rightarrow N(X) \cong \text{solid torus} \cong S^1$

$\Rightarrow \tilde{H}_0(X) = 0 = \tilde{H}^2(S^3 \setminus X)$

$\tilde{H}_1(X) = \mathbb{Z} = \tilde{H}^1(S^3 \setminus X)$

$\tilde{H}_2(X) = 0 = \tilde{H}^0(S^3 \setminus X)$

embedding

so the homology of a knot complement does not tell knots apart (always same)

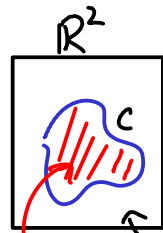
Theorem (Jordan curve Theorem)

e.g. by stereographic projection $S^2 \cong \mathbb{C} \cup \infty$

$C \cong S^1$ closed curve in $\mathbb{R}^2 \subseteq S^2$

$\Rightarrow \mathbb{R}^2 \setminus C$ has 2 path-components (= connected components)

Similarly for $S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$.



Alexander duality

Pf $S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1} \Rightarrow \mathbb{Z} \cong \tilde{H}_n(S^n) \cong \tilde{H}^0(S^{n+1} \setminus C)$

$\Rightarrow H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2$

$\Rightarrow S^{n+1} \setminus C$ has 2 path components. \square

Proof Alexander duality Abbreviate $N = N(X)$ (nbhd of X which is $\simeq X$)

$$Y := S^n \setminus N \simeq S^n \setminus X$$

for $* < n-1$

$$\begin{aligned} \tilde{H}^{n-*} (Y) &= H^{n-*} (Y) \\ &\cong_{\text{Lefschetz}} H_{*+1} (Y, \partial Y) \\ &\cong_{\text{exc.}} H_{*+1} (S^n, \bar{N}) \\ &\cong_{\text{LES using } * < n-1} \tilde{H}_* (\bar{N}) \underset{X}{=} \end{aligned}$$

for $* = n-1$

$$\begin{aligned} \tilde{H}^0 (Y) \oplus \mathbb{Z} &\cong H^0 (Y) \\ &\cong_{\text{Lef.}} H_n (Y, \partial Y) \\ &\cong_{\text{exc.}} H_n (S^n, \bar{N}) \\ &\cong \tilde{H}_{n-1} (\bar{N}) \underset{X}{=} \oplus \mathbb{Z} \end{aligned}$$

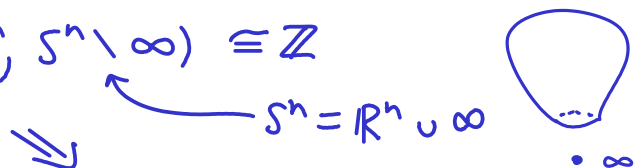
Explanation of ↗:

LES: $0 \rightarrow \tilde{H}_n (S^n) \rightarrow H_n (S^n, \bar{N}) \rightarrow \tilde{H}_{n-1} (\bar{N}) \rightarrow 0$ is SES

⊛ $\tilde{H}_n (\bar{N}) = H_n (\bar{N}) = 0$

(see Cor. to Poincaré-Lefschetz, using: each (path-) connected component of the manifold \bar{N} has non-empty boundary)

$\cong \downarrow$ quotient $H_n (S^n, S^n \setminus \infty) \cong \mathbb{Z}$



Hence that quotient map gives a splitting of the SES.

for $* = n$ $H^{n-*} (Y) = H^{-1} (Y) = 0$

$H_n (X) \cong H_n (N) \cong_{\text{Lefschetz duality}} H^0 (N, \partial N) = 0. \square$ ↖ see ⊛