M2: Analysis II - Continuity and Differentiability

BY ZHONGMIN QIAN

Hilary Term 2022

Contents

The general advice for the use of lecture notes is that, you should read the notes in advance, and take notes from lectures. Let me quote what Nobel laureate William Faulkner (1897-1962), who grow up in Oxford (USA), said when an interviewer asked that “Mr. Faulkner, some of your readers claim they still cannot understand your work after reading it two or three times. What approach would you advise them to adopt?” Faulkner answered, “ Read it a fourth time.” This advice applies to these notes and books on analysis too – you need to come back and read them again and again.

The structure of the lecture notes for Analysis II (Oxford Edition) was based on the hand-written notes by Professor Heath-Brown. I have tried to maintain the precise, rigor and simplicity style. Thanks must also go to the previous lecturers of the course who have made substantial improvement over the past years. While there are many excellent textbooks you should use for a comprehensive and systematic account. I should recommend two classics, one by W. Rudin: Principles of Mathematical Analysis (3rd Edition), the other by T. M. Apostol: Mathematical Analysis (Second Edition).

I do not implement a numbering system in lectures, however, if necessary, I may quote statements with numbers referring to the lecture notes.

4. Several notations I will use frequently through the lectures:

- $\mathbb{C}$: the set of all complex numbers – the complex plane
- $\mathbb{R}$: the set of real numbers – the real line; $\mathbb{R} \subset \mathbb{C}$.
- $\mathbb{Q}$: the set of rational numbers, $\mathbb{Q} \subset \mathbb{R}$.
- $\forall$: “for all”, “for every one”, “whenever”
- $\exists$: “there exist(s)”, “there is (are)”,
- iff stands for “if and only if”

If $z = x + iy$ is a complex number, then its $|z| = \sqrt{x^2 + y^2}$ is called the *absolute value* of z (also called the modulus of z).

Comments will be put in square brackets $[\cdots]$ giving further information.

Please send any comments you may have, or any typos and errors you may spot while you are enjoying your reading to `qianz@maths.ox.ac.uk`

Chapter 1

Function Limits and Continuity

In this chapter, we are going to

1) introduce the definition of limits for functions, including left-hand side and right-hand side limits for functions on intervals, and some variations of function limits;

2) derive essential properties of functions limits, and establish relationship between function limits and limits for sequences;

3) introduce the concepts of continuity and uniform continuity for functions;

4) prove several important theorems about continuous functions on intervals, such as the intermediate value theorem, boundedness and bounds of continuous functions on closed and bounded intervals, uniform continuity of continuous functions on closed and bounded intervals;

5) study the continuity of monotone functions on intervals, and establish the inverse function theorem (continuity part) for strictly monotone functions on intervals;

6) discuss the uniform convergence of series of functions, and prove that the continuity is preserved under uniform convergence.

1.1 Function Limits

Let us recall several facts about limits for sequences, which were covered in your Analysis I.

Limits for sequences and completeness

Definition 1.1.1 *1) A sequence (z_n) of real (or complex) numbers has a limit l, denoted by $z_n \to l$ as $n \to \infty$, or $\lim_{n\to\infty} z_n = l$, if for every $\varepsilon > 0$, there is a real number N such that for every $n > N$, $|z_n - l| < \varepsilon$. [*Some authors require N being a positive integer*, but we do not demand for this].*

2) A sequence (z_n) of numbers converges if it has a limit l for some number l.

3) A sequence (z_n) of numbers is called a Cauchy sequence *if for every $\varepsilon > 0$ there exists real number N such that for any $n,m > N$*

$$|z_n - z_m| < \varepsilon.$$

That is, (z_n) is Cauchy if

$$|z_n - z_m| \to 0 \text{ as } n,m \to \infty.$$

Here $|x_n - x_m| \to 0$ as $n,m \to \infty$ means that for any given $\varepsilon > 0$ there is N such that $|x_n - x_m| < \varepsilon$ whenever $n,m \geq N$.

Remark 1.1.2 *We may use symbol $\forall$ to mean "for every"; "whenever"; "for all", and use notation symbol $\exists$ to mean "there exist(s)"; "there is (are)".*

s. t. is the abbreviation of "such that", "iff" stands for "if and only if" and "resp." for "respectively".

Remark 1.1.3 *According to definition, a sequence (z_n)* does not *converge to l [that is, either (z_n) diverges or $z_n \to a \neq l$], if and only if there exists $\varepsilon > 0$, for every natural number k, there is at least one $n_k > k$ such that*

$$|z_{n_k} - l| \geq \varepsilon \, .$$

In general, **to formulate a *contra-positive proposition*:** Replace symbol $\forall$ ("for every") by $\exists$ ("there exist(s)"), and $\exists$ by $\forall$, and negate the statement.

Theorem 1.1.4 ***(Cauchy's Criterion, The General Principle for Convergence)*** *A sequence (z_n) of (real or complex) numbers converges if and only if it is a Cauchy sequence.*

In this sense, the real line $\mathbb{R}$ and the complex plane $\mathbb{C}$ are *complete [as metric spaces. We will study this topic in Paper A2 in your second year].*

Remark 1.1.5 *According to Cauchy's criterion, (z_n) diverges [i.e. (z_n) does not converge to a* finite *limit], if and only if there is $\varepsilon > 0$, such that for every $k \in \mathbb{N}$, there are integers n_{k_1}, $n_{k_2} > k$ such that*

$$|z_{n_{k_1}} - z_{n_{k_2}}| \geq \varepsilon \, .$$

A sequence (a_n) of real numbers is increasing (or called *non-decreasing*) if $a_{n+1} \geq a_n$ for $n = 1, 2, 3, \cdots$. An increasing sequence (a_n) has a finite limit if it is bounded from above, or $a_n \to \infty$. In fact

$$a_n \to \sup\{a_k : k \geq 1\} = \sup\{a_k : k \geq m\}$$

as $n \to \infty$ (for any m) with the convention that $\sup\{a_k\} = \infty$ if the sequence (a_n) is unbounded from above. Similarly, if (a_n) is decreasing (or called *non-increasing*), then

$$a_n \to \inf\{a_k : k \geq 1\} = \inf\{a_k : k \geq m\}$$

as $n \to \infty$ (for any m) with the convention that $\inf\{a_k\} = -\infty$ if the sequence (a_n) is unbounded from below.

For a bounded sequence (a_n) of real numbers, its upper limit

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$

and its lower limit

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$$

respectively.

Compactness

The following theorem demonstrates the "compactness" of a bounded subset.

Theorem 1.1.6 ***(Bolzano-Weierstrass' Theorem)*** *A bounded sequence in $\mathbb{R}$ (or in $\mathbb{C}$) has a sub-sequence which converges to some number. That is, a bounded sequence of numbers possesses a convergent sub-sequence.*

Proof. First we prove this for real sequences. Suppose (a_n) is a bounded sequence of real numbers. Then the tail supremum

$$c_n = \sup\{a_k : k \geq n\}$$

exists for every $n = 1, 2, \cdots$, and (c_n) is bounded and decreasing. Therefore (c_n) has a limit denoted by c. For each $k = 1, 2, \cdots$, there is $n_k \geq k$ such that a_{n_k}

$$c_k - \frac{1}{k} < a_{n_k} \leq c_k$$

and clearly we can choose n_k so that $k \to n_k$ strictly increasing. Then (a_{n_k}) is a sub-sequence of (a_n), and by Sandwich lemma, $a_{n_k} \to c$. This proves the case of real sequences.

Suppose $z_n = a_n + b_n i$ is a bounded sequence of complex numbers. then both (a_n) and (b_n) are bounded sequences. Hence we can extract a convergent sub-sequence (a_{n_k}). Now (b_{n_k}) is a bounded real sub-sequence from which one can extract a convergent sub-sequence $(b_{n'_k})$. Then $(z_{n'_k})$ is a convergent sub-sequence of (z_n). ■

We will use frequently the following consequence of the Bolzano-Weierstrass theorem.

Corollary 1.1.7 *A* bounded *sequence (z_n) in $\mathbb{R}$ (or in $\mathbb{C}$) converges to a limit l if and only if every convergent sub-sequence of (z_n) has the same limit.*

Proof. [$\Longrightarrow$; "only if " part; Necessity] Proved in Analysis I: any sub-sequence of a convergent sequence tends to the same limit.

[$\Longleftarrow$; "if" part; Sufficiency] Let us argue by contradiction [*If you cannot prove a statement directly, then formulate the contra-positive, and prove it is wrong*]. Suppose (z_n) were divergent. Since (z_n) is bounded, according to Bolzano-Weierstrass' Theorem, one can extract a sub-sequence (z_{n_k}) from (z_n) which converges to some number l_1. Let $(y_n) \equiv (z_n) \setminus (z_{n_k})$ [the sub-sequence of (z_n) with all z_{n_k} removed] which must be a sub-sequence otherwise (z_n) converges to l_1. If (y_n) did not tend to l_1, then there is $\varepsilon > 0$ such that for every $j \in \mathbb{N}$, there is an integer $n_j > j$ such that

$$|y_{n_j} - l_1| \geq \varepsilon \ .$$

[*which is the contra-positive to that $y_n \to l_1$*]. Since (y_{n_j}) is bounded, according to Bolzano-Weierstrass' Theorem, there is a convergent sub-sequence (z'_{n_k}) of (y_{n_j}), so that $\lim z'_{n_k} = l_2$ for some l_2. Since

$$|z'_{n_k} - l_1| \geq \varepsilon \qquad \forall k,$$

which yields that

$$\lim_{k\to\infty} |z'_{n_k} - l_1| = |l_2 - l_1| \geq \varepsilon > 0 \ .$$

[Here we have used the fact that if $a_n \to a$ then $|a_n| \to |a|$: you should be able to prove this by using definition of sequence limits]. Therefore $l_1 \neq l_2$. Thus we have found two sub-sequences of (z_n) converging to distinct limits, which is a contradiction to the assumption. ■

Limit points

Definition 1.1.8 *Let $E \subseteq \mathbb{R}$ (resp. $\mathbb{C}$). $p \in \mathbb{R}$ (resp. $\mathbb{C}$) is called a* limit point *(or an* accumulation point*, a* cluster point *) of E, if for every $\varepsilon > 0$, there is $z \in E$* other than *p, i.e. $z \neq p$, such that*

$$|z - p| < \varepsilon.$$

A point of E which is not a limit point of E is called an isolated point *of E.*

Proposition 1.1.9 *$p \in \mathbb{R}$ is a limit point of an interval $[a,b]$ ($(a,b]$, $[a,b)$ or (a,b)) if and only if $p \in [a,b]$, where a,b are two numbers.*

[Exercise]

Concept of function limits

Mathematicians call a mapping from a subset E of $\mathbb{R}$ (or $\mathbb{C}$) to $\mathbb{R}$ (or $\mathbb{C}$) a function with domain E. That is, a real (resp. complex) valued function f on $E \subset \mathbb{R}$ (or $E \subset \mathbb{C}$) is a correspondence (i.e. a mapping) which assigns each x of E to a unique real (resp. complex) number $f(x)$. E is called the domain of f. $f(E) = \{f(x) : x \in E\}$ is called the range of f. That is, $f(E)$ is the image of E under the mapping f.

Example 1.1.10 *$f(x) = \sqrt{1-x^2}$ with domain $E = [-1,1]$. What is its graph? Its graph looks continuous, and $f(E) = [0,1]$.*

Example 1.1.11 *Let*

$$f(x) = \begin{cases} \frac{1}{q+p}, & \text{if } x = \frac{p}{q} \in (0,1], \text{ and } (p,q) = 1 , \\ 0 , & \text{if } x \in (0,1] \text{ is irrational,} \end{cases} .$$

The domain of f is $(0,1]$. It is not easy to sketch the graph of f.

Example 1.1.12 *$f(x) = x \sin \frac{1}{x}$ with its domain $\mathbb{R} \backslash \{0\}$. As x tends to 0, f oscillates but tends to 0, so that f has limit 0 as x goes to 0.*

Definition 1.1.13 *Let $E \subseteq \mathbb{R}$ (or $\mathbb{C}$), and $f : E \to \mathbb{R}$ (or $\mathbb{C}$) be a real (or complex) function. Let p be a* limit point *of E [but p is not necessary in E], and l be a number. If for every $\varepsilon > 0$ there is $\delta > 0$ [which may depend on p and ε] such that for every $x \in E$ with $0 < |x - p| < \delta$ we have*

$$|f(x) - l| < \varepsilon,$$

then we say f tends to l as x goes to p [along E], written as

$$\lim_{x \to p} f(x) = l$$

or $f(x) \to l$ as $x \to p$ [along E]. In this case we also say f (or $f(x)$) has limit l, or say $f(x)$ converges to l as $x \to p$.

[Do a sketch to demonstrate the meaning of the definition]. To underscore that we are taking limit along E, we also write the limit as

$$\lim_{x \in E, x \to p} f(x) = l.$$

This will be the case for side limits which will be introduced shortly.

Remark 1.1.14 *f* doesn't converge *to l as $x \to p$ [that is, either f has no limit or $f(x) \to a \neq l$ as $x \to p$], then there is $\varepsilon > 0$, for every $\delta > 0$ there exists $x \in E$ such that $0 < |x - p| < \delta$ but $|f(x) - l| \geq \varepsilon$.*

Example 1.1.15 *Let $f(x) = |x|^\alpha \sin \frac{1}{x}$ for $x \neq 0$, where $\alpha > 0$ is a constant. [$E = \mathbb{R} \backslash \{0\}$]. Show that $f(x) \to 0$ as $x \to 0$.*

Proof. Since $\left|x^\alpha \sin\frac{1}{x}\right| \le |x|^\alpha$ for any $x \neq 0$, therefore, for every $\varepsilon > 0$, we may choose $\delta = \varepsilon^{1/\alpha}$. Then

$$\left|x^\alpha \sin\frac{1}{x} - 0\right| \le |x|^\alpha < \varepsilon$$

whenever $0 < |x-0| < \delta$. According to definition, $|x|^\alpha \sin\frac{1}{x} \to 0$ as $x \to 0$.

Proposition 1.1.16 *Let $f : E \to \mathbb{R}$ (or $\mathbb{C}$) and p be a limit point of E. If f has a limit as $x \to p$, then the limit is unique.*

Proof. [Proof by contradiction]. Suppose $f(x) \to l_1$ and also $f(x) \to l_2$ as $x \to p$, where $l_1 \neq l_2$. Then $\frac{1}{2}|l_1 - l_2| > 0$, so that, according to definition of function limits, there is $\delta_1 > 0$ such that

$$|f(x) - l_1| < \frac{1}{2}|l_1 - l_2| \ \ \forall x \in E \text{ s. t. } 0 < |x-p| < \delta_1 \ ,$$

and there exists $\delta_2 > 0$ such that

$$|f(x) - l_2| < \frac{1}{2}|l_1 - l_2| \ \ \forall x \in E \text{ s. t. } 0 < |x-p| < \delta_2 \ .$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Since p is a limit point of E, there is $x \in E$ such that $0 < |x-p| < \delta$, and therefore

$$\begin{aligned} |l_1 - l_2| &= |f(x) - l_2 - f(x) + l_1| \quad \text{[+1 and -1 technique]} \\ &\le |f(x) - l_1| + |f(x) - l_2| \quad \text{[Triangle Ineq.]} \\ &< \frac{1}{2}|l_1 - l_2| + \frac{1}{2}|l_1 - l_2| \\ &= |l_1 - l_2| \end{aligned}$$

which is impossible. Thus we have completed the proof. ■

Theorem 1.1.17 *[*Function limits via limits for sequences.*] Let $f : E \to \mathbb{R}$ (or $\mathbb{C}$) where $E \subseteq \mathbb{R}$ (or $\mathbb{C}$), p be a limit point of E, and $l \in \mathbb{C}$. Then $\lim_{x\to p} f(x) = l$ if and only if for any sequence (p_n) in E such that $p_n \neq p$ and $p_n \to p$, we have*

$$\lim_{n\to\infty} f(p_n) = l \ .$$

[$\lim_{x\to p} f(x) = l$ if and only if f tends to the same limit l along any sequence in E converging to p.]

Proof. [*Necessity*] Suppose $\lim_{x\to p} f(x) = l$. Then for every $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \qquad \forall x \in E \text{ with } 0 < |x-p| < \delta \ .$$

Let $p_n \in E$ be a sequence such that $p_n \to p$ and $p_n \neq p$. Then, according to the definition for sequence limits, there is $N \in \mathbb{N}$ such that for every $n > N$, $|p_n - p| < \delta$. Since $p_n \neq p$ for every n, we also have $|p_n - p| > 0$, and therefore

$$0 < |p_n - p| < \delta \quad \forall n > N.$$

Hence, for every $n > N$

$$|f(p_n) - l| < \varepsilon.$$

According to definition of sequence limits, $f(p_n) \to l$ as $n \to \infty$.

[*Sufficiency*] Let us argue by contradiction. If $\lim_{x\to p} f(x) = l$ were not true, then there is $\varepsilon > 0$, for each $n = 1, 2, \cdots$ [with$\delta = \frac{1}{n}$] there is [at least] one point $x_n \in E$, such that $0 < |x_n - p| < \frac{1}{n}$ but

$$|f(x_n) - l| \geq \varepsilon .$$

Therefore we have constructed a sequence (x_n) which converges to p, $x_n \neq p$, but $(f(x_n))$ does not tend to l, which is a contradiction. ■

Proposition 1.1.18 *[Algebra of limits] Let p be a limit point of E, and f, g be two real (or complex) functions on E. Suppose* $\lim_{x\to p} f(x) = A$ *and* $\lim_{x\to p} g(x) = B$. *Then*

1) $\lim_{x\to p} (f(x) \pm g(x)) = A \pm B$;

2) $\lim_{x\to p} f(x)g(x) = AB$;

3) if $B \neq 0$,

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{A}{B} .$$

Proof. Using AOL for sequence limits together with Theorem 1.1.17. [Exercise]. ■

Example 1.1.19 *Show that* $\lim_{x\to 0} \sin \frac{1}{x}$ *does not exist.*

Proof. Let $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{2\pi n + \pi/2}$. Then $x_n \to 0$ and $y_n \to 0$, but

$$\lim_{n\to\infty} \sin \frac{1}{x_n} = 0 \text{ and } \lim_{n\to\infty} \sin \frac{1}{y_n} = 1 .$$

So that $\lim_{x\to 0} \sin \frac{1}{x}$ doesn't exist according to Theorem 1.1.17. ■

Example 1.1.20 *[*A very useful fact about function limits*] If* $\lim_{x\to p} f(x) = l \neq 0$, *then there is* $\delta > 0$, *such that for* $x \in E$ *with* $0 < |x - p| < \delta$ *we have*

$$|f(x)| \geq \frac{|l|}{2} .$$

In particular, $|f(x)| > 0$ *for all* $x \in E$ *such that* $0 < |x - p| < \delta$.

Proof. Since $\lim_{x\to p} f(x) = l$ and $|l| > 0$, applying the definition of function limits to f at p with $\varepsilon = |l|/2$ which is positive, there is $\delta > 0$, for $x \in E$ such that $0 < |x - p| < \delta$ we have

$$|f(x) - l| < \frac{|l|}{2}$$

Using triangle inequality we then deduce that

$$\begin{aligned} |f(x)| &= |l + (f(x) - l)| \\ &\geq |l| - |f(x) - l| \\ &> |l| - \frac{|l|}{2} = \frac{|l|}{2} \end{aligned}$$

for every $x \in E$ such that $0 < |x - p| < \delta$. ■

Left-hand limits and right-hand limits limits for functions on intervals

For functions defined on an interval, we may talk about right-hand and left-hand limits, which however are special cases of our definition for function limits.

Definition 1.1.21 *1) Let f be a real or complex function in $[a,b)$ and $p \in [a,b)$. Then we say the right-hand limit of f at p exists and equals l, written as $\lim_{x\to p+} f(x) = l$ (or $\lim_{x\downarrow p} f(x) = l$, or $\lim_{x>p, x\to p} f(x) = l$), if for every $\varepsilon > 0$, there is $\delta > 0$, for any $x \in [a,b)$ such that $0 < x - p < \delta$ one has*

$$|f(x) - l| < \varepsilon.$$

2) Let $f : (a,b] \to \mathbb{R}$ (or $\mathbb{C}$), and let $p \in (a,b]$. Then we say the left-hand limit of f at p exists and equals l, written as $\lim_{x\to p-} f(x) = l$ (or $\lim_{x\uparrow p} f(x) = l$, or $\lim_{x<p, x\to p} f(x) = l$), if for every $\varepsilon > 0$, there is $\delta > 0$, such that for any $x \in (a,b]$, $0 < p - x < \delta$ one has

$$|f(x) - l| < \varepsilon.$$

For simplicity, the left-hand limit (resp. the right-hand limit) is denoted by $f(p-)$ (resp. $f(p+)$).

Obviously, $\lim_{x\to p} f(x)$ exists if and only if both the left-hand and the right-hand side limits at p exist and equal.

There are some variations of function limits which are quite useful as well.

Definition 1.1.22 *1) Let f be a real or complex function defined on (a,∞) (resp. $(-\infty,b)$). We say $f(x) \to l$ as $x \to \infty$ (resp. $x \to -\infty$), written as $\lim_{x\to\infty} f(x) = l$ (resp. $\lim_{x\to-\infty} f(x) = l$), if every $\varepsilon > 0$, there is N, such that $x > N$ (resp. $x < -N$)*

$$|f(x) - l| < \varepsilon.$$

2) Let f be a real or complex function defined on $\{z : |z| > R\}$ for some $R > 0$. Then $f(z) \to l$ as $z \to \infty$, if for every $\varepsilon > 0$, there is $N > 0$, such that for any $|z| > N$ we have

$$|f(z) - l| < \varepsilon.$$

One can generalize the definition of limits at ∞ (resp. $-\infty$) for function f with domain E, such that ∞ (resp. $-\infty$) is a limit point of E.

Exercise 1.1.23 *1) Give definitions of $\lim_{x\to x_0} f(x) = \infty$, $\lim_{x\to x_0} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$ and etc.*

2) Formulate a statement that f does not tend to l as $x \to \infty$.

Example 1.1.24 *Show that $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x$ exists.*

We will develop a powerful tool, the L'Hoptial rules, in the later part of the course to evaluate this kind of limits. Here we prove this based on sequence limits.

Let $a_n = \left(1 + \frac{1}{n}\right)^n$. Then

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots \\ &\quad + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right), \end{aligned}$$

so that a_n is increasing. Moreover

$$\begin{aligned} 0 &\leq a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &\leq 2 + \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)n} \\ &< 3. \end{aligned}$$

Hence $\{a_n\}$ is increasing and bounded, so that $\lim_{n\to\infty} a_n = \sup_n \left(1+\frac{1}{n}\right)^n$ exists. This limit is denoted by e.

If $x>0$, we use $[x]$ to denote the integer part of x. Obviously $[x] \geq x-1 \to \infty$ as $x\to\infty$. Since

$$\begin{aligned}\left(1+\frac{1}{x}\right)^x &\geq \left(1+\frac{1}{[x]+1}\right)^{[x]} \\ &= \left(1+\frac{1}{[x]+1}\right)^{[x]+1} \frac{[x]+1}{[x]+2} \to e\end{aligned}$$

and

$$\left(1+\frac{1}{x}\right)^x \leq \left(1+\frac{1}{[x]}\right)^{[x]+1} = \left(1+\frac{1}{[x]}\right)^{[x]} \frac{[x]+1}{[x]} \to e$$

the Sandwich Rule (or called the Squeezed Lemma) [Analysis I. You should formulate a version for function limits and prove it !] implies that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\ .$$

For negative x, we set $y=-x>0$. Then \leq

$$\begin{aligned}\left(1+\frac{1}{x}\right)^x &= \left(1-\frac{1}{y}\right)^{-y} \\ &= \left(\frac{y-1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^{y} \\ &= \left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right) \to e\ .\end{aligned}$$

We will show that $e=\sum_{n=0}^{\infty}\frac{1}{n!}$ and study the exponential function exp after we establish powerful tools.

1.2 Continuity of functions

In the definition of $\lim_{x\to p} f(x)$, the point p may not belong to the domain E of f. Even $f(p)$ is well-defined, the limit of f at p may not coincide with its value $f(p)$.

Definition 1.2.1 *Let $f: E\to\mathbb{R}$ (or $\mathbb{C}$), where $E\subseteq\mathbb{R}$ (or $\mathbb{C}$), and $p\in E$ [so p belongs to the domain of f]. If for every $\varepsilon>0$ there is $\delta>0$, such that for every $x\in E$ with $|x-p|<\delta$ we have*

$$|f(x)-f(p)|<\varepsilon,$$

then we say that f is continuous at p.

According to definition, f is continuous at *any isolated point* of E.

If p is a limit point of E, then f is continuous at p, if and only if

1. p belongs to the domain of f, i.e. $f(p)$ is well defined,

2. $\lim_{x\to p} f(x)$ exists,

3. and $\lim_{x\to p} f(x)$ equals the value of f at p.

Example 1.2.2 *Let $\alpha > 0$ be a constant. The function $f(x) = |x|^\alpha \sin\frac{1}{x}$ is not continuous at $x = 0$ as f is not well-defined. Redefine the function to be*

$$g(x) = \begin{cases} |x|^\alpha \sin\frac{1}{x} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then g is continuous at $x = 0$.

Example 1.2.3 *Let $f : (0,1] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} \frac{1}{q}\,, & \text{if } x = \frac{p}{q} \text{ and } (p,q) = 1, \\ 0\,, & \text{if } x \text{ is irrational.} \end{cases}$$

(here $(p,q) = 1$ means that p and q are co-prime, i.e, p,q have no common factor). Then f is continuous at irrationals of $(0,1]$, and is not continuous at rationales.

Proof. Suppose that $x_0 \in (0,1)$ is irrational, so by definition of f, $f(x_0) = 0$, hence

$$|f(x) - f(x_0)| = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ and } (p,q) = 1\,. \end{cases}$$

For every $\varepsilon > 0$, there are only finite many pairs of positive integers p and q such that $p \leq q$ and $q \leq \frac{1}{\varepsilon}$, so that

$$\delta \equiv \min\left\{ \left| x_0 - \frac{p}{q} \right| : p \leq q \text{ and } q \leq \frac{1}{\varepsilon} \right\} > 0.$$

If $|x - x_0| < \delta$, then x is either irrational and $f(x) = 0$, or x is rational but $0 \leq f(x) < \varepsilon$, so that

$$|f(x) - f(x_0)| < \varepsilon.$$

By definition, this shows that f is continuous at irrational number x_0.

If $x_0 = \frac{p}{q} \in (0,1]$ is rational, then, for $\varepsilon = \frac{1}{2q} > 0$ and for whatever how small $\delta > 0$, there is an irrational number $\tilde{x} \in (0,1]$ such that $|\tilde{x} - \frac{p}{q}| < \delta$ [Here we use the fact that rational numbers are dense in $\mathbb{R}$, a fact proved in Analysis I in MT], so that

$$|f(\tilde{x}) - f(x_0)| = \frac{1}{q} > \varepsilon\,.$$

f is not continuous at rational numbers. ■

Proposition 1.2.4 *If f and g are continuous at p, so are $f \pm g$; fg and f/g (provided $g(p) \neq 0$).*

[Definition + Algebra of function limits].

Example 1.2.5 *Let $f : \mathbb{C} \to \mathbb{C}$ (or $\mathbb{R} \to \mathbb{R}$) be a polynomial. Then f is continuous in $\mathbb{C}$ (or $\mathbb{R}$).*

Theorem 1.2.6 *If $f : E \to \mathbb{C}$ and $g : f(E) \to \mathbb{C}$, and $h : E \to \mathbb{C}$ is the composition function of g and f defined by*

$$h(x) = (g \circ f)(x) \equiv g(f(x)) \qquad \text{for } x \in E.$$

If f is continuous at $p \in E$ and g is continuous at $f(p)$, then h is continuous at p.

[Composition of two continuous functions is continuous.]

Proof. For any $\varepsilon > 0$, since g is continuous at $f(p)$, there is $\delta_1 > 0$ such that for any $y \in f(E)$ with $|y - f(p)| < \delta_1$ we have

$$|g(y) - g(f(p))| < \varepsilon,$$

so that for $x \in E$ such that $|f(x) - f(p)| < \delta_1$, then

$$|g(f(x)) - g(f(p))| < \varepsilon.$$

Since$f$ is continuous at p, so there is $\delta > 0$, for any $x \in E$ such that $|x - p| < \delta$, we have

$$|f(x) - f(p)| < \delta_1.$$

Therefore

$$|g(f(x)) - g(f(p))| < \varepsilon$$

for any $x \in E$ such that $|x - p| < \delta$. By definition h is continuous at p. ■

Let f be a real or complex function on $[a,b)$ (resp. $(a,b]$) and $p \in [a,b)$ (resp. $p \in (a,b]$). We say f is right (resp. left) continuous at p if $f(p+) = f(p)$ (resp. $f(p-) = f(p)$) [i.e. the right-hand (or the left-hand) limit of f at p exists and equals $f(p)$]. According to definition, f is continuous at $p \in (a,b)$ if and only if $f(p+) = f(p-) = f(p)$.

Example 1.2.7 *Consider function*

$$f(x) = \begin{cases} x & \text{if } x \geq 0\,, \\ x+1 & \text{if } x < 0\,. \end{cases}$$

Then $f(0+) = 0$ *and* $f(0-) = 1$. f *is not continuous at* 0.

1.3 Continuous functions on intervals

In this part we are going to prove several important results about continuous functions on intervals.

Intervals are simple but important subsets of the real line $\mathbb{R}$. Some authors insist that an interval is bounded, in this course however an interval may be bounded or unbounded. Hardly we need a definition of intervals though – one can either list all possible intervals, or give a formal definition.

Definition 1.3.1 *We say a non-empty subset $E \subseteq \mathbb{R}$ possesses the interval property, if $x, y \in E$, then E contains any real number z between x and y, that is, $[x,y] \subseteq E$ (or $[y,x] \subseteq E$ if $y \leq x$).*

Proposition 1.3.2 *Assume that $E \subseteq \mathbb{R}$ is non-empty and possesses the interval property.*

(i) If E is unbounded from above and is also unbounded from below, then $E = (-\infty, \infty)$.

(ii) If E is unbounded from below but bounded from above, then $E = (-\infty, b]$ or $E = (-\infty, b)$, where $b = \sup E$.

(iii) If E is unbounded from above but bounded from below, then $E = [a, \infty)$ or $E = (a, \infty)$, where $a = \inf E$.

(iv) If E is bounded, then $E = (a,b)$, $E = (a,b]$, $E = [a,b)$ or $E = [a,b]$, where $a = \inf E$ and $b = \sup E$.

Proof. Let us prove (ii). The proofs of the others are similar. If E is unbounded from below, and bounded above, then $b = \sup E$ exists. Let us show that $(-\infty, b) \subseteq E$. Suppose $x < b$, then by definition of $\sup E$, there is $x_0 \in E$ such that $b \geq x_0 > x$. Since E is unbounded from below, there is $A \in E$ such that $A < x$. Therefore $A, x_0 \in E$ and $A < x < x_0$, and since E possesses the interval property, so $x \in E$ too. Thus $(-\infty, b) \subseteq E$. On the other hand $E \subseteq (-\infty, b]$ by definition of b. Therefore $E = (-\infty, b]$ or $E = (-\infty, b)$ depending on whether $b \in E$ or not. The proof is complete. ■

In the rest of the course we will only deal with functions on intervals.

By definition, a real or complex valued function f is continuous on a (bounded) closed interval $[a, b]$ (where a and b are two real numbers), by definition, if f is continuous at every $x_0 \in [a, b]$. That is, for every $x_0 \in (a, b)$,

$$f(x_0) = f(x_0+) = f(x_0-) = \lim_{x \to x_0} f(x_0),$$

$$f(a) = f(a+) = \lim_{x > a, x \to a} f(x)$$

and

$$f(b) = f(b-) = \lim_{x < b, x \to b} f(x).$$

In terms of $\varepsilon - \delta$, for any given $\varepsilon > 0$, for every $x_0 \in (a, b)$, there is $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \text{ for every } x \in (x_0 - \delta, x_0 + \delta)$$

and there are $\delta_a > 0$ and $\delta_b > 0$, such that

$$|f(x) - f(a)| < \varepsilon \text{ for any } x \in [a, a + \delta_a)$$

and

$$|f(x) - f(b)| < \varepsilon \text{ for any } x \in (b - \delta_b, b].$$

These properties of a continuous function f on $[a, b]$ will be used in our arguments below.

1.3.1 Intermediate Value Theorem

Intermediate Value Theorem (in short, IVT) is one of the most important result about continuous functions on intervals, which lies in the foundation for many concepts you will meet in your Part A to Part C. The concept of connectivity of topological spaces (Paper A2 and Paper A5) has its origin in IVT.

Theorem 1.3.3 ***(Intermediate Value Theorem (IVT)).*** *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let C be a number between $f(a)$ and $f(b)$. Then there is $\xi \in [a, b]$ such that $f(\xi) = C$. Therefore $[f(a), f(b)] \subset f([a, b])$ (or $[f(b), f(a)] \subset f([a, b])$ if $f(b) < f(a)$), where $f([a, b]) = \{f(x) : x \in [a, b]\}$.*

Proof. We may assume that $f(a) \leq f(b)$, otherwise consider the function $-f(x)$ instead. If $f(a) = f(b)$, or $C = f(a)$ or $f(b)$, then the conclusion is clearly true with $\xi = a$ or b. We may further assume that $C = 0$ otherwise consider $f(x) - C$ instead. Therefore we assume that $f(a) < 0 < f(b)$, and want to show that there is $\xi \in (a, b)$ such that $f(\xi) = 0$.

Do a sketch of the graph of f, which is a continuous curve, and observe that the first crossing point through the x-axis of the curve must be a zero of f. Therefore we define

$$\xi = \inf\{x \in [a, b] : f(x) > 0\},$$

where $\{x \in [a,b] : f(x) > 0\}$ denotes the subset of $[a,b]$ consisting of all $x \in [a,b]$ such that $f(x) > 0$. Since $f(b) > 0$, so that

$$\{x \in [a,b] : f(x) > 0\}$$

is non-empty and bounded, thus its infinimum ξ exists by the completeness axiom of real numbers. We prove that $f(\xi) = 0$. To this end, we first show that $\xi \in (a,b)$ by using the continuity of f at a and at b. In fact, since $f(a) < 0$ and $f(b) > 0$, and f is continuous at a and at b, there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - f(a)| < -\frac{f(a)}{2} \text{ for } x \in [a, a+\delta_1)$$

[Here we have applied the definition of continuity to f at a with $\varepsilon = -f(a)/2$ which is positive], and

$$|f(x) - f(b)| < \frac{f(b)}{2} \text{ for } x \in (b-\delta_2, b]$$

[Similarly here we have used the definition of continuity for f at b with $\varepsilon = f(b)/2 > 0$]. Therefore

$$f(x) < \frac{f(a)}{2} < 0 \text{ for } x \in [a, a+\delta_1)$$

and

$$f(x) > \frac{f(b)}{2} > 0 \text{ for } x \in (b-\delta_2, b].$$

By definition of ξ, the inequalities above yield that $\xi \geq a + \delta_1 > a$ and that $\xi \leq b - \delta_2 < b$. Therefore $\xi \in (a,b)$.

We next show that $f(\xi) = 0$ by using continuity of f at ξ. By definition of ξ, $f(x) \leq 0$ for every x such that $a \leq x < \xi$, since f is continuous at ξ, so that

$$f(\xi) = f(\xi-) = \lim_{x<\xi, x\to\xi} f(x) \leq 0.$$

We next show that $f(\xi)$ can't be negative. If $f(\xi) < 0$, then since f is continuous at ξ, there is $\delta > 0$ such that

$$|f(x) - f(\xi)| < -\frac{f(\xi)}{2} \text{ for } x \in (\xi - \delta, \xi + \delta)$$

[Here using the definition of continuity for f at ξ with $\varepsilon = -f(\xi)/2$ – which were positive by contradiction assumption], so that

$$f(x) < \frac{f(\xi)}{2} < 0 \text{ for } x \in (\xi - \delta, \xi + \delta)$$

and therefore $f(x) \leq 0$ for all $x \in [a, \xi + \delta)$. Hence we must have $\xi \geq \xi + \delta$, which is a contradiction. Hence $f(\xi) = 0$. The proof is complete. ■

In the previous proof, $\xi = \inf\{x \in [a,b] : f(x) > 0\}$ is the first x-coordinate at which the graph of f crosses the x-axis, but ξ is not necessary the first root of $f(x) = 0$ greater than a. Of course we may locate the first zero of the function f on $[a,b]$, which is given by $\eta = \inf\{x \in [a,b] : f(x) \geq 0\}$. Under the conditions that f is continuous on $[a,b]$ and $f(a) < 0 < f(b)$, one can show that $f(\eta) = 0$. This gives a slightly different proof of the IVT.

Proof. (*Proof of IVT – a constructive proof*) The case that $C = f(a)$ or $C = f(b)$ is trivial, so we assume that $C \neq f(a)$ or $f(b)$. Therefore $f(a) < C < f(b)$ or $f(b) < C < f(a)$. Let $g(x) = f(x) - C$. Then $g(a)$ and $g(b)$ have different sign, $g(a)g(b) < 0$. Let $x_1 = a$ and $y_1 = b$. Divide the interval

$[x_1,y_1]$ at its center $\frac{1}{2}(x_1+y_1)$ into two equal parts. If $g(\frac{1}{2}(x_1+y_1))=0$ then $\xi=\frac{1}{2}(x_1+y_1)$ will do. Otherwise, we choose $x_2=x_1$ and $y_2=(\frac{1}{2}(x_1+y_1)$ if $g(\frac{1}{2}(x_1+y_1))>0$, or $x_2=\frac{1}{2}(x_1+y_1)$ and $y_2=y_1$ if $g(\frac{1}{2}(x_1+y_1))<0$. Then $g(x_2)g(y_2)<0$; $[x_2,y_2]\subset[x_1,y_1]$ and

$$|y_2-x_2|=\frac{1}{2}(b-a)\ .$$

Apply the previous argument to $[x_2,y_2]$ instead of $[a,b]$, to obtain $[x_3,y_3]\subset[x_2,y_2]$, such that

$$|y_3-x_3|=\frac{1}{2}|y_2-x_2|=\frac{1}{2^2}(b-a)$$

and $g(x_3)g(y_3)\le 0$. By repeating the same procedure, we may find some $[x_k,y_k]\subset[a,b]$, $g(x_k)=0$ or $g(y_k)=0$ then $\xi=x_k$ or y_k will do.

Otherwise, we may construct two sequences (x_n) and (y_n) such that $g(x_n)g(y_n)\le 0$, $[x_n,y_n]\subset[x_{n-1},y_{n-1}]$ for any $n=2,\cdots$, and

$$\begin{aligned}|y_n-x_n| &= \frac{1}{2}|y_{n-1}-x_{n-1}|\\ &= \cdots=\frac{1}{2^{n-1}}|y_1-x_1|\\ &= \frac{b-a}{2^{n-1}}\ .\end{aligned}$$

Obviously, (x_n) is a bounded increasing sequence, and (y_n) is a bounded decreasing sequence, thus $x_n\to\xi$ and $y_n\to\xi'$ for some ξ, $\xi'\in[a,b]$ [Analysis I: bounded monotone sequences converge]. Since

$$\lim_{n\to\infty}|y_n-x_n|=\lim_{n\to\infty}\frac{1}{2^{n-1}}(b-a)=0,$$

so $\xi=\xi'$. Since g is continuous at ξ,

$$0\ge\lim_{n\to\infty}g(x_n)g(y_n)=\lim_{n\to\infty}g(x_n)\lim_{n\to\infty}g(y_n)=g(\xi)^2,$$

which yields that $g(\xi)^2=0$, and therefore $g(\xi)=0$ [As $g(\xi)$ is a real number], so that $f(\xi)=C$. ■

Remark 1.3.4 *Given C between $f(a)$ and $f(b)$, ξ may be not unique. From the proof we can see that, if $[x_n,y_n]$ is a decreasing net of closed intervals (i.e. $[x_n,y_n]\subset[x_{n+1},y_{n+1}]$ for each n) such that the length $y_n-x_n\to 0$, then $\cap_{n=1}^{\infty}[x_n,y_n]$ exactly contains one point (and in particular is not empty).*

Remark 1.3.5 *The proof of the IVT also provides a method of finding roots to $f(\xi)=c$, but other methods may find roots faster if additional information about f (e.g. that f is differentiable) is available.*

The following corollary is the general form of IVT for real valued functions of one real variable.

Theorem 1.3.6 *Let $E\subseteq\mathbb{R}$ be an interval, and f be real-valued and continuous on E. Then $f(E)\equiv\{f(x):x\in E\}$ is an interval too.*

Proof. If E is empty, then there is nothing to prove, so we assume that E is a non-empty interval. Then $f(E)$ is non-empty. Let $A,B\in f(E)$. We prove that for every real number C between A and B, C also belongs to $f(E)$. Let $a,b\in E$ such that $f(a)=A$ and $f(b)=B$. Since E is an interval, by Proposition 1.3.2, $[a,b]\subset E$ (or $[b,a]\subset E$ if $b<a$), f is continuous on $[a,b]$ (or $[b,a]$). By IVT applying to f on $[a,b]$ (or $[b,a]$), there is ξ between a and b such that $f(\xi)=C$, which implies that $C\in f(E)$. According to Proposition 1.3.2, $f(E)$ is an interval. ■

Theorem 1.3.7 *If f is a real valued function which is continuous on $\mathbb{R}$, then f maps an interval to an interval, that is, if $E \subseteq \mathbb{R}$ is an interval, then so is its image $f(E) = \{f(x) : x \in E\}$.*

In Paper A2, we will show that the only *connected subsets* of $\mathbb{R}$ are intervals, so the previous Corollary may be stated as the following

Theorem 1.3.8 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous (i.e. f is continuous at every $x \in \mathbb{R}$), and if $E \subseteq \mathbb{R}$ is connected, then so is $f(E)$.*

1.3.2 Boundedness

A real or complex function f is bounded on E, if the image $f(E)$ of E under the function f, which is the subset $\{f(x) : x \in E\}$, is bounded. That is, there is non-negative constant M such that

$$|f(x)| \leq M \qquad \forall\, x \in E\ .$$

Theorem 1.3.9 *If $f : [a,b] \to \mathbb{R}$ (or $\mathbb{C}$) is continuous, where $a \leq b$ are two real numbers, then f is bounded on $[a,b]$.*

Proof. Let us prove this theorem by contradiction. Suppose f were unbounded, then for every $n \in \mathbb{N}$, there is [at least one] $x_n \in [a,b]$ such that $|f(x_n)| \geq n$. According to Bolzano-Weierstrass' Theorem, we may extract a convergent sub-sequence (x_{n_k}) from (x_n). Let $x_{n_k} \to p$. Since $[a,b]$ contains all its limiting points, so that $p \in [a,b]$. Since f is continuous on $[a,b]$, so it is continuous at p, thus according to Theorem 1.1.17,

$$\lim_{n\to\infty} f(x_{n_k}) = f(p).$$

Therefore $(f(x_{n_k}))$ must be bounded [from Analysis I: any convergent sequence is bounded], which is a contradiction to the assumption that $|f(x_{n_k})| \geq n_k \geq k$ for every k. Therefore f is bounded, and the proof is complete. ■

In order to state the next important theorem about continuous functions on closed intervals, we introduce the following notations.

Let $f : E \to \mathbb{R}$ be a *real-valued* function on E, where E is non-empty. Then $f(E) = \{f(x) : x \in E\}$ is a non-empty subset of $\mathbb{R}$. If $f(E)$ is bounded from above, that is, $f(E)$ has an upper bound, then $\sup_{x\in E} f(x)$ (or denoted by $\sup_E f$) is the least upper bound of $f(E)$, called the supremum of f on E, that is,

$$\sup_{x\in E} f(x) = \sup\{f(x) : x \in E\}\,.$$

Similarly, if $f(E)$ is bounded from below, that is, $f(E)$ has a lower bound, then $\inf_{x\in E} f(x)$ denotes the greatest lower bound of $f(E)$, the infimum of f on E, so that

$$\inf_{x\in E} f(x) = \inf\{f(x) : x \in E\}\,.$$

The existence of the least and the greatest bounds for a *bounded* real function f is guaranteed by the completeness of the real number system.

Suppose f is a real valued function which bounded from above on E. Then $M = \sup_{x\in E} f(x)$ if and only if $f(z) \leq M$ [so M is an *upper bound* on E] and for any given $\varepsilon > 0$ there is $z_\varepsilon \in E$ such that $f(z_\varepsilon) > M - \varepsilon$ [that is, any real which is smaller than M can not be a upper bound of f on E]. Similarly, if f is bounded from below on E, then $m = \inf_{x\in E} f(x)$ if and only if $f(z) \geq m$ [so m is a *lower bound* on E] and for every $\varepsilon > 0$ there is $z_\varepsilon \in E$ such that $f(z_\varepsilon) < m + \varepsilon$ [that is, any real which is greater than m is not a lower bound of f on E].

Theorem 1.3.10 *If $f : [a,b] \to \mathbb{R}$ is continuous, then f attains its bounds on $[a,b]$. That is, there are $x_1, x_2 \in [a,b]$ such that*

$$f(x_1) = \sup_{x\in[a,b]} f(x) \quad \text{and} \quad f(x_2) = \inf_{x\in[a,b]} f(x)$$

respectively.

Proof. [*That is, sup and inf are attained. Note that x_1, x_2 are not necessary unique. In short, we may say "a continuous function on a closed bounded interval is bounded and attains its bounds".*] We give two different proofs for this important theorem.

(1st Proof) According to Theorem 1.3.9, f is bounded on $[a,b]$, so that $m \equiv \inf_{x\in[a,b]} f(x)$ exists by the completeness of the real number system [Analysis I]. By definition, $f(x) \geq m$ for all $x \in [a,b]$, and for every $n = 1, 2, \cdots$, there is an $x_n \in [a,b]$ such that

$$m \leq f(x_n) \leq m + \frac{1}{n}.$$

Since (x_n) is bounded, according to Bolzano-Weierstrass' Theorem, we may extract a convergent subsequence $(x_{n_k}) : x_{n_k} \to p$. Then $p \in [a,b]$. Since f is continuous at p, $\lim_{x\to p} f(x) = f(p)$, so that $f(x_{n_k}) \to f(p)$ according to Theorem 1.1.17. While

$$m \leq f(x_{n_k}) \leq m + \frac{1}{n_k} \tag{1.3.1}$$

for all k, so by letting $k \to \infty$ in the previous inequality (1.3.1) we obtain that

$$m \leq \lim_{k\to\infty} f(x_{n_k}) = f(p) \leq \lim_{k\to\infty} \left(m + \frac{1}{n_k} \right) = m$$

[or by Sandwich lemma for sequence limits] which implies that $f(p) = m = \inf_{x\in[a,b]} f(x)$.

(2nd Proof) [More elegant proof – again argue by contradiction.] Let us prove that the supremum of f is attained by contradiction. Let $M = \sup_{[a,b]} f$. Suppose M were not attained on $[a,b]$, so that

$$f(z) < M \qquad \forall z \in [a,b].$$

Then

$$g(x) = \frac{1}{M - f(x)}$$

is positive and continuous on $[a,b]$. Therefore, according to Theorem 1.3.9, g is bounded on $[a,b]$. Hence there is a positive number M_0 such that

$$g(x) = \frac{1}{M - f(x)} \leq M_0$$

for every $x \in [a,b]$. It follows that

$$f(x) \leq M - \frac{1}{M_0} < M$$

for all $x \in [a,b]$, which is a contradiction to the assumption that M is the least upper bound of f on $[a,b]$. ■

Remark 1.3.11 *The proofs of the previous two theorems rely on the following facts:*

1) $[a,b]$ is bounded;

2) $[a,b]$ is closed (i.e. $[a,b]$ contains all limit points of $[a,b]$);

3) f is continuous.

Remark 1.3.12 *In Paper A2 in your second year, we will study the concepts of open/closed subsets, compact spaces and compact subsets. A subset A of $\mathbb{R}$ (or $\mathbb{C}$) is called closed if A contains all its limit points. A subset A of $\mathbb{R}$ or $\mathbb{C}$ is compact if and only if A is bounded and closed.*

In terms of compact subsets, we have

Theorem 1.3.13 *1) If f is a continuous real or complex valued function on a compact subset E, then $f(E)$ is also a compact subset.*

2) If f is a continuous real valued function on a compact subset $E \subseteq \mathbb{R}$ *or on a compact subset $E \subseteq \mathbb{C}$, then f attains its bounds, that is, there are x_1, $x_2 \in E$ such that*

$$f(x_1) \leq f(x) \leq f(x_2) \text{ for every } x \in E,$$

so that $f(x_1) = \inf_{x \in E} f(x)$ and $f(x_2) = \sup_{x \in E} f(x)$.

Remark 1.3.14 *The proofs of Theorem 1.3.20, 1.3.9, 1.3.10 rely on the compactness of the closed interval $[a,b]$ [via Bolzano-Weierstrass' theorem], and the proof of IVT relies on the fact that $[a,b]$ is unbroken, i.e. $[a,b]$ is "connected". For details about "connectedness", see W. Rudin's Principles, page 93, Theorem 4.22 and Theorem 4.23.*

As a consequence we have the following important

Corollary 1.3.15 *Let $f : [a,b] \to \mathbb{R}$ be continuous, $M = \sup_{x \in [a,b]} f(x)$ and $m = \inf_{x \in [a,b]} f(x)$. Then for any $c \in [m,M]$ there is at least one $\xi \in [a,b]$ such that $f(\xi) = c$. Therefore*

$$f([a,b]) = [m,M] .$$

Proof. Let $E = [a,b]$ an bounded and closed interval. Since f is continuous on E, so f is bounded, thus $m = \inf f(E)$ and $M = \sup f(E)$ exist. By definition $f(E) \subset [m,M]$. On the other hand, $f(E)$ is an interval too (Theorem 1.3.6) and $m, M \in f(E)$ by Theorem 1.3.10, $[m,M] \subset f(E)$. Therefore $f(E) = [m,M]$. ■

Example 1.3.16 *Suppose $f : [0,1] \to [0,1]$ is continuous, then there is a fixed point of on $[0,1]$, that is, there is $\xi \in [0,1]$ such that $f(\xi) = \xi$. In fact, $g(x) = f(x) - x$ is continuous on $[0,1]$, and $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$, so, by IVT, there is $\xi \in [0,1]$, such that $f(\xi) = \xi$.*

1.3.3 Uniform Continuity

Recall that f with its domain E is continuous at $x_0 \in E$, if for any given $\varepsilon > 0$ one can find a number $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

holds for all $x \in E$ satisfying that $|x - x_0| < \delta$. In general, the positive number δ depends not only on ε but also on x_0, and the dependence of δ on ε and x_0 measures the degree of "continuity" of f on E.

Example 1.3.17 *Show that for every $x_0 \neq 0$, $\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$. Therefore $\frac{1}{x}$ is continuous at any $x \neq 0$.*

Proof. Since

$$\left|\frac{1}{x}-\frac{1}{x_0}\right| = \frac{|x-x_0|}{|x||x_0|} ,$$

thus, if $|x-x_0| < \frac{|x_0|}{2}$ [so we need to choose δ smaller than $\frac{|x_0|}{2}$], then

$$|x| \geq |x_0| - |x-x_0| > \frac{|x_0|}{2} \quad \text{[by using the triangle inequality]}$$

so that

$$\left|\frac{1}{x}-\frac{1}{x_0}\right| = \frac{|x-x_0|}{|x||x_0|} \leq \frac{2}{|x_0|^2}|x-x_0| .$$

[Thus in order to ensure that $\left|\frac{1}{x}-\frac{1}{x_0}\right| < \varepsilon$ we only need $\frac{2}{|x_0|^2}|x-x_0| < \varepsilon$ and $|x-x_0| < \frac{|x_0|}{2}$]. Choose $\delta = \min\left\{\frac{|x_0|}{2}, \frac{\varepsilon|x_0|^2}{2}\right\}$ [which is positive as $x_0 \neq 0$]. Then

$$\left|\frac{1}{x}-\frac{1}{x_0}\right| < \varepsilon$$

whenever $|x-x_0| < \delta$. Hence $\frac{1}{x} \to \frac{1}{x_0}$ as $x \to x_0$. Note that δ depends on ε and also on x_0 as well, so that the degree of "continuity" of $f(x) = \frac{1}{x}$ is not uniform in $x \in (0,\infty)$. ■

Example 1.3.18 *Suppose that f is* Lipschitz continuous *in E in the sense that there is a constant M such that*

$$|f(x)-f(y)| \leq M|x-y|$$

for any $x,y \in E$. Then f is continuous at any $x_0 \in E$.

Proof. Let $x_0 \in E$. For every $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M+1}$ [which depends only on ε but not on $x_0 \in E$]. Then

$$\begin{aligned} |f(x)-f(x_0)| &\leq M|x-x_0| \\ &\leq M\left(\frac{\varepsilon}{M+1}\right) < \varepsilon \end{aligned}$$

whenever $x \in E$ such that $|x-x_0| < \delta$. Therefore for a given $\varepsilon > 0$ we can find a number $\delta > 0$ that works for all $x_0 \in E$, so that f is *uniformly* continuous on E.

For example, $f(x) = \sqrt{x}$ is Lipschitz continuous on $[1,\infty)$:

$$|f(x)-f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq |x-y|$$

for all $x,y \geq 1$, so that $\sqrt{x}$ is *uniformly* continuous on $[1,\infty)$. ■

Definition 1.3.19 *Let $f : E \to \mathbb{R}$ (or $\mathbb{C}$). f is uniformly continuous on E, if for every $\varepsilon > 0$, there is $\delta > 0$, such that for all $y,x \in E$ with $|y-x| < \delta$ we have*

$$|f(y)-f(x)| < \varepsilon .$$

The following theorem is important in the theory of Riemann integrals, which will be the analysis topic in Trinity Term.

Theorem 1.3.20 *If $f : [a,b] \to \mathbb{R}$ (or $\mathbb{C}$) is continuous, then f is uniformly continuous on $[a,b]$.*

Proof. [This theorem says that a continuous function on a *closed interval* (or in general on a *compact space,* i.e. a bounded and closed subset of $\mathbb{R}$ or $\mathbb{C}$, see W. Rudin's Principles, Theorem 4.19, page 91) is uniformly continuous.]

Let us argue by contradiction. Suppose that f were not uniformly continuous, then, $\exists\ \varepsilon > 0$, such that for any n [with $\delta = \frac{1}{n}$], $\exists$ a pair of points x_n, $y_n \in [a,b]$, $|x_n - y_n| < \frac{1}{n}$ but

$$|f(x_n) - f(y_n)| \geq \varepsilon \ .$$

[which is the contra-positive to the uniform continuity]. Since (x_n) is bounded, by Bolzano-Weierstrass' Theorem, we may extract a convergent subsequence (x_{n_k}) from (x_n) which converges to some p. p must be a limit point of $[a,b]$, so that $p \in [a,b]$. Since

$$\begin{aligned} |y_{n_k} - p| &\leq |x_{n_k} - y_{n_k}| + |x_{n_k} - p| \\ &< \frac{1}{n_k} + |x_{n_k} - p| \to 0 \end{aligned}$$

Thus $x_{n_k} \to p$ and $y_{n_k} \to p$. Since f is continuous at p,

$$0 < \varepsilon \leq \lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = |f(p) - f(p)| = 0$$

which is impossible. Here we have used again the following fact about sequence limits: $a_n \to a$ as $n \to \infty$ implies that $|a_n| \to |a|$ as $n \to \infty$. ∎

Proposition 1.3.21 *If f is a real or complex valued function which is uniformly continuous on $E \subseteq \mathbb{R}$ or $\mathbb{C}$, then f maps a Cauchy sequence in E to a Cauchy sequence. That is, if (x_n) is a Cauchy sequence, where $x_n \in E$ for $n = 1, 2, \cdots$, then $(f(x_n))$ is also a Cauchy sequence.*

Proof. For any given $\varepsilon > 0$, since f is uniformly continuous on E, there is $\delta > 0$, whenever $x, y \in E$ such that $|x - y| < \delta$ we have

$$|f(x) - f(y)| < \varepsilon.$$

Since (x_n) is Cauchy, there is $N > 0$ such that for all $n, m \geq N$, $|x_n - x_m| < \delta$. Hence

$$|f(x_n) - f(x_m)| < \varepsilon$$

for all $n, m \geq N$. Therefore $(f(x_n))$ is a Cauchy sequence. ∎

This is the best for what we can say about Cauchy sequences for a function on a general domain. Actually the converse of the previous proposition is not true in general as the following example shows.

Example 1.3.22 $f(x) = x^2$ *is continuous on* $[0, \infty)$ *but not uniformly in* $[0, \infty)$. *While f maps a Cauchy sequence to a Cauchy sequence.*

In fact, for every $n = 1, 2, \cdots$, let $x_n = n + \frac{1}{n}$ and $y_n = n$ then $x_n - y_n = \frac{1}{n}$ tends to zero as $n \to \infty$, but

$$|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} > 2,$$

so f is not uniformly continuous. We claim that f maps a Cauchy sequence into a Cauchy sequence. In fact, if (a_n) is a Cauchy sequence of $[0, \infty)$, then (a_n) must be bounded, thus there is $A > 0$ such that all $a_n \in [0, A]$, and therefore (a_n) is a Cauchy sequence in $[0, A]$. Since f is uniformly continuous on $[0, A]$ by Theorem 1.3.20, so $(f(a_n))$ is also a Cauchy sequence by Proposition 1.3.21 applying to f on the closed interval $[0, A]$. Therefore $f(x) = x^2$ maps a Cauchy sequence into a Cauchy sequence, but is not uniformly continuous on E.

While the converse is true if the domain E is bounded.

Proposition 1.3.23 *Let E be a* bounded subset *of $\mathbb{R}$ or $\mathbb{C}$. Then a real or complex valued function f is uniformly continuous on E if and only if f maps Cauchy sequences of E into Cauchy sequences.*

Proof. By Proposition 1.3.21 we only need to show the sufficiency. That is, we prove that if E is bounded, and if f maps a Cauchy sequence to a Cauchy sequence, then f must be uniformly continuous on E. Suppose f were not uniformly continuous on E, so by definition there is an $\varepsilon > 0$ such that for every n there is a pair x_n, y_n in E such that $|x_n - y_n| < \frac{1}{n}$ but

$$|f(x_n) - f(y_n)| \geq \varepsilon.$$

Since $(x_n) \subset E$, so it is a bounded sequence, therefore by Bolzano-Weierestrass' theorem, we may extract a convergent sequence (x_{n_k}). Hence (x_{n_k}) is a Cauchy sequence, and, by AOL,

$$y_{n_k} = x_{n_k} + (y_{n_k} - x_{n_k})$$

tends to the same limit of (x_{n_k}). Now define $a_{2k+1} = x_{n_k}$ and $a_{2k} = y_{n_k}$. Then (a_k) is convergent, so it is a Cauchy sequence of E, while

$$|f(a_{2k+1}) - f(a_{2k})| \geq \varepsilon$$

for all k, so the image $(f(a_k))$ is not Cauchy. This is a contradiction. ■

Example 1.3.24 $f(x) = \sqrt{x}$ *is uniformly continuous in* $[0,\infty)$.

Proof. For every $\varepsilon > 0$, since $\sqrt{x}$ is continuous on $[0,1]$, according to Theorem 1.3.20, it is uniformly continuous the closed interval $[0,1]$. Hence $\exists \delta_1 > 0$, $\forall x, y \in [0,1]$ such that $|x - y| < \delta_1$ we have

$$|\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{2}. \tag{1.3.2}$$

On $[1,\infty)$, the function $\sqrt{x}$ is Lipschitz. In fact, for $x, y \geq 1$,

$$\left|\sqrt{x} - \sqrt{y}\right| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}|x-y|$$

and therefore $\sqrt{x}$ is uniformly continuous on $[1,\infty)$.

[In fact we can prove that $\sqrt{x}$ is Lipschitz continuous on $[a,\infty)$ for any positive number a, but it is not Lipschitz continuous on $[0,\infty)$].

Thus $\exists \delta_2 > 0$, $\forall x, y \geq 1$ such that $|x - y| < \delta_2$ we have

$$|\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{2}. \tag{1.3.3}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Let $x, y \in [0,\infty)$ such that $|x - y| < \delta$. If both x and y belong to $[0,1]$ or both in $[1,\infty)$, then

$$|\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{2} < \varepsilon\ .$$

If $x \in [0,1]$ and $y \geq 1$, since $|x - y| < \delta$, so that $|x - 1| < \delta$ and $|y - 1| < \delta$, and therefore

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &\leq |\sqrt{x} - \sqrt{1}| + |\sqrt{y} - \sqrt{1}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence

$$|\sqrt{x} - \sqrt{y}| < \varepsilon$$

whenever x, $y \in [0,\infty)$ such that $|x - y| < \delta$. By definition, $f(x) = \sqrt{x}$ is uniformly continuous in the *unbounded interval* $[0,\infty)$. ■

1.3.4 Monotonic Functions and Inverse Function Theorem

We study in this part the continuity of monotone functions on intervals.

A function $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$ is a subset, is increasing (or called non-decreasing) on E if $x, y \in E$ and $x \le y$ implies that $f(x) \le f(y)$. Similarly we may define decreasing (or called non-increasing) functions on E. A function on E is monotone if it is increasing on E or it is decreasing on E. A function f is strictly monotone (resp. strictly increasing) on E if f is monotone (resp. increasing) on E and f is also 1-1. If $f : E \to \mathbb{R}$ is 1-1, then f defines an inverse function f^{-1} with its domain $f(E) = \{f(x) : x \in E\}$.

Definition 1.3.25 *Let f be a real valued function on $E \subseteq \mathbb{R}$.*

1) If $f(x) \le f(y)$ (resp. $f(x) \ge f(y)$) whenever $x < y$ and $x, y \in E$, then we say f is increasing (resp. decreasing) in E.

2) A function is called monotone on E if it is increasing on E or decreasing on E.

3) If $x < y$ implies that $f(x) < f(y)$ (resp. $f(x) > f(y)$) then f is said to be strictly increasing (resp. strictly decreasing) on E.

Theorem 1.3.26 *Let f be a monotone function on (a,b), and $x_0 \in (a,b)$. Then*

1) The right-hand limit $f(x_0+)$ and left-hand limit $f(x_0-)$ exist, $f(x_0)$ lies between $f(x_0-)$ and $f(x_0+)$.

2) f is continuous at x_0 if and only if $f(x_0+) = f(x_0-)$. In general, the difference $f(x_0+) - f(x_0-)$ is the "jump" or "increment" of f at x_0.

Proof. We may assume that f is increasing (i.e. non-decreasing) on (a,b), otherwise we consider $-f$ instead. Let $x_0 \in (a,b)$. Then $\{f(x) : a < x < x_0\}$ is clearly a non-empty subset of $\mathbb{R}$. Since f is non-decreasing, this subset is bounded from above by $f(x_0)$, so that

$$l = \sup_{a<x<x_0} f(x) \equiv \sup\{f(x) : a < x < x_0\}$$

exists. By definition of l, for every $\varepsilon > 0$, there is $x_\varepsilon < x_0$ such that

$$l - \varepsilon < f(x_\varepsilon) \le l.$$

Let $\delta = x_0 - x_\varepsilon$. Then for every $x \in (x_0 - \delta, x_0)$, $x_0 > x > x_\varepsilon$, so that

$$l - \varepsilon < f(x_\varepsilon) \le f(x) \le l,$$

which implies that

$$|f(x) - l| < \varepsilon.$$

By definition of left-hand side limits

$$f(x_0-) = \sup_{a<x<x_0} f(x).$$

Similarly we have

$$f(x_0+) = \inf_{x_0<x<b} f(x) \equiv \inf\{f(x) : x_0 < x < b\}.$$

Since f is increasing, we have

$$f(x_0-) \le f(x_0) \le f(x_0+).$$

Finally, by definition, $\lim_{x\to x_0} f(x)$ exists if and only if $f(x_0+) = f(x_0-)$. Since $f(x_0)$ is sandwich between $f(x_0-)$ and $f(x_0+)$, so this is equivalent to that f is continuous at x_0. ∎

There are similar results for monotone functions on other types of intervals.

Theorem 1.3.27 *Let f be a monotone function on an interval $E \subseteq \mathbb{R}$. Then f is continuous on E if and only if $f(E) = \{f(x) : x \in E\}$ is an interval.*

Proof. If f is continuous, and E is an interval, then by IVT, $f(E)$ is an interval too.

Suppose f is monotone and $f(E)$ is an interval, let us show that f is continuous on E. We may assume that f is increasing on E otherwise consider $-f$ instead. We may assume $E = (a,b)$, as the other cases may be reduced to this case. For example, if $E = [a,b)$, where a is a number, then we define $f(x) = f(a) + (x-a)$ for $x < a$, so f is increasing and continuous on $(-\infty, b)$, so we consider this extension of f on $(-\infty, b)$ instead.

If there were $x_0 \in (a,b)$ such that f were not continuous at x_0, we may deduce a contradiction. In fact, according to Theorem 1.3.26, $(f(x_0-), f(x_0))$ or/and $(f(x_0), f(x_0+))$ is non-empty. Suppose $(f(x_0-), f(x_0))$ is non-empty for example, then we can choose a number $C \in (f(x_0-), f(x_0))$, and choose $x_1 \in (a, x_0)$ and $x_2 \in (x_0, b)$. Then

$$f(x_1) \leq f(x_0-) < C < f(x_0) < f(x_2)$$

so C is between $f(x_1)$ and $f(x_2)$, but $C \notin f(E)$. Therefore, by Proposition 1.3.2, $f(E)$ can't be an interval, which contradicts to the assumption. ■

Lemma 1.3.28 *Let $E \subseteq \mathbb{R}$ be an interval. Suppose $f : E \to \mathbb{R}$ is continuous and 1-1 on E, then f must be strictly monotone on E.*

Proof. We may assume that $E = [a,b]$ (where $a < b$) is a bounded and closed interval without losing generality, as any interval E can be written as

$$E = \bigcup_{n=1}^{\infty} [a_n, b_n]$$

where (a_n) is decreasing and (b_n) is increasing.

We may assume that $f(a) < f(b)$ otherwise consider $-f$ instead. We prove that f is strictly increasing on $[a,b]$.

To this end, we first show that $f(a) < f(x) < f(b)$ for every $x \in (a,b)$. If for some $x \in (a,b)$, $f(x) < f(a)$, then by IVT applying to continuous function f on $[x,b]$, there is a $\xi \in [x,b]$ such that $f(a) = f(\xi)$. Since $a < x \leq \xi$, this is a contradiction to the assumption that f is 1-1. Hence $f(x) > f(a)$ for every $x \in (a,b)$. Similarly, we can show that $f(x) < f(b)$ for any $x \in (a,b)$. If $a < x < y < b$, then considering continuous function f on $[a,y]$, since $f(a) < f(y)$, and f is 1-1 on $[a,y]$, so that $f(a) < f(x) < f(y)$, which implies that f is strictly increasing on $[a,b]$. ■

Now we are going to prove the inverse function theorem. The first part of this theorem is about the continuity of inverse functions, the second part is about the differentiability of inverse functions which will be dealt with in the next chapter.

Theorem 1.3.29 ***(Inverse Function Theorem).*** *Let $E \subseteq \mathbb{R}$ be an interval, and $f : E \to \mathbb{R}$ be continuous and 1-1 on E. Then the inverse function f^{-1} is continuous on $f(E)$, where $f(E) = \{f(x) : x \in E\}$.*

Proof. By IVT (Theorem 1.3.6), $f(E)$ is an interval, and according to Lemma 1.3.28, f is strictly monotone on E. Therefore f^{-1} is well-defined on the interval $f(E)$ and is strictly monotone too. By definition $f^{-1}(f(E)) = E$ which is an interval, so by Theorem 1.3.27 applying to f^{-1} on $f(E)$, we may deduce that f^{-1} is continuous on $f(E)$. This completes the proof. ■

Proof. [*Another proof* via ε-δ definition.] By Lemma 1.3.28, under the assumptions, f is strictly monotone on E. We may assume that f is strictly increasing otherwise study $-f$ instead. Without losing generality we may assume that $E=(a,b)$ is open, otherwise, for example if $E=[a,b)$, we may extend the definition of f continuously to $(-\infty,b)$ by setting $f(x)=f(a)+(x-a)$ for $x<a$ which is continuous and 1-1 on $(-\infty,b)$.

Let f^{-1} be the inverse of f, with its domain $f(E)=\{f(x):a<x<b\}$. Since f is continuous, according to IVT (Theorem 1.3.6), $f(E)$ is again an interval. Since f is strictly increasing, $f(E)=(c,d)$ is also an open interval, where

$$c=\lim_{x\downarrow a}f(x) \quad \text{and } d=\lim_{x\uparrow b}f(x).$$

[Note that c can be $-\infty$, and d can be ∞]. Let $y_0\in(c,d)$. We are going to show that f^{-1} is continuous at y_0. Let $x_0=f^{-1}(y_0)\in(a,b)$. For every $\varepsilon>0$, we may choose $0<\varepsilon_1<\varepsilon$ such that

$$(x_0-\varepsilon_1,x_0+\varepsilon_1)\subseteq(a,b).$$

Since f is strictly increasing,

$$\delta\equiv\min\{f(x_0+\varepsilon_1)-y_0,y_0-f(x_0-\varepsilon_1)\}$$

is positive, and

$$(y_0-\delta,y_0+\delta)\subseteq(c,d).$$

For every y such that $|y-y_0|<\delta$, since f is strictly increasing

$$f^{-1}(y)=x\in(x_0-\varepsilon_1,x_0+\varepsilon_1)$$

which implies that

$$\left|f^{-1}(y)-f^{-1}(y_0)\right|<\varepsilon_1<\varepsilon$$

so by definition f is continuous at y_0. Since $y_0\in f(E)$ is arbitrary, so f^{-1} is continuous on $f(E)$. Thus we have completed the proof. ■

Theorem 1.3.30 ***(Inverse Function Theorem** for functions on closed intervals) Let f be a strictly increasing and continuous real function on $[a,b]$. Then the inverse function f^{-1} is well defined on $[f(a),f(b)]$ and is continuous.*

Proof. [*There is a similar result for decreasing functions.*] In this case $f(a)$ and $f(b)$ are the minimum and the maximum of f respectively, so that $f([a,b])=[f(a),f(b)]$. Therefore f^{-1} is well-defined on $[f(a),f(b)]$. The continuity of f^{-1} follows from Theorem 1.3.29 now. ■

We are now able to give a complete picture about monotone continuous functions on intervals.

Theorem 1.3.31 *Let E be an interval and $f:E\to\mathbb{R}$ be a real valued function. Then the following statements are equivalent:*

(i) f is 1-1 and continuous on E;

(ii) f is continuous, and f is strictly increasing on E or strictly decreasing on E;

(iii) f is 1-1, monotone on E, and $f(E)\equiv\{f(x):x\in E\}$ is an interval.

If f satisfies any of conditions (i)-(iii), then f is continuous on interval E, $f(E)$ is an interval, f maps E one-to-one and onto $f(E)$, and the inverse function f^{-1} is continuous on $f(E)$. Moreover $f(E)=(c,d)$, $[c,d)$, $(c,d]$ or $[c,d]$, where $c=\inf_E f$ and $d=\sup_E f$, with the convention that if f is unbounded from below then $\inf_E f=-\infty$, and similarly if f is unbounded from above then $\sup_E f=\infty$.

Theorem 1.3.32 *If $f:(a,b)\to\mathbb{R}$ is monotone, then f is continuous on (a,b) except* at most countably many *points.*

Proof. Suppose f is increasing in (a,b), and $A\subset(a,b)$ denotes the collection of discontinuous points of f. If $x<y$, $x,y\in(a,b)$, then, since f is increasing

$$f(x+)=\inf_{t>x}f(t)=\inf_{y>t>x}f(t)\le\sup_{x<t<y}f(t)=\sup_{t<y}f(t)=f(y-).$$

Hence

$$f(x-)\le f(x+)\le f(y-)\le f(y+)$$

which implies that

$$(f(x-),f(x+))\cap(f(y-),f(y+))=\emptyset \tag{1.3.4}$$

for any $x\ne y$, $x,y\in(a,b)$. By Theorem 1.3.26, $x\in A$ if and only if $f(x-)<f(x+)$, that is, the open interval $(f(x-),f(x+))$ is non-empty. For any $x\in(a,b)$ at which f is discontinuous, then $(f(x-),f(x+))$ is non-empty, so that we may choose a rational number $r_x\in(f(x-),f(x+))$ [using the fact that rationales are dense in $\mathbb{R}$]. By (1.3.4) r_x are different for different x, thus $x\to r_x$ is injective from A to $\mathbb{Q}$. Therefore A is at most countable. ∎

Example 1.3.33 *Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n$ converges. Let (x_n) be a sequence of distinct numbers in (a,b) [For example all rationales in (a,b)]. Consider*

$$f(x)=\sum_{n:x_n<x}c_n \qquad (a<x<b)\,,$$

where the summation takes over those indices n for which $x_n<x$. If there are no $x_n<x$, then the sum is assumed value zero. [Exercise: f is well defined on (a,b)]. Then f is increasing on (a,b), discontinuous *at each x_n with an jump $f(x_n+)-f(x_n-)=c_n$, and is continuous at any other point of (a,b). Moreover f is a* left-continuous *at x_n: $f(x_n-)=f(x_n)$.*

To study this function, which looks like a step function with infinitely steps, we may consider its partial sum sequence

$$f_n(x)=\sum_{k\le n,x_k<x}c_k$$

where we do the sum over only those indices k which fulfill two constraints that $k\le n$ and also that $x_k<x$. By assumption we have

$$|f(x)-f_n(x)|=\left|\sum_{k>n,x_k<x}c_k\right|\le\sum_{k=n+1}^{\infty}c_k.$$

Note the right-hand side in the inequality is independent of x, so that

$$\sup_x|f(x)-f_n(x)|\le\sum_{k=n+1}^{\infty}c_k\to 0$$

as $n\to\infty$, hence $f_n\to f$ *uniformly* in (a,b), a concept we are going to introduce shortly. Let $A=\{x_k:k=1,2,\cdots\}$. Then for every n, f_n is continuous at every $x\in(a,b)\setminus A$, and is left continuous at every x_k, so as the uniform limit of f_n, f is continuous at every $x\in(a,b)\setminus A$, and is left continuous at every x_k, see the big theorem below which we are going to prove for a general case.

Exercise 1.3.34 *Modify the definition of f in the example so that f is right-continuous at each x_n.*

1.4 Uniform Convergence

Let E be a subset of $\mathbb{R}$ or $\mathbb{C}$, and $f : E \to \mathbb{C}$ be continuous at $p \in E$. Then

$$\lim_{x\to p} f(x) = f(p) = f(\lim_{x\to p} x) ,$$

that is, we may interchange the function operation f and the limiting process $\lim_{x\to p}$. In many situations, we would like to understand if the *order* of performing two (or more) operations is relevant or not.

Consider a sequence (f_n) of functions defined on E ($\subset \mathbb{R}$ or $\mathbb{C}$). If for every $x \in E$, the sequence $f_n(x) \to f(x)$, then we say that f_n converges (to f) on E, and f is the limit function, written $\lim_{n\to\infty} f_n = f$ in E or $f_n \to f$ on E. We are interested in the following question: can we exchange the order of taking two limits $\lim_{n\to\infty}$ and $\lim_{x\to p}$:

$$\lim_{x\to p}\lim_{n\to\infty} f_n(x) \quad \text{and} \quad \lim_{n\to\infty}\lim_{x\to p} f_n(x)\ ?$$

In particular, if all f_n are continuous at p, is the limit function $\lim_{n\to\infty} f_n$ continuous at p as well?

We may ask the same question for series of functions. If the sequence of partial sums

$$s_n(x) \equiv \sum_{k=1}^{n} f_k(x) \qquad \forall x \in E$$

converges for every $x \in E$, then we will use

$$\sum_{n=1}^{\infty} f_n$$

to denote the limit function of (s_n), called the sum of the series $\sum_{n=1}^{\infty} f_n$. Can we exchange the summation $\sum_{n=1}^{\infty}$ [which by definition is understood as $\lim_{n\to\infty}\sum_{k=1}^{n}$] and $\lim_{x\to p}$:

$$\lim_{x\to p}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}\lim_{x\to p} f_n(x)\ \ ?$$

In other words, can we work out the limit $\lim_{x\to p}$ of the infinite sum $\sum_{n=1}^{\infty} f_n$ term by term?

Example 1.4.1 *Consider the sequence of functions [sketch their graphs!]*

$$f_n(x) = \begin{cases} 0 & \text{if } x \geq \frac{1}{n}\ ; \\ -nx+1 & \text{if } 0 \leq x < \frac{1}{n}\ . \end{cases}$$

Then

$$\lim_{n\to\infty} f_n(x) = f(x) \equiv \begin{cases} 0 & \text{if } x \neq 0\ ; \\ 1 & \text{if } x = 0\ . \end{cases}$$

$f_n(x)$ converges to $f(x)$ for every $x \in [0,1]$ [but not uniformly, see definition below]. The limit function f is not continuous at 0*, although all f_n are continuous on* $[0,1]$. *Indeed*

$$\lim_{x\to 0}\lim_{n\to\infty} f_n(x) = \lim_{x\to 0} f(x) = 0$$

while

$$\lim_{n\to\infty}\lim_{x\to 0} f_n(x) = \lim_{n\to\infty} 1 = 1$$

so that

$$\lim_{x\to 0}\lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty}\lim_{x\to 0} f_n(x)\ .$$

Definition 1.4.2 *Let f_n and f be real (or complex) functions on E, where $n = 1, 2, \cdots$.*
1) If for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $x \in E$ and for all $n > N$

$$|f_n(x) - f(x)| < \varepsilon,$$

then we say f_n converges to f uniformly on E, written as $f_n \to f$ uniformly on E as $n \to \infty$.
2) Define the sequence of partial sums

$$s_n(x) \equiv \sum_{k=1}^{n} f_k(x) \qquad \forall x \in E$$

If $s_n \to s$ uniformly on E, then we say the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

By definition, $f_n \to f$ uniformly on E implies point-wise convergence that

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad\qquad \forall x \in E.$$

Theorem 1.4.3 *Let $f_n, f : E \to \mathbb{R}$ (or $\mathbb{C}$). Then $f_n \to f$ uniformly on E if and only if*

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0\,.$$

Proof. Recall the notation used here:

$$\sup_{x\in E} |f_n(x) - f(x)| = \sup\{|f_n(x) - f(x)| : x \in E\}$$

which is the supermum of the function $|f_n - f|$ over E, or ∞ if the function $|f_n - f|$ is unbounded on E.

"$\Longrightarrow$". Suppose $f_n \to f$ *uniformly on* E, then for any given $\varepsilon > 0$ there is N such that $\forall x \in E$ and $n > N$ we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

[That is, $\frac{\varepsilon}{2}$ is an upper bound of $\{|f_n(x) - f(x)| : x \in E\}$]. Hence $\forall n > N$

$$\begin{aligned} \sup_{x\in E} |f_n(x) - f(x)| &\le \frac{\varepsilon}{2} \qquad \text{[Think about why we have " $\le$ ", not " $<$ " ?]} \\ &< \varepsilon. \end{aligned}$$

According to definition, $\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$.

"$\Longleftarrow$". Suppose $\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$, then $\forall \varepsilon > 0\ \exists N$ such that $\forall n > N$

$$\sup_{x\in E} |f_n(x) - f(x)| < \varepsilon.$$

Therefore for all $x \in E$ and $n > N$

$$|f_n(x) - f(x)| \le \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon.$$

By definition $f_n \to f$ *uniformly on* E. ■

Exercise 1.4.4 *Prove that $f_n \to f$ uniformly in E if and only if for any sequence (x_n) in E*

$$\lim_{n\to\infty} |f_n(x_n) - f(x_n)| = 0\,.$$

[Hint: Formulate the contrapositive to that $f_n \to f$ uniformly in E].

Theorem 1.4.5 ***(Cauchy's Criterion for Uniform Convergence)*** *Let $f_n : E \to \mathbb{R}$ (or $\mathbb{C}$). Then f_n converges uniformly on E, if and only if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for $n,m > N$ we have*

$$\sup_{x\in E} |f_n(x) - f_m(x)| < \varepsilon. \tag{1.4.1}$$

Proof. "$\Longrightarrow$". Suppose f_n converges uniformly on E with limit function f, then $\forall\ \varepsilon > 0$, $\exists N$ such that $\forall n > N$

$$\sup_{x\in E} |f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

Since

$$|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f_m(x) - f(x)|$$

so that for any $n,m > N$,

$$\begin{aligned} \sup_{x\in E} |f_n(x) - f_m(x)| &\le \sup_{x\in E} |f_n(x) - f(x)| + \sup_{x\in E} |f_m(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

"$\Longleftarrow$". Conversely, suppose (1.4.1) holds. Then for any $x \in E$, $(f_n(x))$ is a Cauchy sequence, so that it is convergent. Let us denote its limit by $f(x)$. For every $\varepsilon > 0$, choose an integer N such that for all n, $m > N$ and $x \in E$ we have

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}.$$

For any fixed $n > N$ and $x \in E$, letting $m \to \infty$ in the above inequality we obtain

$$\begin{aligned} |f_n(x) - f(x)| &= \lim_{m\to\infty} |f_n(x) - f_m(x)| \\ &\le \frac{\varepsilon}{2} \qquad \text{[Think about why " $\le$ ", not " $<$ " ?]} \\ &< \varepsilon\ . \end{aligned}$$

According to definition, $f_n \to f$ uniformly on E. ■

Remark 1.4.6 *[Cauchy's criterion of uniform convergence for series] A series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent in E if and only if for every $\varepsilon > 0$, there is N such that for $n > m \ge N$*

$$\sup_{x\in E} \left| \sum_{k=m+1}^{n} f_k(x) \right| < \varepsilon\ .$$

[Apply Cauchy's criterion to the partial sum sequence (s_n): $s_n = \sum_{k=1}^{n} f_k$].

As a consequence, we prove the following simple but useful test for uniform convergence of series.

Theorem 1.4.7 ***(Weierstrass M-Test*** *[for Uniform Convergence of Series]) Let (f_n) be a sequence of (real or complex) functions defined on E. If there is a sequence of real numbers (M_n) such that*

$$|f_n(x)| \le M_n \text{ for all } x \in E$$

[i.e. M_n is an upper bound *of $|f_n|$ on E] for $n = 1, 2, \cdots$, and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E. Moreover*

$$\left| \sum_{n=1}^{\infty} f_n(x) \right| \le \sum_{n=1}^{\infty} |f_n(x)| \le \sum_{n=1}^{\infty} M_n$$

Proof. The proof of the last inequality, though obvious, is left as an exercise. By Cauchy's criterion for series of numbers, for every $\varepsilon > 0$, there exists an integer N such that

$$\sum_{k=m+1}^{n} M_k < \varepsilon \quad \text{for all } n > m \geq N.$$

Let $s_n = \sum_{k=1}^{n} f_k$ be the partial sum sequence of $\sum_{n=1}^{\infty} f_n$. Then for any $n > m \geq N$ and for every $x \in E$

$$\begin{aligned} |s_n(x) - s_m(x)| &= \left| \sum_{k=m+1}^{n} f_k(x) \right| \\ &\leq \sum_{k=m+1}^{n} |f_k(x)| \qquad \text{[Triangle Inequality]} \\ &\leq \sum_{k=m+1}^{n} M_k \, . \end{aligned}$$

That is, $|s_n - s_m|$ is bounded above by $\sum_{k=m+1}^{n} M_k$ and therefore

$$\sup_{x\in E} |s_n(x) - s_m(x)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon \, .$$

Hence, according to Cauchy's criterion for uniform convergence, (s_n) converges uniformly in E. ■

Example 1.4.8 *Let* $E = [0,1]$ *and*

$$f_n(x) = \frac{x}{1+n^2x^2} \, .$$

Then $\lim_{n\to\infty} f_n(x) = 0$ *for every* $x \in E$. *Since*

$$0 \leq f_n(x) = \frac{1}{2n}\frac{2nx}{1+n^2x^2} \leq \frac{1}{2n} \to 0$$

so that $f_n \to f$ *uniformly on* $[0,1]$.

Example 1.4.9 *Let*

$$f_n(x) = \frac{nx}{1+n^2x^2} \qquad \textit{for } x \in [0,1].$$

Then $\lim_{n\to\infty} f_n(x) = 0$ *for every* $x \in [0,1]$. *While* $f_n(1/n) = 1/2$, *so that*

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq \frac{1}{2} \nrightarrow 0 \textit{ as } n \to \infty$$

and therefore f_n *converges point-wise but not uniformly in* $[0,1]$.

Example 1.4.10 $\sum_{n=0}^{\infty} x^n$ *converges to* $\frac{1}{1-x}$ *for* $x \in (-1,1)$, *but not uniformly. [*$\sum_{n=0}^{\infty} x^n$ *converges uniformly on* $[-r,r]$ *for any* $0 < r < 1$, *see also Theorem 2.1.15 below].*

Indeed, $s_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ tends to $\frac{1}{1-x}$ for any $|x| < 1$. On the other hand

$$\left| s_n(x) - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{|1-x|}$$

so that

$$\begin{aligned}\sup_{x\in(-1,1)}\left|s_n(x)-\frac{1}{1-x}\right| &\geq \frac{\left(\frac{n+1}{n+2}\right)^{n+1}}{|1-\frac{n+1}{n+2}|}\\ &= \frac{n+2}{\left(1+\frac{1}{n+1}\right)^{n+1}}\to\infty\,.\end{aligned}$$

Hence $\sum_{n=0}^{\infty} x^n$ does not converge uniformly in $(-1,1)$.

Theorem 1.4.11 *Let f_n, $f : E \to \mathbb{R}$ (or $\mathbb{C}$), and $f_n \to f$ uniformly in E. Suppose all f_n are continuous at $x_0 \in E$, then the limit function f is also continuous at x_0. Therefore*

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x)\,.$$

[The uniform limit of continuous functions is continuous.]

Proof. For $\varepsilon > 0$, since $f_n \to f$ uniformly on E, there is N, such that for every $n > N$ and $x \in E$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

Since f_{N+1} is continuous at x_0, there is $\delta > 0$ (depending on x_0 and ε) such that for $x \in E$ satisfying $|x - x_0| < \delta$, we have

$$|f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\varepsilon}{3}\,.$$

Hence, for every $x \in E$ such that $|x - x_0| < \delta$, by using the Triangle Inequality,

$$\begin{aligned}|f(x)-f(x_0)| &\leq |f(x)-f_{N+1}(x)|+|f(x_0)-f_{N+1}(x_0)|\\ &\quad +|f_{N+1}(x)-f_{N+1}(x_0)|\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\,.\end{aligned}$$

According to definition, f is continuous at x_0. ■

Remark 1.4.12 *[Version for series] If $\sum_{n=1}^{\infty} f_n$ converges uniformly on E and every f_n is continuous at $x_0 \in E$, then*

$$\lim_{x\to x_0}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n(x_0).$$

In particular, if f_n is continuous on E for all n and $\sum_{n=1}^{\infty} f_n$ converges uniformly on E, then $\sum_{n=1}^{\infty} f_n$ is continuous on E.

Corollary 1.4.13 *Suppose the convergence radius of the power series $\sum_{n=1}^{\infty} a_n x^n$ is $0 < R \leq \infty$, then for every $0 \leq r < R$, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on the closed disk $\{x : |x| \leq r\}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ is continuous on the open ball $\{x : |x| < R\}$.*

Proof. According to the definition of convergence radius, $\sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent for $|x| < R$. In particular, $\sum_{n=1}^{\infty} |a_n| r^n$ is convergent. Since for any x such that $|x| \leq r$

$$|a_n x^n| \leq |a_n| r^n$$

therefore, by Weierstrass M-test, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $\{x : |x| \leq r\}$. It follows that, according to Theorem 1.4.11, as the uniform limit of continuous functions, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $\{x : |x| < r\}$ for any $0 \leq r < R$. Suppose $|x_0| < R$, then we may choose r such that $|x_0| < r < R$, so that $f(x)$ is continuous at x_0. Since $x_0 \in \{x : |x| < R\}$ is arbitrary, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $\{x : |x| < R\}$. ■

In general a power series $\sum_{n=0}^{\infty} a_n x^n$ is not uniformly convergent on the disk $\{x : |x| < R\}$, where R is its convergence radius, but the previous corollary implies that it is continuous on $\{x : |x| < R\}$. The end points R and $-R$ need to be handled differently.

Theorem 1.4.14 *(Abel's theorem) If the series $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0,1]$. Therefore, $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $[0,1]$, and*

$$\lim_{x \uparrow 1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \,.$$

Proof. Let $s_n(x) = \sum_{l=0}^{n} a_l x^l$ be the partial sum sequence associated with the power series $\sum a_n x^n$. We want to show that (s_n) satisfies the uniform Cauchy principle on $[0,1]$. We have already seen that for $n > m$ we have

$$|s_n(x) - s_m(x)| = \left| \sum_{k=m+1}^{n} a_k x^k \right|$$

and we want to control the right-hand side uniformly in $x \in [0,1]$.

Since $\sum a_n$ is convergent, its partial sum sequence $\sum_{k=0}^{n} a_k$ is a Cauchy sequence, according to the General Principle of Convergence Sequences, from Analysis I. Thus, for every $\varepsilon > 0$, there is N such that, for every $n > m > N$ we have

$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon \,. \tag{1.4.2}$$

Fix $m > N$, set

$$c_k = \sum_{j=m+1}^{k} a_j \quad \text{for} \quad k \geq m+1, \quad c_m = 0 \,.$$

[We may use the following observation – at this stage, from now on, we will only deal with the series with the terms $a_k x^k$ for $k \geq m+1$, while these terms for $k \leq m$ will not play any role in our argument afterwards. Thus we can employ a trick that we can simply assume that all $a_k = 0$ for $k \leq m$!].

Then (1.4.2) implies that $|c_k| < \varepsilon$ whenever $k \geq m$, and $a_k = c_k - c_{k-1}$. We have

$$\begin{aligned}
\sum_{k=m+1}^{n} a_k x^k &= \sum_{k=m+1}^{n} (c_k - c_{k-1}) x^k \\
&= \sum_{k=m+1}^{n} c_k x^k - \sum_{k=m+1}^{n} c_{k-1} x^k \\
&= \sum_{k=m+1}^{n-1} c_k \left(x^k - x^{k+1} \right) + c_n x^n
\end{aligned}$$

[The last equality is called the Abel's summation formula – which is a discrete version of integration by parts]. Hence, for every $x \in [0,1]$,

$$\begin{aligned}\left|\sum_{k=m+1}^{n} a_k x^k\right| &\leq \sum_{k=m+1}^{n-1} |c_k| \left(x^k - x^{k+1}\right) + |c_n| x^n \\ &< \varepsilon \sum_{k=m+1}^{n-1} \left(x^k - x^{k+1}\right) + \varepsilon x^n \\ &= \varepsilon x^{m+1} \leq \varepsilon \,.\end{aligned}$$

According to definition, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0,1]$. Therefore $\sum_{n=0}^{\infty} a_n x^n$ continuous on $[0,1]$. In particular

$$\lim_{x \uparrow 1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \,.$$

■

The following Dini's theorem is interesting, but not examinable in paper M2.

Theorem 1.4.15 ***(Dini's Theorem).*** *Let f_n be a sequence of real continuous functions on $[a,b]$. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in [a,b]$, where f is a continuous function on $[a,b]$, and suppose that*

$$f_n(x) \geq f_{n+1}(x) \qquad \forall\, n \text{ and } \forall x \in [a,b] \,,$$

then $f_n \to f$ uniformly in $[a,b]$.

Proof. Let $g_n(x) = f_n(x) - f(x)$. Then g_n is continuous for every n, $g_n \geq 0$ and $\lim_{n\to\infty} g_n(x) = 0$ for any $x \in [a,b]$. Suppose (g_n) were not uniformly convergent on $[a,b]$. Then there is an $\varepsilon > 0$, such that for each k there are a natural number $n_k > k$ and a point $x_k \in [a,b]$ such that

$$|g_{n_k}(x_k)| = g_{n_k}(x_k) \geq \varepsilon \,.$$

[*which is the contra-positive to that (g_n) converges to 0 uniformly on $[a,b]$*]. We may choose n_k so that $k \to n_k$ is increasing, and may assume that $x_k \to p$. [Otherwise we may argue with a convergent subsequence of (x_k), according to Bolzano-Weierstrass' Theorem]. Then $p \in [a,b]$. Since $g_n(x)$ is decreasing in n for every $x \in [a,b]$, thus for every k fixed, for all $l > k$, we have

$$\varepsilon \leq g_{n_l}(x_l) \leq g_{n_k}(x_l) \,. \qquad (1.4.3)$$

Letting $l \to \infty$ in the above inequality, we obtain

$$\varepsilon \leq \lim_{l\to\infty} g_{n_k}(x_l) = g_{n_k}(p) \qquad \text{[since } g_{n_k} \text{ is continuous at } p\text{]},$$

which is a contradicts with the assumption that $\lim_{k\to\infty} g_{n_k}(p) = 0$. ■

Corollary 1.4.16 *Suppose the series of functions $\sum_{n=1}^{\infty} g_n(x)$ converges to its sum $S(x)$ for $x \in [a,b]$, suppose $g_n(x) \geq 0$ for every n and every $x \in [a,b]$, and suppose all g_n and its limit function S are continuous on $[a,b]$, then $\sum_{n=1}^{\infty} g_n$ converges to S uniformly on $[a,b]$.*

Proof. Apply Dini's Theorem to $f_n = S - \sum_{k=1}^{n} g_k$ to conclude that $f_n \downarrow 0$ uniformly on $[a,b]$. ■

Example 1.4.17 *Let $f_n(x) = \frac{1}{1+nx}$ for $x \in (0,1)$. Then $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in (0,1)$, f_n is decreasing in n, but f_n does not converge uniformly. Dini's theorem does not apply for this case, since $(0,1)$ is not compact.*

The proofs of the following two theorems related to the concept of uniform convergence will be given in the Trinity term.

Theorem 1.4.18 *If $f_n \to f$ uniformly in $[a,b]$ and if every f_n is continuous in $[a,b]$, then*

$$\int_a^b f = \int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n .$$

Similarly, if the series $\sum_{n=1}^{\infty} f_n$ converges uniformly in $[a,b]$ and if all f_n are continuous, then we may integrate the series term by term

$$\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n .$$

Let us however immediately point out that the notion of uniform convergence is not *the right condition* for integrating a series term by term: we may exchange the order of integration $\int_a^b$ (which involves a limiting procedure) and $\lim_{n\to\infty}$ under much weaker conditions. The search for correct conditions for term-by-term integration led to the discovery of Lebesgue's integration [Second year A4 paper: Integration]. For details, see W. Rudin's Principles, Chapter 11 (page 300).

Theorem 1.4.19 *Let $f_n \to f$ in (a,b) (convergence point-wisely). Suppose f_n' exists and is continuous on (a,b) for every n, and if $f_n' \to g$ uniformly in (a,b). Then f' exists and is continuous in (a,b), and*

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x) .$$

Similarly, if $\sum f_n$ converges in (a,b), if every f_n' exists and is continuous in (a,b), and if $\sum f_n'$ converges uniformly in (a,b), then

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n' .$$

Chapter 2

Differentiability

In this chapter, we are going to

1) give the definition of the derivative of a function of a real variable and differentiability, and prove important properties of derivatives such as algebra of derivatives, the chain rule and differentiability of polynomials and inverse functions;

2) state the theorem that the derivative of a function defined by a power series is given by the derived series, whose proof is given in the notes too but the proof is not examinable in paper M2;

3) prove Fermat's theorem about vanishing of the derivative at a local maximum or minimum, and as its application prove Darboux' intermediate value theorem and Rolle's Theorem;

4) establish the most important result in this course, the Mean Value Theorem (MVT), together with simple applications: the identity theorem and a study of monotone functions;

5) give a definition of π and give a study of exponential and trigonometric functions;

5) prove Cauchy's (generalized) Mean Value Theorem and l'Hôpital's rules;

6) establish Taylor's Theorem with remainder in Lagrange's form by using MVT, and give examples of Taylor's Theorem and the binomial expansion with arbitrary index.

The whole chapter is about the Mean Value Theorem and its substantial applications.

2.1 The concept of differentiability

In this course we study the differentiability of real (or complex)-valued functions on E, where E is a subset of the real line $\mathbb{R}$. The study of differentiation of complex functions on the complex plane $\mathbb{C}$ is a totally different story from the real case here. The existence of complex coordinates or the complex structure has a completely different meaning, so that it requires another theory – Complex Analysis [**Second year A2 paper: Metric Spaces and Complex Analysis**].

2.1.1 Derivatives, basic properties

Let us begin with the definition of differentiability of a function, and derivatives.

Definition 2.1.1 *1) Let $(a,b) \subseteq \mathbb{R}$ be an open interval, f be a real or complex valued function defined on (a,b), and $x_0 \in (a,b)$. If*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists (a real or complex number), then the limit is called the derivative *of f at x_0 and is denoted by $f'(x_0)$ or $\frac{df}{dx}(x_0)$.*

2) If $f : (a,b] \to \mathbb{R}$ (or $\mathbb{C}$) and $x_0 \in (a,b]$, then the left-derivative of f at x_0 is defined by

$$f'(x_0-) = \lim_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0} ,$$

provided the limit exists. Similarly, if $f : [a,b) \to \mathbb{R}$ (or $\mathbb{C}$) and $x_0 \in [a,b)$, then the right-derivative of f at x_0 is defined by

$$f'(x_0+) = \lim_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} ,$$

provided the limit exists.

3) If $f : D \to \mathbb{C}$ where $D \subset \mathbb{C}$, $z_0 \in D$ such that there is a (small $\delta > 0$) $D(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| < \delta\} \subseteq D$, then the [complex] derivative of f at z_0 is defined to be

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided the limit exists.

Remark 2.1.2 *Let $y = f(x)$. There are other notations for derivatives*

$\frac{dy}{dx}$ *or* $\frac{df(x_0)}{dx}$ *[used by G. W. Leibnitz]*

y' *or* $f'(x_0)$ *[introduced by J. L. Lagrange]*

Dy *or* $Df(x_0)$ *[used by A. L. Cauchy, in particular for vector-valued functions of several variables].*

Remark 2.1.3 *1) According to definition, $f'(x_0)$ exists if and only if both side derivatives $f'(x_0-)$ and $f'(x_0+)$ exist, and $f'(x_0-) = f'(x_0+)$. If $f : (a,b) \to \mathbb{C}$ and $f'(x_0)$ exists, then we say f is differentiable at x_0.*

2) f is differentiable on (a,b) if it is differentiable at every point in (a,b).

3) f is differentiable on $[a,b]$ if it is differentiable on (a,b) and both $f'(a+)$ and $f'(b-)$ exist.

Remark 2.1.4 *Here we have abused the notations $f'(x_0+)$ and $f'(x_0-)$. Recall that if g is a function defined in (a,b) and $x_0 \in (a,b)$, then $g(x_0+)$ and $g(x_0-)$ represent the right-hand limit and the left-hand limit of g at x_0:*

$$g(x_0+) = \lim_{x \downarrow x_0} g(x) \quad \text{and} \quad g(x_0-) = \lim_{x \uparrow x_0} g(x) ,$$

respectively. According to definition here, if f is differentiable in (a,b) [so that the derivative function f' of f is a well defined on (a,b)], $f'(x_0+)$ and $f'(x_0-)$ do not mean the right-hand and the left-hand limits of the derivative function f' at x_0! However, we will show *that,* if $\lim_{x \downarrow x_0} f'(x)$ exists, then the right-hand limit of f'; $\lim_{x \downarrow x_0} f'(x)$; does coincide with $f'(x_0+)$ *we have defined here. A similar statement holds for $f'(x_0-)$ as well.*

Here is a simple example to show the difference. Consider $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$. Then we can show, by using definition of derivatives, that $f'(0) = 0$ [Exercise] and

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \qquad \text{for } x \neq 0.$$

Therefore $f'(0+) = f'(0-) = f'(0) = 0$, but the right-hand and left-hand limits of f' at 0: neither of $\lim_{x \downarrow 0} f'(x)$ and $\lim_{x \uparrow 0} f'(x)$ exists!

Exercise 2.1.5 *1) If $f'(x_0-) > 0$ (resp. $f'(x_0-) < 0$), then there is a number $\delta > 0$ such that $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for every $x \in (x_0 - \delta, x_0]$.*

2) If $f'(x_0+) > 0$ (resp. $f'(x_0+) < 0$), then there is $\delta > 0$ such that $f(x) \geq f(x_0)$ (resp. $f(x) \leq f(x_0)$) for any $x \in [x_0, x_0 + \delta)$.

3) If $f'(x_0) > 0$ (resp. $f'(x_0) < 0$), then there is $\delta > 0$ such that

$$(f(x) - f(x_0))(x - x_0) \geq 0$$

(resp.

$$(f(x) - f(x_0))(x - x_0) \leq 0 \text{)}$$

for all $x \in (x_0 - \delta, x_0 + \delta)$.

If f is differentiable at x_0, i.e. $f'(x_0)$ exists, then

$$\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \to 0 \text{ as } x \to x_0$$

and therefore the increment of f near x_0 can be expressed as

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x, x_0)$$

so that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x, x_0),$$

where $o(x, x_0)$ is a function of x and x_0 satisfying that

$$\lim_{x \to x_0} \frac{o(x, x_0)}{x - x_0} = 0 \,.$$

The part of the increment $f(x) - f(x_0)$ linear in $x - x_0$, namely $f'(x_0)(x - x_0)$, is called *the differential* of f at x_0, a concept we will not study further in this course. The linear part of $f(x)$ near x_0:

$$f(x_0) + f'(x_0)(x - x_0)$$

is called the *linear approximation* of the function $f(x)$ about x_0. The linear function

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is the equation of the tangent line of f at $(x_0, f(x_0))$, which has been defined in your A-level course.

We next prove several standard facts about differentiability.

Theorem 2.1.6 *Let $f : (a, b) \to \mathbb{R}$ (or $\mathbb{C}$). If f is differentiable at $x_0 \in (a, b)$, then f is continuous at x_0.*

Proof. Since

$$\begin{aligned} \lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0}(x - x_0) \\ &= f'(x_0) \times 0 \\ &= 0 \end{aligned}$$

where the second equality follows from the algebra of limits. Therefore $\lim_{x \to x_0} f(x) = f(x_0)$, thus according to definition f is continuous at x_0. ■

Theorem 2.1.7 *If f, $g:(a,b)\to\mathbb{R}$ (or $\mathbb{C}$) are differentiable at $x_0\in(a,b)$, then*
1) $(f\pm g)'(x_0)=f'(x_0)\pm g'(x_0)$,
2) (Product rule) $(fg)'(x_0)=f'(x_0)g(x_0)+f(x_0)g'(x_0)$ [This means that the mapping $f\to f'$ is a derivation*],*
3) and if in addition $g(x_0)\neq 0$

$$\left(\frac{f}{g}\right)'(x_0)=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}\,.$$

Proof. 1) follows from AOL for limits. 2) Let $h=fg$. Then we can write

$$h(x)-h(x_0)=g(x_0)\left(f(x)-f(x_0)\right)+f(x)\left(g(x)-g(x_0)\right).$$

Dividing both sides by $x-x_0$, and taking limit $x\to x_0$ to obtain

$$\begin{aligned}\lim_{x\to x_0}\frac{h(x)-h(x_0)}{x-x_0} &= g(x_0)\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}+\lim_{x\to x_0}f(x)\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\\ &= f'(x_0)g(x_0)+f(x_0)g'(x_0)\ \text{[Algebra of limits]}\end{aligned}$$

where we have used the fact that $g(x)\to g(x_0)$ as $x\to x_0$ [Theorem 2.1.6].

To prove 3), we need to show f/g is well defined near x_0. Since g is continuous at x_0, for $\varepsilon=\frac{|g(x_0)|}{2}$ which is positive as $g(x_0)\neq 0$, there is $\delta>0$, for any $x\in(a,b)$ such that $|x-x_0|<\delta$ we have

$$|g(x)-g(x_0)|<\frac{|g(x_0)|}{2}\,.$$

It follows that

$$\begin{aligned}|g(x)| &\geq |g(x_0)|-|g(x)-g(x_0)| \qquad \text{[Triangle Inequality]}\\ &> \frac{|g(x_0)|}{2}>0 \qquad \forall\,.\end{aligned}$$

for all $x\in(a,b)$ such that $|x-x_0|<\delta$. Let $h=\frac{f}{g}$ on $(a,b)\cap(x_0-\delta,x_0+\delta)$. Then

$$\frac{h(x)-h(x_0)}{x-x_0}=\frac{1}{g(x)g(x_0)}\left[g(x_0)\frac{f(x)-f(x_0)}{x-x_0}-f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right]\,.$$

Letting $x\to x_0$ we prove 3). ■

Theorem 2.1.8 *(The chain rule for derivatives) Suppose $f:(a,b)\to\mathbb{R}$ is differentiable at $x_0\in(a,b)$, $g:(c,d)\to\mathbb{R}$ is differentiable at $y_0=f(x_0)\in(c,d)$, and $f((a,b))\subseteq(c,d)$, then $h=g\circ f$ is differentiable at x_0 and*

$$h'(x_0)=g'(y_0)f'(x_0)\,.$$

Proof. Let

$$v(y)=\frac{g(y)-g(y_0)}{y-y_0}-g'(y_0)\qquad \forall y\neq y_0$$

and $v(y_0)=0$. Since g is differentiable at y_0, $v(y)\to 0=v(y_0)$ as $y\to y_0$, and therefore v is continuous at y_0. We may write the increment

$$g(y)-g(y_0)=(y-y_0)\left(g'(y_0)+v(y)\right)$$

which is valid for every $y \in (c,d)$. In particular

$$g(f(x)) - g(f(x_0)) = (f(x) - f(x_0))\left(g'(y_0) + v(f(x))\right)$$

for any $x \in (a,b)$, so that

$$\frac{h(x) - h(x_0)}{x - x_0} = g'(y_0)\frac{f(x) - f(x_0)}{x - x_0} + v(f(x))\frac{f(x) - f(x_0)}{x - x_0} \ . \tag{2.1.1}$$

for all $x \neq x_0$. Since f is differentiable at x_0, f continuous at x_0 [Theorem 2.1.6], and therefore $f(x) \to y_0$ as $x \to x_0$, which in turn yields that $v(f(x)) \to 0$ as $x \to x_0$. Letting $x \to x_0$ in (2.1.1) we obtain

$$\begin{aligned}
\lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} &= g'(y_0) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&\quad + \lim_{x \to x_0} v(f(x)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= g'(y_0) f'(x_0) + 0 \times f'(x_0) \\
&= f'(x_0) g'(y_0) \ .
\end{aligned}$$

■

Theorem 2.1.9 *Let f be real valued continuous and 1-1 function on (a,b), and $x_0 \in (a,b)$. If f is differentiable at x_0 and $f'(x_0) \neq 0$, then the inverse function f^{-1} is differentiable at $y_0 = f(x_0)$ and the derivative of f^{-1} at y_0 is given by*

$$\frac{d}{dy} f^{-1}(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

Proof. According to IVT, since f is continuous on (a,b), $f((a,b))$ is an interval. Since f is 1-1, by Lemma 1.3.28, f is strictly monotone (i.e. strictly increasing on (a,b), or is strictly decreasing on (a,b)), hence $f((a,b))$ must be an open interval (Theorem 1.3.31), denoted by (c,d), where

$$c = \lim_{x \downarrow a} f(x) \quad \text{ and } d = \lim_{x \uparrow b} f(x).$$

According to the Inverse Function Theorem (continuity part), the inverse function f^{-1} is continuous on (c,d). Hence $y_0 = f(x_0) \in (c,d)$. If $y \to y_0$, where $y \neq y_0$ and $y \in (c,d)$, then since f^{-1} continuous,

$$x = f^{-1}(y) \to f^{-1}(y_0) = x_0$$

and $x \neq x_0$ as f is 1-1, and $x \in (a,b)$. Therefore, by AOL

$$\begin{aligned}
\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} &= \lim_{y \to y_0} \frac{x - x_0}{f(x) - f(x_0)} \\
&= \lim_{y \to y_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \\
&= \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} \\
&= \frac{1}{f'(x_0)}
\end{aligned}$$

exists, so that f^{-1} is differentiable at y_0 and

$$\frac{d}{dy}f^{-1}(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

which completes the proof. ■

Example 2.1.10 *Consider function*

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0\,; \\ 0 & \text{if } x = 0\,, \end{cases}$$

which is continuous on $\mathbb{R}$. *Since*

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\sin\frac{1}{x}$$

doesn't exist, f *is not differentiable at* 0. f *is differentiable at any other point, and*

$$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} \qquad \forall x \neq 0\,.$$

Note that $\lim_{x\to 0} f'(x)$ *does not exist [Why ?]*

Example 2.1.11 *Let* $f(x) = x^2\sin\frac{1}{x}$ $(x \neq 0)$ *and* $f(0) = 0$. *Then*

$$\begin{aligned} f'(0) &= \lim_{x\to 0}\frac{f(x)-f(0)}{x} \\ &= \lim_{x\to 0} x\sin\frac{1}{x} = 0 \end{aligned}$$

and

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \qquad \forall x \neq 0\,.$$

Therefore f *is differentiable everywhere, the derivative function* f' *is not continuous at* 0: $\lim_{x\to 0} f'(x)$ *doesn't exist.*

Example 2.1.12 $f(x) = |x|$ *is continuous but not differentiable at* 0. *But the left (right)-derivative of* f *at* 0 *exists, and* $f'(0-) = -1$ *and* $f'(0+) = 1$. *Note that* $\lim_{x\downarrow 0} f'(x) = f'(0+)$ *and* $\lim_{x\to\uparrow 0} f'(x) = f'(0-)$.

Definition 2.1.13 *If* f *is differentiable on* (a,b), *then the second-order derivative*

$$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}$$

if the limit exists, which is denoted also by $f^{(2)}(x)$. *Inductively define* $f^{(n+1)}(x)$ *to be the derivative of* $f^{(n)}$ *for any* n, *as long as the derivative exists.*

Theorem 2.1.14 ***(Leibnitz Formula)*** *If* $F = fg$, *then*

$$F^{(n)}(x) = \sum_{j=0}^{n}\binom{n}{j} f^{(j)}(x)g^{(n-j)}(x)\,.$$

2.1.2 Differentiability of power series

Power series are important class of differentiable functions.

Theorem 2.1.15 *Consider the power series*

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n \\ &= a_0 + a_1 z + \cdots + a_n z^n + \cdots . \end{aligned} \tag{2.1.2}$$

Let R be its convergence radius, and assume that $0 < R \leq \infty$. Then

1) The power series obtained by differentiating f term by term

$$\begin{aligned} g(z) &= \sum_{n=1}^{\infty} n a_n z^{n-1} \\ &= a_1 + 2a_2 z \cdots + n a_n z^{n-1} + \cdots . \end{aligned} \tag{2.1.3}$$

has the same convergence radius R.

2) The [complex] derivative

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for every z satisfying that $|z| < R$, and $f'(z) = g(z)$. That is

$$\frac{d}{dz} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \textit{for any } |z| < R\,. \tag{2.1.4}$$

Proof. [This theorem says that we may differentiate a power series term by term. *Proof is not examinable in Prelims Paper II – this theorem will be revisited in Paper A2.*]

1) Let $|z| < R$. Set $r = \frac{1}{2}(|z| + R)$ (or $r = 2|z| + 1$ if $R = \infty$). Then $|z| < r < R$ and $q \equiv \frac{|z|}{r} \in [0, 1)$. We have the following facts:

(a) $\sum_{n=0}^{\infty} |a_n| r^n < \infty$ [Analysis 1: a power series converges absolutely inside its convergence disk],

(b) $\{nq^{n-1}\}$ is bounded. [Indeed $\sum nq^{n-1}$ converges (by the ratio test), so that $\lim_{n\to\infty} nq^{n-1} = 0$: but we don't need these stronger results here].

Let $b_n = nq^{n-1}$. Then

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{n} q$$

which is smaller than 1 for n large enough. Thus (b_n) is decreasing for large n, so that $\lim_{n\to\infty} b_n$ exists, and therefore (nq^{n-1}) is bounded. Let $nq^{n-1} \leq M$ for some $M > 0$, for every n.

(c) $\sum_{n=1}^{\infty} n a_n z^{n-1}$converges absolutely. Indeed

$$\begin{aligned} |n a_n z^{n-1}| &\leq n|a_n||z|^{n-1} = nq^{n-1}|a_n| r^{n-1} \\ &\leq \frac{M}{r} |a_n| r^n \qquad \forall n \geq 1 \end{aligned}$$

so that, by the comparison test [Analysis 1]

$$\sum_{n=1}^{\infty} n|a_n||z|^{n-1} \leq \frac{M}{r} \sum_{n=1}^{\infty} |a_n| r^n < \infty.$$

Similarly we may prove that the convergence radius of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ can not be greater than that of $\sum_{n=0}^{\infty} a_n z^n$.

2) We are going to show that the complex derivative $f'(z)$ exists and equals $g(z)$ at every point z such that $|z| < R$. Let $r = \frac{1}{2}(|z| + R)$ (or $r = |z| + 1$ if $R = \infty$). Then $r < R$, and $|z| < r$. For any point $w \neq z$ such that $|w| < r$, consider

$$\begin{aligned} \frac{f(w) - f(z)}{w - z} - g(z) &= \sum_{n=1}^{\infty} a_n \left(\frac{w^n - z^n}{w - z} - nz^{n-1} \right) \\ &= \sum_{n=2}^{\infty} a_n \left(\frac{w^n - z^n}{w - z} - nz^{n-1} \right) \ ; \end{aligned} \tag{2.1.5}$$

where we have added the series $f(w)$, $f(z)$ and $g(z)$ term by term, which is justified as all these series are absolutely convergent [Analysis 1: a power series converges absolutely inside the convergence disk]. Our aim is to show that

$$\frac{f(w) - f(z)}{w - z} - g(z) \to 0 \qquad \text{as } w \to z \, .$$

To this end we use the following identity

$$\frac{w^n - z^n}{w - z} = z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1}$$

[*Exercise*. Apply the geometric series

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x} \qquad \forall n \geq 1$$

to $x = w/z$ or z/w]. Therefore, for any $w \neq z$ and $n \geq 2$

$$\begin{aligned} \frac{w^n - z^n}{w - z} - nz^{n-1} &= z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1} \\ &\quad -z^{n-1} - z^{n-1} - \cdots - z^{n-1} - z^{n-1} \\ &= \sum_{k=1}^{n-1} \left(z^{n-1-k} w^k - z^{n-1} \right) \\ &= \sum_{k=1}^{n-1} z^{n-1-k} \left(w^k - z^k \right) \, . \end{aligned}$$

Let

$$h_n(w) = a_n \sum_{k=1}^{n-1} z^{n-1-k} \left(w^k - z^k \right) \ ; \quad n = 2, 3, \cdots .$$

Then

$$\frac{f(w) - f(z)}{w - z} - g(z) = \sum_{n=2}^{\infty} h_n(w)$$

All h_n are continuous in $\mathbb{C}$ (polynomials in w), and $h_n(z) = 0$ (for all $n \geq 2$). We claim that $\sum_{n=2}^{\infty} h_n(w)$ converges uniformly in $|w| \leq r$. In fact

$$\begin{aligned} |h_n(w)| &\leq |a_n| \sum_{k=1}^{n-1} |z|^{n-1-k} \left(|w|^k + |z|^k \right) \\ &\leq 2n|a_n| r^{n-1} \, . \end{aligned}$$

By 1), $\sum n|a_n|r^{n-1} < \infty$, so that $\sum_{n=2}^{\infty} h_n(w)$ converges uniformly in closed disk $\{w : |w| \leq r\}$ [Weierstrass M-test, Chapter 2]. Hence $\sum_{n=2}^{\infty} h_n(w)$ is continuous in the disk $|w| \leq r$ [Theorem 1.4.11: the uniform limit of continuous functions is continuous]. Therefore

$$\lim_{w \to z} \sum_{n=2}^{\infty} h_n(w) = \sum_{n=2}^{\infty} h_n(z) = 0$$

so that

$$\begin{aligned} \lim_{w \to z} \frac{f(w) - f(z)}{w - z} &= \lim_{w \to z} \left(\frac{f(w) - f(z)}{w - z} - g(z) \right) + g(z) \\ &= \lim_{w \to z} \sum_{n=2}^{\infty} h_n(w) + g(z) \\ &= g(z) . \end{aligned}$$

This completes the proof. ■

Now we are in a position to study several important elementary functions.

Example 2.1.16 *It was the great mathematician Gauss who studied the exponential function* exp *as a function on the complex plane, and made the link between* exp *and trigonometric functions* sin *and* cos. *The modern approach we present as the following is essentially due to him.*

The exponential function is defined by the power series

$$\exp z = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2!} + \cdots$$

which converges everywhere in $\mathbb{C}$ (that is, its convergence radius is ∞). Substituting z by iz or $-iz$, and using the fact that $i^{2n} = (-1)^n$ we obtain that

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

and

$$e^{-iz} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

which allows to define the trigonometric functions sin and cos in terms of the exponential function exp, namely

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots$$

and

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

which have infinite convergence radius, and therefore both are differentiable. It follows immediately the Euler formula

$$e^{iz} = \cos z + i \sin z$$

which is valid for every complex number z.

According to Theorem 2.1.15 exp is differentiable in $\mathbb{C}$ and its derivative may be calculated by differentiate term by term. Hence

$$\frac{\mathrm{d}}{\mathrm{d}z}\exp z = \sum_{n=1}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{z^n}{n!} = \exp z.$$

Similarly

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} = \cos z$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z}\cos z = \sum_{n=1}^{\infty}(-1)^n 2n\frac{z^{2n-1}}{(2n)!} = -\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!} = -\sin z.$$

Example 2.1.17 *Let us consider* exp *as the function on* $\mathbb{R}$. *Then* $\exp 0 = 1$, $x \to \exp x$ *is strictly increasing on* $[0,\infty)$. *Since* $\exp x \geq 1 + x$ *for every* $x \geq 0$, *so that* exp *maps* $[0,\infty)$ *one-to-one and onto* $[1,\infty)$. *[Indeed it is strictly increasing from* $(-\infty,\infty)$ *one to one and onto* $(0,\infty)$, *see below Corollary 2.3]. Let* $\ln : [1,\infty) \to [0,\infty)$ *denote the inverse function of* $f = \exp$ *on* $[0,\infty)$. *By definition,* $\ln 1 = 0$ *and* ln *is strictly increasing on* $[1,\infty)$. *Since* $\frac{d}{dx}\exp x = \exp x > 0$ *for all* $x \geq 0$, *according to Theorem 2.1.9,* f^{-1} *is differentiable on* $[1,\infty)$ *and, for every* $y \in [1,\infty)$

$$\frac{d}{dy}\ln y = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f(f^{-1}(y))} = \frac{1}{y}.$$

Therefore ln *is differentiable and*

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

for all $x \geq 1$.

We will study exp on $(-\infty,\infty)$ and its inverse ln, which is defined on $(0,\infty)$, after we establish the important Mean Value Theorem.

2.1.3 Van der Vaerden's example

The following example of a continuous function on $\mathbb{R}$ which is nowhere differentiable was constructed by **B. L. Van der Waerden** [For your reading – I don't think I'll have time to work through this example].

Let us begin with a simple continuous function

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1; \\ 2-x & \text{if } 1 \leq x \leq 2 \end{cases}$$

and extend h to be a periodic function with period 2, i.e. $h(x+2) = h(x)$ for $x \in \mathbb{R}$. Then h is continuous on $\mathbb{R}$. Consider the series

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n h(4^n x)\ .$$

By the Weierstrass M-test, $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n h(4^n x)$ converges uniformly in $\mathbb{R}$, thus f is continuous on $\mathbb{R}$ [Theorem 1.4.11] and

$$|f(x)| \leq \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = 4 \quad \text{for every } x \in \mathbb{R}\ .$$

Let $x \in \mathbb{R}$, $m \in \mathbb{N}$ and set $k = [4^m x]$ the integer part of $4^m x$: k is the unique integer such that

$$k \leq 4^m x < k+1 \, .$$

Let $\alpha_m = 4^{-m}k$ and $\beta_m = 4^{-m}(k+1)$. Obviously

$$\alpha_m \leq x < \beta_m$$

and

$$\beta_m - \alpha_m = \frac{1}{4^m} \to 0 \quad \text{as } m \to \infty \, .$$

In particular, $\lim_{m\to\infty} \alpha_m = \lim_{m\to\infty} \beta_m = x$. We are going to show that

$$\lim_{m\to\infty} \frac{f(\beta_m) - f(\alpha_m)}{\beta_m - \alpha_m}$$

does not exist, so that f is not differentiable at x. Since x is arbitrary, f is nowhere differentiable.

If $n > m$, then $4^n\beta_m - 4^n\alpha_m$ is an even number, and if $n \leq m$ then there is no integer between $4^n\beta_m$ and $4^n\alpha_m$. Therefore

$$|h(4^n\beta_m) - h(4^n\alpha_m)| = \begin{cases} 0, & \text{if } n > m; \\ 4^{n-m}, & \text{if } n \leq m. \end{cases}$$

Hence

$$\begin{aligned} f(\beta_m) - f(\alpha_m) &= \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n (h(4^n\beta_m) - h(4^n\alpha_m)) \\ &= \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n (h(4^n\beta_m) - h(4^n\alpha_m)) \end{aligned}$$

so that

$$\begin{aligned} |f(\beta_m) - f(\alpha_m)| &\geq \left(\frac{3}{4}\right)^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n |h(4^n\beta_m) - h(4^n\alpha_m)| \\ &= \left(\frac{3}{4}\right)^m - \sum_{n=0}^{m-1} 4^{n-m} \left(\frac{3}{4}\right)^n \\ &= \left(\frac{3}{4}\right)^m - \frac{1}{4^m}\frac{3^m - 1}{2} \\ &= \frac{1}{2}\left(\frac{3}{4}\right)^m + \frac{1}{2}\frac{1}{4^m} \, . \end{aligned}$$

Therefore

$$\frac{|f(\beta_m) - f(\alpha_m)|}{\beta_m - \alpha_m} \geq \frac{3^m + 1}{2} \to \infty \ \text{ as } m \to \infty$$

and it follows that $\lim \frac{f(\beta_m) - f(\alpha_m)}{\beta_m - \alpha_m}$ does not exist. Hence f is not differentiable at any point x.

2.2 Mean Value Theorem (MVT)

Next we are going to study functions by using the tools we have developed, namely function limits and derivatives.

2.2.1 Local maxima and minima

If $f : E \to \mathbb{R}$ is a real function on E, then $a \in E$ is a local maximum (resp. local minimum) if there is a $\delta > 0$, such that $(a-\delta, a+\delta) \subseteq E$ and for every $x \in (a-\delta, a+\delta)$.

$$f(x) \leq f(a) \quad \text{(resp. } f(x) \geq f(a)).$$

A local maximum or minimum is called a local extremum.

Lemma 2.2.1 *Let $\delta > 0$ and $a \in \mathbb{R}$.*
1) If $f(x) \leq f(a)$ for every $x \in (a-\delta, a]$ and if $f'(a-)$ exists, then $f'(a-) \geq 0$. Similarly
2) If $f(x) \leq f(a)$ for every $x \in [a, a+\delta)$ and if $f'(a+)$ exists, then $f'(a+) \leq 0$.

Proof. 1) Since

$$f(x) - f(a) \leq 0, \qquad \text{for all } x \in (a-\delta, a)$$

so that

$$\frac{f(x) - f(a)}{x-a} \geq 0 \qquad \text{for all } x \in (a-\delta, a)$$

and therefore, as $f'(a-)$ exists,

$$f'(a-) = \lim_{x \uparrow a} \frac{f(x) - f(a)}{x-a} \geq 0\ .$$

The proof of 2) is similar, as for $x \in (a, a+\delta)$, $(x) - f(a) \leq 0$ so that

$$\frac{f(x) - f(a)}{x-a} \leq 0 \qquad \text{for all } x \in (a, a+\delta)$$

and therefore

$$f'(a+) = \lim_{x \downarrow a} \frac{f(x) - f(a)}{x-a} \leq 0.$$

■

Similarly we have

Lemma 2.2.2 *Let $\delta > 0$ and $a \in \mathbb{R}$.*
1) If $f(x) \geq f(a)$ for every $x \in (a-\delta, a]$ and if $f'(a-)$ exists, then $f'(a-) \leq 0$. Similarly
2) If $f(x) \geq f(a)$ for every $x \in [a, a+\delta)$ and if $f'(a+)$ exists, then $f'(a+) \geq 0$.

As a consequence we have the following important

Theorem 2.2.3 ***(Fermat's Theorem)*** *Let $f : E \to \mathbb{R}$. Suppose that a is a local extremum of f, and f is differentiable at a. Then $f'(a) = 0$.*

Since f is differentiable at a we have

$$f'(a+) = f'(a-) = f'(a)$$

so according to the previous lemma, $f'(a) = 0$.

[Fermat's theorem says that a local extremum must be a stationary point.]

As an interesting application, we show the following Intermediate Value Theorem for derivative functions.

Theorem 2.2.4 ***(Darboux' Intermediate Value Theorem)*** *If $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$, and $f'(a) < A < f'(b)$, then there exists a point $\xi \in (a,b)$ such that $f'(\xi) = A$.*

Proof. Let $g(x) = f(x) - Ax$. Then g is differentiable in $[a,b]$, so that g is continuous in $[a,b]$. Therefore g attains its bounds. Moreover

$$g'(x) = f'(x) - A$$

so that $g'(a) = f'(a) - A < 0$ and $g'(b) = f'(b) - A > 0$. Since $g'(a) < 0$ there exists $\delta_1 > 0$ such that $g(x) < g(a)$ for $x \in (a, a+\delta_1)$. Similarly, since $g'(b) > 0$, there is $\delta_2 > 0$ such that $g(x) < g(b)$ for $x \in (b - \delta_2, b)$. Therefore a or b cannot be the minimum of g on $[a,b]$, so that g must have its minimum (though not necessary unique) $\xi \in (a,b)$, which is thus a local minimum of g. By Fermat's theorem, $g'(\xi) = 0$. ∎

Example 2.2.5 *Consider $f(x) = x^2 \sin \frac{1}{x}$ if $x \neq 0$, and $f(0) = 0$. f is differentiable everywhere, but the derivative function*

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

is not continuous at 0, and thus IVT [Chapter 1: IVT for continuous functions on closed intervals] does not apply to f' on $[-1,1]$ for example, but f' attains all values between $f'(-1)$ and $f'(1)$, according to the Darboux IVT.

Theorem 2.2.6 ***(Rolle's Theorem, 1691)*** *If $f : [a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then there exists a point $\xi \in (a,b)$ such that $f'(\xi) = 0$.*

Proof. If f is constant on $[a,b]$, then $f'(x) = 0$ for every $x \in (a,b)$, so that any point $\xi \in (a,b)$ will do. Since f is continuous, f attains its maximum and minimum on $[a,b]$. That is, there are $x_1, x_2 \in [a,b]$ such that $f(x_1) = \min_{x \in [a,b]} f(x)$ and $f(x_2) = \sup_{x \in [a,b]} f(x)$. If f is not constant, then $f(x_1) \neq f(x_2)$. Since $f(a) = f(b)$, at least one (denoted by ξ) of x_1 and x_2 belongs to (a,b). ξ must be a local extremum and therefore, by Fermat's Theorem, $f'(\xi) = 0$. ∎

Corollary 2.2.7 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then between any two distinct roots of $f(x) = 0$ there is a root of $f'(x) = 0$.*

Example 2.2.8 *$f(x) = \sin x$ and $f'(x) = \cos x$. Study the zeros of f and f'.*

2.2.2 Mean Value Theorems

Theorem 2.2.9 ***(Mean Value Theorem, MVT)*** *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, and f is differentiable on (a,b), then there is a point $\xi \in (a,b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} .$$

Proof. The idea is to rotate the graph to the level position, so we can apply Rolle's theorem. Analytically, observe that the line equation of the chord through $(a, f(a))$ and $(b, f(b))$ is given by

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

where the ratio $(f(b)-f(a))/(b-a)$ is the slope of the chord. The idea is to apply Rolle's theorem to the function

$$F(x)=f(x)-\left[f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right].$$

Clearly F is continuous on $[a,b]$ and is differentiable on (a,b),

$$F'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$$

and $F(a)=0=F(b)$. According to Rolle's Theorem, there is $\xi\in(a,b)$ such that $F'(\xi)=0$, that is $f'(\xi)=\frac{f(b)-f(a)}{b-a}$. ■

In applications, we often write MVT as

$$f(b)-f(a)=f'(\xi)(b-a)$$

for some $\xi\in(a,b)$. Since $\xi\in(a,b)$, ξ can be written as $\xi=a+\theta(b-a)$ for some $\theta\in(0,1)$. Therefore, if we set $h=b-a$, then $b=a+h$, so that the MVT becomes

$$f(a+h)-f(a)=f'(a+\theta h)h$$

or in the form:

$$f(a+h)=f(a)+f'(a+\theta h)h$$

[which is a special case of Taylor's Theorem], for some $\theta\in(0,1)$.

Theorem 2.2.10 ***(Cauchy's Mean Value Theorem)*** *Suppose f and $g:[a,b]\to\mathbb{R}$ are continuous, f and g are differentiable on (a,b), and $g'\neq 0$ on (a,b), then there is a point $\xi\in(a,b)$ such that*

$$\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. First we show that $g(b)\neq g(a)$. In fact, if $g(a)=g(b)$, then by Rolle's Theorem, there is $x_0\in(a,b)$, $g'(x_0)=0$, which is a contradiction to the assumption.

We employ the same idea as in the proof for MVT, and apply Rolle's Theorem to the following function

$$F(x)=f(x)-\left[f(a)+\frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))\right].$$

Then F is continuous on $[a,b]$, and differentiable in (a,b),

$$F'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x)$$

and $F(a)=F(b)=0$. According to Rolle's Theorem, there is a point $\xi\in(a,b)$ such that $F'(\xi)=0$, that is

$$f'(\xi)=\frac{f(b)-f(a)}{g(b)-g(a)}g'(\xi).$$

Since $g'(\xi)\neq 0$, so that, by dividing $g'(\xi)$ both sides,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}.$$

■

Corollary 2.2.11 *Suppose $f:(a,b)\to\mathbb{R}$ is differentiable at every $x\in(a,b)$. Then*
*1) [*Identity Theorem*] f is constant on (a,b) if and only if $f'=0$ on (a,b).*
*2) [*Monotone property*] f is increasing (resp. decreasing) on (a,b) if and only if $f'(x)\geq 0$ (resp. $f'(x)\leq 0$) for all $x\in(a,b)$.*
3) If $f'(x)>0$ for all $x\in(a,b)$, then f is strictly increasing on (a,b).

Proof. 1) Suppose $f'(x)=0$ for every $x\in(a,b)$, then by applying MVT to f on $[x,y]$, where x, y are any two points in (a,b). Then $f(x)-f(y)=f'(\xi)(x-y)$ for some number ξ between x and y. Since $f'(\xi)=0$, so that $f(x)=f(y)$. Therefore f is constant in (a,b). The proofs of 2) and 3) are similar. ∎

Proposition 2.2.12 *Let f be differentiable on (a,b), and $f'(x)>0$ for every $x\in(a,b)$. Then f is strictly increasing on (a,b) and its inverse function f^{-1} is differentiable on (c,d), and*

$$\frac{d}{dy}f^{-1}(y)=\frac{1}{f'(f^{-1}(y))}$$

for every $y\in(c,d)$, where $c=\lim_{x\downarrow a}f(x)$ and $d=\lim_{x\uparrow b}f(x)$.

Proof. Since f is differentiable on (a,b), so it is continuous on (a,b). Since $f'(x)>0$ for every $x\in(a,b)$, so f is strictly increasing on (a,b). The conclusion now follows immediately from Theorem 2.1.9. ∎

Example 2.2.13 *Show that the general solution for $f'(x)=f(x)\ ;\ \ x\in(0,\infty)$, is $f(x)=A\exp(x)$ where A is a constant.*

Proof. Let $g(x)=\frac{f(x)}{\exp(x)}$ which is differentiable as $\exp x\neq 0$ and both f and exp are differentiable. Then

$$\begin{aligned} g'(x) &= \frac{f'(x)\exp(x)-f(x)\exp'(x)}{\exp(x)^2} \\ &= \frac{f(x)\exp(x)-f(x)\exp(x)}{\exp(x)^2} \quad \text{[Use the facts: } \exp'=\exp \text{ and } f'=f\text{]} \\ &= 0 \end{aligned}$$

so that $g=A$ on $(0,\infty)$ for some constant [Identity Theorem]. Therefore $f(x)=A\exp(x)$ for all $x\in(0,\infty)$. ∎

Now we are in a position to study the exponential function $\exp x$ for $x\in(-\infty,\infty)$ and its inverse the logarithm function ln.

Proposition 2.2.14 *1)* $\exp(a+b)=\exp(a)\exp(b)$ *for all $a,b\in\mathbb{R}$.*
2) $\exp(x)>0$ *for any $x\in(-\infty,\infty)$, and $x\to$* $\exp(x)$ *is strictly increasing,* $\exp(x)\to\infty$ *as $x\to\infty$ and* $\exp(x)\to 0$ *as $x\to-\infty$. Therefore the inverse function of* exp *exists, called the logarithm function, denoted by* $\ln x$ *for $x\in(0,\infty)$.*
3) $\ln:(0,\infty)\to(-\infty,\infty)$ *is differentiable, and* $\frac{d}{dx}\ln x=\frac{1}{x}$.

Proof. 1) For any (fixed real) c, consider $g(x)=\exp(x)\exp(c-x)$. Then

$$\begin{aligned} g'(x) &= \exp'(x)\exp(c-x)-\exp(x)\exp'(c-x) \\ &= \exp(x)\exp(c-x)-\exp(x)\exp(c-x) \\ &= 0 \end{aligned}$$

so that g is constant [Identity Theorem]. Clearly $\exp 0 = 1$, so that $g(x) = g(0) = \exp c$ for every x and c. That is

$$\exp(x)\exp(c-x) = \exp(c) \qquad \forall x\ .$$

Setting $x = a$ and $c = a+b$ we obtain

$$\exp(a+b) = \exp(a)\exp(b)\ .$$

2) If $x \geq 0$ then

$$\begin{aligned}\exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\ &\geq 1\end{aligned}$$

and if $x < 0$, then

$$1 = \exp(x-x) = \exp(-x)\exp(x)$$

so that

$$0 < \exp(x) = \frac{1}{\exp(-x)} \leq 1 \qquad \forall x < 0\ .$$

In particular, by using MVT, since $\exp'(x) = \exp(x) > 0$ for every $x \in (-\infty,\infty)$, $\exp(x)$ is strictly increasing on $(-\infty,\infty)$. Since $\lim_{x\to\infty}\exp(x) = \infty$, and $\exp(x) \to 0$ as $x \to -\infty$, by IVT, exp maps $(-\infty,\infty)$ 1-1 and onto $(0,\infty)$. Thus exp has a continuous inverse $\exp^{-1}$ defined on $(0,\infty)$, which is denoted by ln. Since the derivative of $\exp'(x) = \exp(x) > 0$, so that, according to Theorem 2.1.9, $\exp^{-1} = \ln$ is differentiable on $(0,\infty)$, and

$$\ln'(y) = \frac{1}{\exp'(\ln(y))} = \frac{1}{\exp(\ln(y))} = \frac{1}{y}\ .$$

That is, $\frac{d}{dx}\ln x = \frac{1}{x}$ for any $x > 0$. ■

Exercise 2.2.15 *Define* $e = \exp(1)$. *Show that (i)* $1 < e < 3$*; (ii)* e *is irrational.*

Example 2.2.16 *For* $x \geq 0$*, we have*
(i) $\exp(-x) \leq 1$*;*
(ii) $\exp(-x) \geq 1 - x$*;*
(iii) $\exp(-x) \leq 1 - x + \frac{x^2}{2}$.
In general we have, for any natural number n,

$$\exp(-x) \leq \sum_{k=0}^{2n}(-1)^k\frac{x^k}{k!} \qquad \textit{and} \qquad \exp(-x) \geq \sum_{k=0}^{2n+1}(-1)^k\frac{x^k}{k!} \tag{2.2.1}$$

for any $x \geq 0$.

Proof. (i) Let $f(x) = \exp(-x)$. Then $f'(x) = -\exp(-x) < 0$, so that f is decreasing in $[0,\infty)$. In particular $f(x) \leq f(0) = 1$ for all $x \geq 0$.

(ii) Let $g(x) = \exp(-x) - 1 + x$. Then $g'(x) = -\exp(-x) + 1 \geq 0$ [By (i)], so that g is increasing, thus $g(x) \geq g(0) = 0$.

(iii) Consider $h(x) = \exp(-x) - 1 + x - \frac{x^2}{2}$. Then

$$h'(x) = -\exp(-x) + 1 - x \leq 0$$

so that h is decreasing in $[0,\infty)$. Hence $h(x) \le h(0) = 0$.

To prove (2.2.1) we use an induction argument on n. We have proven the case where $n = 0$. Suppose (2.2.1) is true for n. Consider

$$f(x) = \exp(-x) - \sum_{k=0}^{2(n+1)} (-1)^k \frac{x^k}{k!}.$$

Then

$$\begin{aligned} f'(x) &= \sum_{k=2(n+1)+1}^{\infty} (-1)^k k \frac{x^{k-1}}{k!} = \sum_{k=2(n+1)+1}^{\infty} (-1)^k \frac{x^{k-1}}{(k-1)!} \\ &= -\sum_{k=2(n+1)+1}^{\infty} (-1)^{k-1} \frac{x^{k-1}}{(k-1)!} = -\sum_{k=2(n+1)}^{\infty} (-1)^k \frac{x^k}{(k-1)!} \\ &= -\left(\exp(-x) - \sum_{k=0}^{2n+1} (-1)^k \frac{x^k}{(k-1)!}\right) \\ &\le 0 \qquad \text{[Induction Assumption]} \end{aligned}$$

so that $f(x)$ is decreasing in $[0,\infty)$. Hence $f(x) \le f(0) = 0$, that is

$$\exp(-x) \le \sum_{k=0}^{2(n+1)} (-1)^k \frac{x^k}{k!}$$

for all $x \ge 0$. A similar argument shows that

$$\exp(-x) \ge \sum_{k=0}^{2(n+1)+1} (-1)^k \frac{x^k}{k!}$$

for all $x \ge 0$. ■

Proposition 2.2.17 *For $x > 0$ and $a \in \mathbb{R}$, define $x^a = \exp(a \ln x)$. Then (i) $x^0 = 1$; (ii) $x^1 = x$; (iii) $x^{a+b} = x^a x^b$ (iv) $x^a y^a = (xy)^a$; (v) $(x^a)^b = x^{ab}$; (vi) $\frac{d}{dx} x^a = a x^{a-1}$. [If n is positive integer, then x^n coincides with the product $x \cdots x$ (n times) as you expect].*

Proof. [Careful arguments based on the definition of x^a are required here.]

(i) By definition for $x > 0$

$$x^0 = \exp(0 \ln x) = \exp 0 = 1.$$

[But be careful, 0^0 is not defined]

(ii) Similarly $x^1 = \exp(\ln x) = x$ for $x > 0$ as ln is the inverse of $\exp : (-\infty,\infty) \to (0,\infty)$.

(iii) By definition for $x > 0$ we have

$$\begin{aligned} x^{a+b} &= \exp((a+b)\ln x) = \exp(a \ln x + b \ln x) \\ &= \exp(a \ln x) \exp(b \ln x) \\ &= x^a x^b. \end{aligned}$$

(iv) Since $\exp(A+B) = \exp A \exp B$, by setting $A = \ln x$ and $B = \ln y$ where $x, y > 0$, we have

$$\exp(\ln x + \ln y) = xy$$

which implies that

$$\ln(xy) = \ln x + \ln y$$

for all $x, y > 0$. Hence

$$\begin{aligned} x^a y^a &= \exp(a \ln x) \exp(a \ln y) = \exp(a(\ln x + \ln y)) \\ &= \exp(a \ln(xy)) = (xy)^a \end{aligned}$$

for any $x, y > 0$.

(iv) For $x > 0$

$$\begin{aligned} (x^a)^b &= (\exp(a \ln x))^b = \exp[b \ln(\exp(a \ln x))] \\ &= \exp(ba \ln x) \\ &= x^{ab}. \end{aligned}$$

(v) According to chain rule, $x^a = \exp(a \ln x)$ is differentiable on $(0, \infty)$, and

$$\begin{aligned} \frac{d}{dx} x^a &= \exp'(a \ln x)(a \ln x)' \\ &= \exp(a \ln x) a \frac{1}{x} \\ &= a x^a \frac{1}{x}. \end{aligned}$$

Since

$$x^{-1} = \exp(-\ln x) = \frac{1}{\exp(\ln x)} = \frac{1}{x}$$

therefore

$$\frac{d}{dx} x^a = a x^a x^{-1} = a x^{a-1}$$

for $x > 0$, as we have expected. ■

2.2.3 π and trigonometric functions

As an application of Mean Value Theorem and Intermediate Value Theorem, we study the exponential function and the trigonometric functions, the approach is credited to the genius *Gauss*.

MVT and IVT allow us to identify the minimal positive period 2π of sin, cos functions, and to derive their important properties.

Good references on this topic are:

1) L. V. Ahlfors: *Complex Analysis*. Chapter 2 Section 3.

2) W. Rudin: *Real and Complex Analysis*. Prologue, pages 1-4.

We have introduced three functions $\exp z$, $\sin z$ and $\cos z$ by means of power series, namely

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \text{and } \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

for $z \in \mathbb{C}$. Let $e = \exp 1$ so $\ln e = 1$ by definition of ln being the inverse function of exp on $\mathbb{R}$. For $x \in \mathbb{R}$, according to the definition of power e^x, we have

$$e^x = \exp(x \ln e) = \exp x.$$

This is the reason why $\exp z$ is also denoted by e^z. From their definitions, $\exp x$, $\sin x$ and $\cos x$ are real for every $x \in \mathbb{R}$. $\cos 0 = 1$, $\sin 0 = 0$, $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$. Moreover

$$\frac{d}{dz}\sin z = \cos z \text{ and } \frac{d}{dz}\cos z = -\sin z.$$

Lemma 2.2.18 *1) We have*

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

and

$$\sin(x+y) = \sin x \cos y + \sin y \cos x.$$

for $x, y \in \mathbb{R}$. [In fact the addition formulas hold for complex numbers x and y too, will do in your A2 paper].

2) $\sin^2 x + \cos^2 x = 1$ for $x \in \mathbb{R}$. [The identity holds well for complex x, A2 paper].

3) $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for every $x \in \mathbb{R}$.

Proof. To show 1) for every $c \in \mathbb{R}$, a fixed number but arbitrary, we apply Identity Theorem to the function

$$f(x) = \cos x \cos(c-x) - \sin x \sin(c-x)$$

for $x \in \mathbb{R}$. Then

$$\begin{aligned} f'(x) &= -\sin x \cos(c-x) + \cos x \sin(c-x) \\ &\quad - \cos x \sin(c-x) + \sin x \cos(c-x) \\ &= 0 \end{aligned}$$

so that, according to Identity Theorem, f is constant on $\mathbb{R}$. Hence $f(x) = f(c)$ for every $x \in \mathbb{R}$. Since $\cos 0 = 1$, so that $f(c) = \cos c$, and therefore

$$\cos c = \cos x \cos(c-x) - \sin x \sin(c-x)$$

for any c and x. Setting $c = x+y$ we obtain the first identity. To obtain the second one, we differentiate both sides of the cos identity in x for any fixed y, and obtain that

$$-\sin(x+y) = -\sin x \cos y - \cos x \sin y$$

which gives the addition formula for sin.

2) Since $\cos 0 = 1$, by setting $y = -x$ in the cos identity and using the facts that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, one obtains the well known equality.

3) follows directly from 2) as $\sin x$ and $\cos x$ are real numbers for any real x. ■

Lemma 2.2.19 *The following inequalities hold for $x \geq 0$: (i) $\sin x \leq x$; (ii) $\cos x \geq 1 - \frac{x^2}{2!}$; (iii) $\sin x \geq x - \frac{x^3}{3!}$ and (iv) $\cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.*

Proof. (i) Consider $f(x) = \sin x - x$. Then $f(0) = 0$ and $f'(x) = \cos x - 1 \leq 0$, so that f is decreasing on $[0, \infty)$, and therefore $f(x) \leq f(0)$ for every $x \geq 0$, that is, $\sin x \leq x$ for $x \geq 0$.

To show (ii) we study function $f(x) = \cos x - 1 + \frac{x^2}{2!}$. Then $f'(x) = -\sin x + x \geq 0$ for $x \geq 0$ according to (i), so that f is increasing on $[0, \infty)$, which yields (ii).

The proofs of (iii) and (iv) are similar, so let us prove (i)-(iv) in one go. Consider function

$$h(x) = \cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!}$$

for $x \geq 0$. Then

$$h(0) = 0, \text{ and } h'(x) = -\sin x + x - \frac{x^3}{3!},$$

$$h'(0) = 0, \text{ and } h''(x) = -\cos x + 1 - \frac{x^2}{2!}$$

$$h''(0) = 0, \text{ and } h^{(3)}(x) = \sin x - x$$

and

$$h^{(3)}(0) = 0, \text{ and } h^{(4)}(x) = \cos x - 1.$$

Now, since $h^{(4)}(x) \leq 0$ for any $x \geq 0$, so that $h^{(3)}$ is decreasing and therefore $h^{(3)}(x) \leq h^{(3)}(0) = 0$ for $x \in [0, \infty)$. This in turn implies that $h^{''}$ is decreasing on $[0, \infty)$, so that $h^{''}(x) \leq h^{''}(0) = 0$ for $x \geq 0$. Hence h' is decreasing on $[0, \infty)$, so that $h'(x) \leq h'(0) = 0$ for $x \geq 0$, which implies that

$$-\sin x + x - \frac{x^3}{3!} \leq 0 \quad \text{for every } x \geq 0.$$

That is (iii). It follows then that h is decreasing on $[0, \infty)$, so that $h(x) \leq h(0) = 0$ for $x \geq 0$, which proves the inequality

$$\cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{for all } x \geq 0.$$

■

Lemma 2.2.20 $\cos 0 = 1$, $\cos 2 < 0$, *and* $\cos$ *is strictly decreasing on* $[0, 2]$ *and* $\sin x > 0$ *for all* $x \in (0, 2]$. *Therefore there is a unique* $\xi \in (0, 2)$ *such that* $\cos \xi = 0$ *and* $\sin \xi = 1$. Define $\pi = 2\xi$.

Proof. By definition $\cos 0 = 1$. By Lemma 2.2.19

$$\cos 2 \leq 1 - 2 + \frac{16}{4!} = -\frac{1}{3} < 0.$$

Since cos is continuous, according to IVT, there is a number $\xi \in (0, 2)$ such that $\cos \xi = 0$. We next show that cos is strictly decreasing on $[0, 2]$. In fact, by Lemma 2.2.19

$$\sin x \geq x - \frac{x^3}{3!} = x\left(1 - \frac{x^2}{3!}\right) = x\left(1 + \frac{x}{\sqrt{6}}\right)\left(1 - \frac{x}{\sqrt{6}}\right) > 0$$

for $x \in (0, \sqrt{6})$. In particular $\sin x > 0$ for $x \in (0, 2]$, and therefore

$$\cos' x = -\sin x < 0 \text{ for } x \in (0, 2],$$

which yields that cos is strictly decreasing on $[0, 2]$, so that cos is 1-1 on $[0, 2]$. Therefore ξ is the unique zero of cos on the interval $[0, 2]$. Since $\sin \xi \geq 0$ and $\cos^2 \xi + \sin^2 \xi = 1$, we must have $\sin \xi = 1$. ■

Lemma 2.2.21 $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$, $\cos \pi = -1$, $\sin \pi = 0$, $\cos \frac{3\pi}{2} = 0$, $\sin \frac{3\pi}{2} = -1$, $\cos(2\pi) = 1$ *and* $\sin(2\pi) = 0$. *Moreover* $\cos$ *and* $\sin$ *are periodic functions with their least positive period* 2π.

Proof. By Lemma 2.2.20, $\cos\frac{\pi}{2}=0$, $\frac{\pi}{2}\in[0,2]$, and $\sin\frac{\pi}{2}=1$. Hence

$$\cos\pi=\cos\frac{\pi}{2}\cos\frac{\pi}{2}-\sin\frac{\pi}{2}\sin\frac{\pi}{2}=-1$$

and it follows that $\sin\pi=0$. Now

$$\cos\frac{3\pi}{2}=\cos\frac{\pi}{2}\cos\pi-\sin\frac{\pi}{2}\sin\pi=0$$

and

$$\cos(2\pi)=\cos\pi\cos\pi-\sin\pi\sin\pi=1.$$

Similarly we may verify that $\sin\frac{3\pi}{2}=-1$ and $\sin(2\pi)=0$. Now using the addition formula again

$$\cos(x+2\pi)=\cos x\cos(2\pi)-\sin x\sin(2\pi)=\cos x.$$

Taking derivative both side we obtain that $\sin(x+2\pi)=\sin x$.

We next show that 2π is the least positive period of cos, and also sin.

Suppose there is a positive number $q>0$ such that $4q$ is a period, that is $\cos(4q+x)=\cos x$ for all x, then by differentiating the last equality we also $\sin(4q+x)=\sin x$, so $4q$ is a period of sin. Suppose $4q<2\pi$. Then $0<q<\frac{\pi}{2}$, so that $\cos q>0$ and $\sin q>0$ by the definition of $\frac{\pi}{2}$ above. On the other hand, by the addition angle identity

$$1=\cos 0=\cos(4q)=\cos^2(2q)-\sin^2(2q)=1-2\sin^2(2q)$$

which yields $\sin(2q)=0$. Using addition angle identity again

$$0=\sin(2q)=2\sin q\cos q>0$$

which is a contradiction. Therefore 2π is the minimal positive period. ■

Lemma 2.2.22 $\sin x=0$ *if and only if* $x=k\pi$ *for* $k\in\mathbb{Z}$. *Similarly* $\cos x=0$ *if and only if* $x=k\pi+\frac{\pi}{2}$ *for* $k\in\mathbb{Z}$.

Proof. By angle addition formula

$$\sin\pi=2\sin\frac{\pi}{2}\cos\frac{\pi}{2}=0$$

and using the periodicity we deduce that $\sin(k\pi)=0$ for every $k\in\mathbb{Z}$, and consequently $\cos(k\pi)=\pm1$.

Suppose $\sin x=0$, then we may write $x=k\pi+q$ for some $k\in\mathbb{Z}$ and $q\in[0,\pi)$. Since

$$\begin{aligned}0=\sin x=\sin(k\pi+q)&=\sin(k\pi)\cos q+\sin q\cos(k\pi)\\&=\sin q\cos(k\pi),\end{aligned}$$

so that $\sin q=0$. Hence $0=\sin\frac{q}{2}\cos\frac{q}{2}$. If $q>0$, then $0<\frac{q}{2}<\frac{\pi}{2}<2$, so that $\sin\frac{q}{2}>0$ and $\cos\frac{q}{2}>0$, which is a contradiction. Hence $q=0$ and the proof is complete for the first part.

Since

$$\begin{aligned}\cos x&=\cos(x-\frac{\pi}{2})\cos\frac{\pi}{2}-\sin(x-\frac{\pi}{2})\sin\frac{\pi}{2}\\&=-\sin(x-\frac{\pi}{2}),\end{aligned}$$

$\cos x=0$ if and only if $\sin(x-\frac{\pi}{2})=0$, and therefore if and only if $x-\frac{\pi}{2}=k\pi$ for $k\in\mathbb{Z}$. ■

Question. Is $\exp z$ for $z\in\mathbb{C}$ a periodic function? If so, what is its period?

Remark 2.2.23 *In general, we have*

$$\cos x \le \sum_{k=0}^{2n} (-1)^k \frac{x^{2k}}{(2k)!},\ \cos x \ge \sum_{k=0}^{2n-1} (-1)^k \frac{x^{2k}}{(2k)!}$$

for $x \in (-\infty, \infty)$*, and*

$$\sin x \le \sum_{k=0}^{2n} (-1)^k \frac{x^{2k+1}}{(2k+1)!},\ \sin x \ge \sum_{k=0}^{2n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

for all $x \in [0, \infty)$*. These inequalities can be proven by using induction (in one induction argument for all 4 inequalities together).*

Example 2.2.24 *(*Several important inequalities*) Let* $0 < x < \frac{\pi}{2}$*. Then*
1) $\sin x < x < \tan x$ *; [which yields that* $\cos x < \frac{\sin x}{x} < 1$*, so that* $\lim_{x\to 0} \frac{\sin x}{x} = 1$*.]*
2) $\frac{2}{\pi} < \frac{\sin x}{x} < 1$*. [1) + 2) implies that* $\max\{\cos x, \frac{2}{\pi}\} < \frac{\sin x}{x} < 1$ *for* $x \in (0, \pi/2)$*].*

Proof. To prove the first inequality, consider $f(x) = \tan x - x$, $x \in [0, \pi/2)$. Then f is differentiable on $(0, \pi/2)$ and

$$f'(x) = \frac{1}{\cos^2 x} - 1 > 0 \quad \forall x \in (0, \pi/2)\ .$$

f is strictly increasing [Apply MVT to any $[x_1, x_2]$, where $x_i \in (0, \pi/2)$]. Thus $f(x) > f(0)$ for any $x \in (0, \pi/2)$ which yields the inequality 1).

2) If $g(x) = x - \sin x$ then $g'(x) = 1 - \cos x > 0$ for any $x \in (0, \pi/2)$. Hence g is strictly increasing on $[0, \pi/2]$, so that $\sin x < x$ for all $x \in (0, \pi/2)$. Now consider

$$h(x) = \frac{\sin x}{x} \qquad x \in (0, \pi/2]\ .$$

Then

$$h'(x) = \frac{\cos x(x - \tan x)}{x^2} < 0 \quad \forall x \in (0, \pi/2)$$

so that h is strictly decreasing, so that $g(x) > g(\pi/2)$ for any $x \in (0, \pi/2)$. ■

Example 2.2.25 *Show that*

$$\frac{t}{1+t} < \ln(1+t) < t \qquad \forall t > 0\,.$$

Proof. In fact, by applying MVT to ln on $[1, 1+t]$, we have

$$\begin{aligned} \ln(1+t) - \ln 1 &= \log'(\xi)(1+t-1) \\ &= \frac{t}{\xi} \end{aligned}$$

for some $\xi \in (1, 1+t)$. Since $1 < \xi < 1+t$, and $t > 0$, we have $\frac{t}{1+t} < \frac{t}{\xi} < t$. Therefore

$$\ln(1+t) = \ln(1+t) - \ln 1 = \frac{t}{\xi}$$

belongs to $(\frac{t}{1+t}, t)$. ■

Example 2.2.26 (Euler's constant) *Let*

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n.$$

Then $\lim_{n\to\infty} \gamma_n$ *exists, the limit is denoted by* γ. γ *is called the Euler constant.*

Proof. In MT, we have demonstrated that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent, and the partial sum $\sum_{k=1}^{n} \frac{1}{k}$, which is increasing in n, grows like $\ln n$. Equipped with MVT, we are now in a position to prove this statement.

Firstly we write

$$\ln n = (\ln n - \ln(n-1)) + \cdots + (\ln 2 - \ln 1)$$

so that

$$\gamma_n = \sum_{k=1}^{n-1} \left(\frac{1}{k} - (\ln(k+1) - \ln k) \right) + \frac{1}{n}.$$

Apply MVT to $\ln x$ on the interval $[k, k+1]$ for each $k = 1, 2, \cdots$. Since ln is differentiable on $[k, k+1]$, there is $\xi_k \in (k, k+1)$ such that

$$\frac{\ln(k+1) - \ln k}{k+1-k} = \frac{1}{\xi_k}$$

that is

$$\ln(k+1) - \ln k = \frac{1}{\xi_k}$$

for some $\xi_k \in (k, k+1)$. Therefore

$$\frac{1}{k} - (\ln(k+1) - \ln k) = \frac{1}{k} - \frac{1}{\xi_k} = \frac{\xi_k - k}{k\xi_k}$$

which yields that

$$0 < \frac{1}{k} - (\ln(k+1) - \ln k) < \frac{1}{k^2}$$

for $k = 1, 2, \cdots$. Since $\sum \frac{1}{k^2}$ is convergent, so by the comparison test for series,

$$\sum_{k=1}^{n-1} \left(\frac{1}{k} - (\ln(k+1) - \ln k) \right)$$

converges as $n \to \infty$. Since $\frac{1}{n} \to 0$ as $n \to \infty$, we may thus conclude, by AOL, that

$$\gamma_n = \sum_{k=1}^{n-1} \left(\frac{1}{k} - (\ln(k+1) - \ln k) \right) + \frac{1}{n}$$

converges as $n \to \infty$, that is $\lim_{n\to\infty} \gamma_n = \gamma$ exists. Moreover

$$0 < \gamma \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

which is however not a good estimate for the Euler constant γ. In fact $\gamma = 0.57721566490\cdots$. ■

Example 2.2.27 *(i) Suppose f is continuous in $[x_0, x_0+\delta]$ and differentiable in $(x_0, x_0+\delta)$ for some $\delta > 0$ and suppose $\lim_{x\downarrow x_0} f'(x)$ exists, then the right-derivative of f at x_0 exists and*

$$f'(x_0+) = \lim_{x\downarrow x_0} f'(x) .$$

[Recall that, here, $f'(x_0+)$ does not mean the right-hand limit of the derivative function f', but the limit

$$\lim_{x\downarrow x_0} \frac{f(x)-f(x_0)}{x-x_0} .$$

It shows that, if the right-hand limit of f' exists, i.e. $\lim_{x\downarrow x_0} f'(x)$ exists, then $\lim_{x\downarrow x_0} f'(x)$ coincides with $f'(x_0+)$, which justify the abuse of notations]. In particular, if $\lim_{x\to x_0} f'(x)$ exists, then f is differentiable at x_0, and $f'(x_0) = \lim_{x\to x_0} f'(x)$ [However, f can be differentiable at x_0, but $\lim_{x\to x_0} f'(x)$ may not exist. Example?]

(ii) Show that $f(x) = x\arcsin x + \sqrt{1-x^2}$ is differentiable on $[-1,1]$. [$\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the inverse of $\sin$, and $\sqrt{x}$ is the inverse of x^2 in $[0,\infty)$].

Proof. (i) Indeed, for any $x \in (x_0, x_0+\delta)$ we apply the MVT to f on $[x_0, x]$

$$f(x) - f(x_0) = f'(\xi_x)(x-x_0) .$$

Clearly, as $x \to x_0$, $\xi_x \to x_0$ so that $\lim_{x\downarrow x_0} f'(\xi_x) = \lim_{x\downarrow x_0} f'(x)$, and therefore

$$\begin{aligned} f'(x_0+) &= \lim_{x\downarrow x_0} \frac{f(x)-f(x_0)}{x-x_0} \\ &= \lim_{x\downarrow x_0} f'(\xi_x) = \lim_{x\downarrow x_0} f'(x) . \end{aligned}$$

(ii) First let us compute the derivative of arcsin on $(-1,1)$. According to Theorem 2.1.9

$$\begin{aligned} \frac{d}{dx}\arcsin x &= \frac{1}{\sin'(\arcsin x)} \\ &= \frac{1}{\cos(\arcsin x)} . \end{aligned}$$

Since sin is increasing in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so its inverse arcsin is continuous on $[-1,1]$ with values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. In particular $\cos(\arcsin x) \geq 0$. Since $\cos^2 + \sin^2 = 1$, so that

$$\begin{aligned} \cos(\arcsin x) &= \sqrt{1-(\sin(\arcsin x))^2} \\ &= \sqrt{1-x^2} . \end{aligned}$$

Therefore [Theorem 2.1.9]

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \forall x \in (-1,1) .$$

[Exercise: Carefully work out the derivative $\frac{d}{dx}\sqrt{x}$ via Theorem 2.1.9]. Hence

$$f'(x) = \arcsin x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

on $(-1,1)$. However $\lim_{x\to\pm 1} f'(x) = \pm\frac{\pi}{2}$ exist, so that $f'(-1+) = -\frac{\pi}{2}$ and $f'(1-) = \frac{\pi}{2}$. f is differentiable in $[-1,1]$. ∎

2.3 L'Hôpital rule

[Theorems of G. F. de l'Hospitales, French mathematician, and Joh. Bernoulli] In this section, all functions are real-valued functions. We state several versions of the technique under the name of L'Hôpital rule.

Theorem 2.3.1 *Suppose f, g are differentiable on $(a, a+\delta)$ (for some $\delta > 0$), and $\lim_{x\downarrow a} f(x) = \lim_{x\downarrow a} g(a) = 0$, then*

$$\lim_{x\downarrow a} \frac{f(x)}{g(x)} = \lim_{x\downarrow a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right-hand side exists.

Proof. Since f, g are differentiable so they are continuous on $(a, a+\delta)$. Let us define $f(a) = g(a) = 0$. Then f, g are continuous on $[a, a+\delta)$. Let

$$l = \lim_{x\downarrow a} \frac{f'(x)}{g'(x)}$$

which exists by the assumption. Therefore for every $\varepsilon > 0$ there is $0 < \delta_1 \leq \delta$ such that

$$\left| \frac{f'(x)}{g'(x)} - l \right| < \varepsilon \quad \text{for every } x \in (a, a+\delta_1).$$

On the other hand, for every $x \in (a, a+\delta_1)$, by applying Cauchy's Mean Value Theorem to f, g on $[a, x]$, there is $\xi_x \in (a, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi_x)}{g'(\xi_x)}.$$

Since $\xi_x \in (a, x) \subseteq (a, a+\delta_1)$,

$$\left| \frac{f(x)}{g(x)} - l \right| = \left| \frac{f'(\xi_x)}{g'(\xi_x)} - l \right| < \varepsilon.$$

By definition we have

$$\lim_{x\downarrow a} \frac{f(x)}{g(x)} = l.$$

■

Similarly

Theorem 2.3.2 *Suppose f, g are differentiable on $(a-\delta, a)$ (for some $\delta > 0$), and $\lim_{x\uparrow a} f(x) = \lim_{x\uparrow a} g(x) = 0$, then*

$$\lim_{x\uparrow a} \frac{f(x)}{g(x)} = \lim_{x\uparrow a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right-hand side exists.

Theorem 2.3.3 ***(L'Hôpital Rule)*** *Suppose f and g are continuous on $(a-\delta, a+\delta)$ (for some $\delta > 0$) and differentiable on $(a-\delta, a+\delta) \setminus \{a\}$, $f(a) = g(a) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right-hand side exists.

Example 2.3.4 *Show that (i)* $\lim_{x\to0}\frac{\sin x}{x}=1$*; (ii)* $\lim_{x\to0}\frac{1-\cos x}{x^2}=\frac{1}{2}$*; (iii)* $\lim_{x\to0}\frac{\ln(1+x)}{x}=1$*; (iv)* $\lim_{x\to0}(1+x)^{\frac{1}{x}}=e$ *; (v) Find* $\lim_{x\to0}\frac{e^x-e^{-x}-2x}{x-\sin x}$.

Solutions. (i) This is a $\frac{0}{0}$ type limit, so we may apply L'Hoptial's rule to evaluate its limit. $\sin x$ and x are differentiable everywhere, both tend to zero as $x\to0$, and

$$\lim_{x\to0}\frac{\sin' x}{x'}=\lim_{x\to0}\cos x=1$$

exists, so that

$$\lim_{x\to0}\frac{\sin x}{x}=\lim_{x\to0}\frac{\sin' x}{x'}=1 \qquad \text{[L'Hôpital Rule].}$$

(ii) This is again a $\frac{0}{0}$ type limit. We have

$$\begin{aligned}\lim_{x\to0}\frac{1-\cos x}{x^2} &= \lim_{x\to0}\frac{\sin x}{2x} \quad \text{[provided this limit exists]}\\ &= \lim_{x\to0}\frac{\cos x}{2} \quad \text{[provided this limit exists]}\\ &= \frac{1}{2}\,.\end{aligned}$$

Here we have used L'Hôpital Rule twice.

(iii) ($\frac{0}{0}$ type) Attempt to apply L'Hôpital Rule. $\ln(1+x)$ is continuous and differentiable for x near 0, and $\log(1+0)=0$, so that we attempt to evaluate the limit by using L'Hôpital Rule.

$$\begin{aligned}\lim_{x\to0}\frac{\ln(1+x)}{x} &= \lim_{x\to0}\frac{\ln'(1+x)}{x'} \quad \text{[provided this limit exists]}\\ &= \lim_{x\to0}\frac{1}{1+x}=1\,.\end{aligned}$$

(iv) (1^∞ type $\Longrightarrow \exp(\frac{0}{0})$ type, then use the continuity of exp) According the definition a^p,

$$(1+x)^{\frac{1}{x}}=\exp\left(\frac{1}{x}\ln(1+x)\right)$$

Since exp is continuous on $\mathbb{R}$, so that [By (iii)]

$$\begin{aligned}\lim_{x\to0}(1+x)^{\frac{1}{x}} &= \lim_{x\to0}\exp\left(\frac{\ln(1+x)}{x}\right)\\ &= \exp\left(\lim_{x\to0}\frac{\ln(1+x)}{x}\right) \quad \text{[exp is continuous at 1]}\\ &= \exp 1=e\,.\end{aligned}$$

Example 2.3.5 $\lim_{x\to0}(1+ax)^{\frac{1}{x}}=\exp a$ *for any* $a\in\mathbb{R}$. *In particular*

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n=\exp a\,.$$

If $a=0$, then $\lim_{x\to 0}(1+ax)^{\frac{1}{x}}=\lim_{x\to 0}1=1=\exp 0$. If $a\neq 0$, then

$$\begin{aligned}
\lim_{x\to 0}(1+ax)^{\frac{1}{x}} &= \lim_{x\to 0}\exp\left(\frac{1}{x}\ln(1+ax)\right) && \text{[By definition]}\\
&= \exp\left(\lim_{x\to 0}\frac{1}{x}\ln(1+ax)\right) && \text{[Continuity of exp]}\\
&= \exp\left(\lim_{x\to 0}\frac{a}{1+ax}\right) && \text{[if the limit exists, L'Hôpital Rule]}\\
&= \exp a\,.
\end{aligned}$$

Theorem 2.3.6 *If $f,g:(a,a+\delta)\to\mathbb{R}$ are differentiable, where $\delta>0$, $g'(x)\neq 0$, $f(x)\to\infty$, $g(x)\to\infty$ as $x\downarrow a$, and $\lim_{x\downarrow a}\frac{f'(x)}{g'(x)}$ exists (or ∞ or $-\infty$), then*

$$\lim_{x\downarrow a}\frac{f(x)}{g(x)}=\lim_{x\downarrow a}\frac{f'(x)}{g'(x)}\,.$$

Proof. Suppose that $\lim_{x\downarrow a}\frac{f'(x)}{g'(x)}=K$ is finite [Otherwise we may consider $\lim_{x\downarrow a}\frac{g(x)}{f(x)}$ instead]. We may assume that $g'\neq 0$ [That $g'\neq 0$ near a is implied in the assumption that $\lim_{x\downarrow a}\frac{f'(x)}{g'(x)}$ exists]. $\forall\varepsilon>0$ there is a number δ_1 ($<\delta$) such that

$$\left|\frac{f'(x)}{g'(x)}-K\right|<\frac{\varepsilon}{2}\qquad\forall x\in(a,a+\delta_1)\,. \tag{2.3.1}$$

Now we choose a number c in $(a,a+\delta_1)$ [c is fixed from now on]. For any $x\in(a,c)$ we apply Cauchy's MVT to f, g on $[x,c]$: there is a number $\xi_x\in(x,c)$ such that

$$\frac{f(c)-f(x)}{g(c)-g(x)}=\frac{f'(\xi_x)}{g'(\xi_x)}\,.$$

Since $\xi_x\in(x,c)\subset(a,a+\delta_1)$, by (2.3.1)

$$\left|\frac{f(x)-f(c)}{g(x)-g(c)}-K\right|=\left|\frac{f'(\xi_x)}{g'(\xi_x)}-K\right|<\frac{\varepsilon}{2}\qquad\forall x\in(a,c)\,. \tag{2.3.2}$$

[However, we cannot conclude from (2.3.2) that $\frac{f(x)-f(c)}{g(x)-g(c)}\to K$ as $x\downarrow a$ (although it does !!), as there is no guarantee that ξ_x will tend to a as $x\downarrow a$]. Now we consider

$$\begin{aligned}
\frac{f(x)}{g(x)}-K &= \frac{f(x)-f(c)+f(c)}{g(x)}-K\\
&= \frac{f(c)}{g(x)}+\frac{f(x)-f(c)}{g(x)-g(c)}\frac{g(x)-g(c)}{g(x)}-K\\
&= \frac{f(c)}{g(x)}+\frac{f(x)-f(c)}{g(x)-g(c)}\left(1-\frac{g(c)}{g(x)}\right)-K\\
&= \frac{f(c)}{g(x)}+\left(\frac{f(x)-f(c)}{g(x)-g(c)}-K\right)\left(1-\frac{g(c)}{g(x)}\right)+K\left(1-\frac{g(c)}{g(x)}\right)-K\\
&= \frac{f(c)}{g(x)}+\left(\frac{f(x)-f(c)}{g(x)-g(c)}-K\right)\left(1-\frac{g(c)}{g(x)}\right)-\frac{Kg(c)}{g(x)}\\
&= \frac{f(c)-Kg(c)}{g(x)}+\left(1-\frac{g(c)}{g(x)}\right)\left(\frac{f(x)-f(c)}{g(x)-g(c)}-K\right)
\end{aligned}$$

[Why we are interested in this? Explained in the lecture], so that

$$\begin{aligned}\left|\frac{f(x)}{g(x)}-K\right| &\leq \left|\frac{f(c)-Kg(c)}{g(x)}\right|+\left|1-\frac{g(c)}{g(x)}\right|\left|\frac{f(x)-f(c)}{g(x)-g(c)}-K\right| \\ &\leq \left|\frac{f(c)-Kg(c)}{g(x)}\right|+\frac{\varepsilon}{2}\left|1-\frac{g(c)}{g(x)}\right|\end{aligned}$$

for any $x \in (a,c)$. Since $g(x) \to \infty$ as $x \downarrow a$ so that

$$\lim_{x\downarrow a}\frac{f(c)-Kg(c)}{g(x)}=0$$

and

$$\lim_{x\downarrow a}\left(1-\frac{g(c)}{g(x)}\right)=1 .$$

[Algebra of limits]. Thus there is $\delta_2 > 0$ [and $\delta_2 < \min\{\delta_1, c-a\}$] such that

$$\left|1-\frac{g(c)}{g(x)}\right|<\frac{4}{3} \quad \text{and} \quad \left|\frac{f(c)-Kg(c)}{g(x)}\right|<\frac{\varepsilon}{3}$$

for every $x \in (a, a+\delta_2)$. Therefore

$$\left|\frac{f(x)}{g(x)}-K\right|<\frac{\varepsilon}{3}+\frac{4}{3}\frac{\varepsilon}{2}=\varepsilon \qquad \forall x \in (a,a+\delta_2) .$$

By definition, $\lim_{x\downarrow a}\frac{f(x)}{g(x)}=K$. ■

Theorem 2.3.7 *Suppose $f,g:(a,\infty)\to\mathbb{R}$ are continuous and differentiable, with $f(x)\to 0$ and $g(x)\to 0$ as $x\to\infty$. If $g'(x)\neq 0$ on (a,∞) and $\frac{f'(x)}{g'(x)}\to l$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=l$.*

Proof. Apply **L'Hôpital Rule** to functions $F(x)=f(\frac{1}{x})$ and $G(x)=g(\frac{1}{x})$. ■

Example 2.3.8 $\lim_{x\to\infty}\frac{\ln x}{x^\mu}=0$ *[$\frac{\infty}{\infty}$ type] and* $\lim_{x\to\infty}\frac{x^\mu}{e^x}=0$ *[$\frac{\infty}{\infty}$ type] for any $\mu>0$.*

Let $g(x)=x^\mu=\exp(\mu\ln x)$. Then $g'(x)=\mu x^{\mu-1}$. By L'Hôpital rule

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln x}{x^\mu} &= \lim_{x\to\infty}\frac{\frac{1}{x}}{\mu x^{\mu-1}} \quad \text{[provided this limit exists]} \\ &= \lim_{x\to\infty}\frac{1}{\mu x^\mu}=0 .\end{aligned}$$

Example 2.3.9 *For any $\mu>0$, $\lim_{x\downarrow 0}x^\mu\ln x=0$. [$0\cdot\infty$ type $\Longrightarrow\frac{\infty}{\infty}$ type]*

Again use L'Hôpital Rule

$$\begin{aligned}\lim_{x\downarrow 0}x^\mu\ln x &= \lim_{x\downarrow 0}\frac{\ln x}{x^{-\mu}} \\ &= \lim_{x\downarrow 0}\frac{\ln' x}{(x^{-\mu})'} \qquad \text{[if this limit exists]} \\ &= \lim_{x\downarrow 0}\frac{\frac{1}{x}}{-\mu x^{-\mu-1}}=\lim_{x\downarrow 0}\frac{x^\mu}{-\mu}=0 .\end{aligned}$$

Example 2.3.10 *Show that*

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} = \frac{1}{\sqrt[3]{e}} .$$

[Idea: first turn 1^∞ type limits into $\exp\left(\frac{0}{0}\text{ type}\right)$ limits, then use the continuity of exp] Since

$$f(x) = \left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$$

is even function, so that we only need to show that $\lim_{x\downarrow 0} f(x) = \frac{1}{\sqrt[3]{e}}$. According to definition

$$\begin{aligned} f(x) &= \exp\left(\frac{1}{1-\cos x}\ln\frac{\sin x}{x}\right) \\ &= \exp\left(\frac{\ln\sin x - \ln x}{1-\cos x}\right) . \end{aligned}$$

By L'Hôpital Rule,

$$\begin{aligned} \lim_{x\downarrow 0}\frac{\ln\sin x - \ln x}{1-\cos x} &= \lim_{x\downarrow 0}\frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{\sin x} \qquad \text{[provided it exists]} \\ &= \lim_{x\downarrow 0}\frac{x\cos x - \sin x}{x\sin^2 x} \\ &= \lim_{x\downarrow 0}\frac{\cos x - x\sin x - \cos x}{\sin^2 x + 2x\sin x\cos x} \qquad \text{[if exists, use L'Hôpital again]} \\ &= -\lim_{x\downarrow 0}\frac{x}{\sin x + 2x\cos x} \\ &= -\lim_{x\downarrow 0}\frac{1}{\cos x + 2\cos x - 2x\sin x} \\ &= -\frac{1}{3} . \end{aligned}$$

Since exp is continuous at $-\frac{1}{3}$, so that

$$\begin{aligned} \lim_{x\downarrow 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} &= \lim_{x\downarrow 0}\exp\left(\frac{\ln\sin x - \ln x}{1-\cos x}\right) \\ &= \exp\left(\lim_{x\downarrow 0}\frac{\ln\sin x - \ln x}{1-\cos x}\right) \qquad \text{[by continuity of exp]} \\ &= \exp\left(-\frac{1}{3}\right) . \end{aligned}$$

There is a discrete version of the L'Hôpital Rule, which was first discovered by O. Stolz.

Theorem 2.3.11 (O. Stolz) *Suppose (x_n) and (y_n) are two sequences of real numbers such that*
(i) $y_n \to \infty$ as $n \to \infty$,
(ii) (y_n) is a strictly increasing sequence (for large n), and
(iii) the limit

$$\lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

exists or tends to ∞ or $-\infty$. Then

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}}.$$

[The case that $y_n = n$ was proved by A. L. Cauchy].

Proof. The proof is similar to the proof of Theorem 2.3.6. Consider the case that $l = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$ is a number. Then for every $\varepsilon > 0$ there is N such that for $n > N$ we have

$$\left| \frac{x_n - x_{n-1}}{y_n - y_{n-1}} - l \right| < \frac{\varepsilon}{2}.$$

Since (y_n) is strictly increasing eventually, so we can choose N big enough such that so that $y_k - y_{k-1} > 0$ for all $k > N$ and therefore

$$-\frac{\varepsilon}{2}(y_k - y_{k-1}) < x_k - y_{k-1} - l(y_k - y_{k-1}) < \frac{\varepsilon}{2}(y_k - y_{k-1}).$$

Adding these inequalities over $k = N+1, \cdots, n$, where $n > N$, we obtain that

$$-\frac{\varepsilon}{2}(y_n - y_N) < x_n - y_N - l(y_n - y_N) < \frac{\varepsilon}{2}(y_n - y_N)$$

which can be written as, since $y_n - y_N > 0$

$$\left| \frac{x_n - x_N}{y_n - y_N} - l \right| < \frac{\varepsilon}{2}$$

for all $n > N$. Next we use the identity (similar to that in the proof of Theorem 2.3.6)

$$\frac{x_n}{y_n} - l = \frac{x_N - l y_N}{y_n} + \left(1 - \frac{y_N}{y_n}\right)\left(\frac{x_n - x_N}{y_n - y_N} - l\right)$$

so that

$$\left| \frac{x_n}{y_n} - l \right| < \left| \frac{x_N - l y_N}{y_n} \right| + \frac{\varepsilon}{2}$$

for every $n > N$. Since $y_n \to \infty$ so that

$$\frac{x_N - l y_N}{y_n} \to 0 \quad \text{as } n \to \infty.$$

Therefore there is $N_1 > N$ such that

$$\left| \frac{x_N - l y_N}{y_n} \right| < \frac{\varepsilon}{2} \quad \text{for } n > N_1$$

and therefore

$$\left| \frac{x_n}{y_n} - l \right| < \left| \frac{x_N - l y_N}{y_n} \right| + \frac{\varepsilon}{2} < \varepsilon$$

for every $n > N_1$. By definition

$$\lim_{n\to\infty} \frac{x_n}{y_n} = l = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

and the proof is complete. ■

As as example, if k is a positive integer, then we can show (Exercise) by Stolz's theorem that

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1}.$$

2.4 Taylor's formula

If f is a function defined on $[a,b]$ (where $a<b$) which has (right-hand) derivatives $f^{(k)}(a)$ at a, where $k=0,1,\cdots,n-1$ ($n\geq 1$ is an integer, with convention that $f^{(0)}=f$), then we may form a polynomial of degree $n-1$:

$$P_{n-1}(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{(n-1)}.$$

$P_{n-1}(x)$ is the unique polynomial of degree $n-1$ whose derivatives at a up to order $n-1$ agree with those of f at a. That is, $P_{n-1}^{(k)}(a)=f^{(k)}(a)$ for all $k\leq n-1$. Here are some examples:

$$\begin{aligned} P_0(x) &= f(a) && \text{[a constant function];} \\ P_1(x) &= f(a)+f'(a)(x-a) && \text{[which is the linear approximation of } f \text{ near } a\text{];} \\ P_2(x) &= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2 && \text{[quadratic approximation about } a\text{];} \\ &\cdots . \end{aligned}$$

Let

$$\begin{aligned} E_n(x,a) &= f(x)-P_{n-1}(x) \\ &= f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned} \tag{2.4.1}$$

be the error between $f(x)$ and $P_{n-1}(x)$.

If f has derivatives at a of any order, then we may form a power series

$$\begin{aligned} P(x) &= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n, \end{aligned} \tag{2.4.2}$$

which is called the *Taylor expansion* of f at a. The following lemma is obvious.

Lemma 2.4.1 *Let $f:[a,b]\to\mathbb{R}$ be differentiable up to any order, i.e. $f^{(n)}(a)$ exists for every n, R be the convergence radius of the Taylor expansion (2.4.2), and $x\in[a,b]$. Then*

$$f(x)=P(x)$$

if and only if $E_n(a,x)\to 0$ as $n\to\infty$. In this case, we must have $|x-a|\leq R$.

It is therefore quite important to derive a useful formula for the error $E_n(a,x)$, which is achieved in the following Taylor's theorem.

Theorem 2.4.2 ***(Taylor's Theorem)*** *Let $f:[a,b]\to\mathbb{R}$ where $b>a$ and $n\in\mathbb{N}$. Suppose $f^{(k)}(x)$ exist for every $x\in[a,b]$ and $f^{(k)}$ are continuous on $[a,b]$ for $k=0,\cdots,n-1$, and $f^{(n)}$ exists on (a,b). Then there is a number $\xi\in(a,b)$ such that*

$$f(b)=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(b-a)^k+\frac{f^{(n)}(\xi)}{n!}(b-a)^n.$$

That is, there is $\xi \in (a,b)$*, the error term*

$$E_n(a,b) = f(b) - P_{n-1}(b) = \frac{f^{(n)}(\xi)}{n!}(b-a)^n$$

(called the remainder in Lagrange form), where $P_{n-1}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

[There is a similar result for $f : [b,a] \to \mathbb{R}$, where *baswell* $< a$.]

Proof. We use the method of "varying a constant" to prove Taylor's theorem. Regard a in the definition of $P_{n-1}(b)$ as a variable. We therefore consider the following function

$$\begin{aligned} F(x) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(b-x)^k \\ &= f(x) + f'(x)(b-x) + \frac{f''(x)}{2!}(b-x)^2 + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1} \end{aligned}$$

for $x \in [a,b]$. Then $F(b) = f(b)$ and $F(a) = P_{n-1}(b)$. F is continuous on $[a,b]$, differentiable on (a,b), and

$$\begin{aligned} F'(x) &= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!}(b-x)^k + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!}(-1)k(b-x)^{k-1} \qquad \text{[Product Rule]} \\ &= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1} \\ &= \frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}. \end{aligned}$$

The idea of the proof is to apply Cauchy's Mean Value Theorem to F and G on $[a,b]$, where G, to be chosen later, is continuous on $[a,b]$, differentiable in (a,b) and $G'(x) \neq 0$ for $x \in (a,b)$. According to Cauchy's MVT, there is a number $\xi \in (a,b)$ such that

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(\xi)}{G'(\xi)} = \frac{\frac{f^{(n)}(\xi)}{(n-1)!}(b-\xi)^{n-1}}{G'(\xi)}.$$

Substituting $F(b)$ by $f(b)$, $F(a) = P_{n-1}(b)$ and rearranging the above equation we obtain

$$f(b) = P_{n-1}(b) + \frac{f^{(n)}(\xi)}{(n-1)!}\frac{(b-\xi)^{n-1}}{G'(\xi)}(G(b)-G(a)).$$

That is to say the error term can be written as

$$E_n(a,b) = \frac{f^{(n)}(\xi)}{(n-1)!}\frac{(b-\xi)^{n-1}}{G'(\xi)}(G(b)-G(a)).$$

This is a general form of the remainder in the Taylor's theorem, where $\xi \in (a,b)$ depends on the function G you have decided to use.

In particular, choosing $G(x) = (b-x)^n$, $G'(x) = -n(b-x)^{n-1}$ and $G(b) - G(a) = -(b-a)^n$, so that

$$E_n(a,b) = \frac{f^{(n)}(\xi)}{n!}(b-a)^n$$

which gives the Lagrange form, and

$$f(b) = P_{n-1}(b) + \frac{f^{(n)}(\xi)}{n!}(b-a)^n .$$

The proof is completed. ∎

Remark 2.4.3 *Choose a function G provided it is continuous in $[a,b]$, differentiable in (a,b), and $G' \neq 0$. According to Cauchy's MVT, there is a number ξ between a and b, such that*

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{\frac{f^{(n)}(\xi)}{(n-1)!}(b-\xi)^{n-1}}{G'(\xi)}$$

so that

$$f(b) = P_{n-1}(b) + \frac{f^{(n)}(\xi)}{(n-1)!}(b-\xi)^{n-1}\frac{G(b)-G(a)}{G'(\xi)}.$$

You may derive Taylor's Theorem with the remainder of different forms. For example, if we choose $G(x) = x-a$, then $\frac{G(b)-G(a)}{G'(\xi)} = b-a$. Thus

$$f(b) = P_{n-1}(b) + \frac{f^{(n)}(\xi)}{(n-1)!}(b-a)(b-\xi)^{n-1}$$

for some $\xi \in (a,b)$. You may for example try $G(x) = (x-a)^m$ for a power $m \geq 1$ to see what kind of Taylor's formula you can get. Of course, if you choose different G, you will have different ξ between a and b.

If we set $b-a=h$ so $b=a+h$ then Taylor's theorem may be stated as

$$f(a+h) = \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}h^k + \frac{f^{(n)}(a+\theta h)}{n!}h^n$$

where $\theta \in (0,1)$ depending on a, h and n in general, and on the function f as well of course. For example, the case that $n=2$, Taylor's theorem says that

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a+\theta h)h^2$$

as long as f' and f'' exist on $[a,a+h]$ or $[a+h,a]$ (if $h<0$), where $\theta \in (0,1)$ depending on h. This formula is a powerful tool to study the stationary points of f.

Given a function f which has derivatives of any order near a, so that you may write down the sequence of $f^{(k)}(a)$ and the power series [called the Taylor expansion of f at a]

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \tag{2.4.3}$$

The power series has convergence radius R, so that (2.4.3) defines a function g on $(a-R,a+R)$ [and in general, you have to use other methods to study the convergence at $a-R$ and $a+R$]. That is

$$g(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \forall x \in (a-R,a+R). \tag{2.4.4}$$

If it happens $R = 0$, then the Taylor expansion (2.4.4) is useless for the study of f. Otherwise, all derivatives of the Taylor expansion (2.4.4) g at a coincide with those of f at a: $g^{(n)}(a) = f^{(n)}(a)$ for any n [Differentiating a power series term by term again and again]. We therefore have high hope that $f(x) = g(x)$ for all $x \in (a-R, a+R)$. However, the Taylor expansion (2.4.4) relies only on the values of f in an arbitrary small neighborhood about a, say $(a-\varepsilon, a+\varepsilon)$ for whatever how small $\varepsilon > 0$, thus there is absolutely no reason why we should have $f(x) = g(x)$ if $x \neq a$, unless $f(x)$ can be determined by the values of f near a [and through the Taylor expansion of course!] This is the concept of *analytic functions* which will be studied in paper A2: Metric Spaces and Complex Analysis.

Example 2.4.4 *Let $f(x) = \exp(-\frac{1}{x^2})$ if $x \neq 0$ and $f(0) = 0$. Then f has derivatives of all order, and $f^{(n)}(0) = 0$ for all n. In fact, for $x \neq 0$, we have*

$$f^{(n)}(x) = Q_n(x)\exp(-\frac{1}{x^2})$$

for some polynomial Q_n of $\frac{1}{x}$, so that $\lim_{x\to 0} f^{(n)}(x) = 0$ for any n [L'Hôpital Rule]. Hence $f^{(n)}(0) = 0$ [Example 2.2.27]. Thus

$$f(0+h) \neq f(0) + f'(0)h + \cdots + \frac{f^{(n)}(0)}{n!}h^n + \cdots$$

for any $h \neq 0$, since the right-hand side is identically zero. The remainder $E_n(0,h) = f(0+h)$ for all n, which does not tend to 0 as $n \to \infty$ for any $h \neq 0$. Thus f is not analytic at 0.

Taylor's Theorem also provides us with an explicit error estimate between $f(x)$ and its Taylor approximation

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Corollary 2.4.5 *Let $f : [a,b] \to \mathbb{R}$ have continuous derivatives of all orders on $[a,b]$, and*

$$E_n = \frac{|b-a|^n}{n!} \sup_{\xi\in[a,b]} |f^{(n)}(\xi)|.$$

Then

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| \leq E_n \quad \textit{for } x \in [a,b] .$$

In particular if $E_n \to 0$ as $n \to \infty$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad \textit{uniformly on } [a,b] .$$

Theorem 2.4.6 *We have*

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} \quad \textit{for } x \in (-1,1] . \tag{2.4.5}$$

In particular

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} .$$

Proof. Consider $f(x) = \ln(1+x)$. Then $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$, so that

$$f(x) = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{x^k}{k} + E_n(x)$$

where, by applying Taylor's Theorem to f at $a = 0$,

$$E_n(x) = \frac{f^{(n)}(\xi_n)}{n!} x^n = (-1)^{n-1} \frac{1}{n} \left(\frac{x}{1+\xi_n} \right)^n$$

for some ξ_n between 0 and x [which depends on x and n]. Since

$$|E_n(x)| = \frac{1}{n} \left| \frac{x}{1+\xi_n} \right|^n .$$

and therefore, if $\left| \frac{x}{1+\xi_n} \right| \leq 1$ for all n, then $|E_n(x)| \leq \frac{1}{n}$, so for such x, $E_n(x) \to 0$.

Since the convergence radius of $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ is 1 [Ratio Test, Analysis I], we must have $|x| \leq 1$ in order that $E_n(x) \to 0$.

Now analyze the condition that $\left| \frac{x}{1+\xi_n} \right| \leq 1$ by keeping in mind the facts that $|x| \leq 1$, $|\xi_n| < 1$ and ξ_n is between 0 and x. The inequality $\left| \frac{x}{1+\xi_n} \right| \leq 1$ is thus equivalent to that $|x| \leq 1 + \xi_n$, that is, $\xi_n \geq |x| - 1$. The last inequality is true if $x \in [-\frac{1}{2}, 1]$. Thus

$$|E_n(x)| \leq \frac{1}{n} \to 0 \quad \text{for } x \in [-\frac{1}{2}, 1],$$

therefore

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \quad \text{for } x \in [-\frac{1}{2}, 1] \tag{2.4.6}$$

and the convergence is uniform on $[-\frac{1}{2}, 1]$.

However we are unable to prove that $E_n(x) \to 0$ for $x \in (-1, -\frac{1}{2})$ (it does tend to zero though!) by using the argument above, because we lack of enough information about ξ_n to make a conclusion. Therefore we employ a different approach. Let us consider the function given by the power series

$$P(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad \forall x \in (-1, 1] .$$

(which has a convergence radius 1). Then $P(x)$ is differentiable on $(-1, 1)$ and $P'(x)$ can be determined by differentiating the power series term by term [Theorem 2.1.15]:

$$\begin{aligned} P'(x) &= \sum_{n=1}^{\infty} (-1)^{n-1} n \frac{x^{n-1}}{n} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} \\ &= \frac{1}{1-(-x)} = \frac{1}{1+x} \qquad \forall x \in (-1, 1). \end{aligned}$$

On the other hand $f'(x) = \frac{d}{dx} \ln(1+x) = \frac{1}{1+x}$ on $(-1, 1)$, thus $f' = P'$ on $(-1, 1)$. By Identity Theorem

$$f(x) - P(x) = \text{constant} = f(0) - P(0) = 0$$

so that

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad \forall x\in(-1,1)\ .$$

Together with (2.4.6) we thus have

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad \forall x\in(-1,1]\ .$$

■

Theorem 2.4.7 *(The Binomial Expansion) Let p be a real number, and let $P(x)$ be the power series*

$$P(x)=1+px+\frac{p(p-1)}{2!}x^2+\cdots+\frac{p(p-1)\cdots(p-n+1)}{n!}x^n+\cdots$$

whose convergence radius $R=1$ unless $p=0$ or $p\in\mathbb{N}$. If $p\in\mathbb{N}$, $P(x)$ is a polynomial of degree p.

1) For any real number p we have

$$(1+x)^p=P(x) \quad \textit{for } x\in(-1,1).$$

2) If $p>0$ then

$$(1+x)^p=P(x) \quad \textit{for } x\in(-1,1].$$

Proof. If $p=0$ or $p\in\mathbb{N}$, $P(x)$ is reduced to a polynomial, 1) and 2) follow immediately from the ordinary binomial formula.

Let us first show that $P(x)$ is the Taylor expansion for the function $f(x)=(1+x)^p$ for $x>-1$ at $a=0$. In fact

$$\begin{aligned} f'(x) &= p(1+x)^{p-1} \quad ; \\ f''(x) &= p(p-1)(1+x)^{p-2}\ ; \\ & \cdots ; \\ f^{(k)}(x) &= p(p-1)\cdots(p-(k-1))(1+x)^{p-k} \end{aligned}$$

so $f^{(k)}(0)=p(p-1)\cdots(p-(k-1))$. Hence the Taylor expansion of $f(x)$ at $a=0$ is by definition given by

$$P(x)=\sum_{k=0}^{\infty}\frac{p(p-1)\cdots(p-(k-1))}{k!}x^k.$$

If $p\neq 0,1,2,\cdots$, then, by ratio test, the convergence radius $R=1$. For convenience, one may introduce notation

$$\binom{p}{k}=\frac{p(p-1)\cdots(p-(k-1))}{k!}$$

so that the Taylor's expansion of $(1+x)^p$ may be written as

$$P(x)=\sum_{k=0}^{\infty}\binom{p}{k}x^k$$

which is a polynomial of order p in the case that p is zero or a positive integer, as if $p\in\mathbb{N}$, then $\binom{p}{k}=0$ for $k>p$. Hence the case that $p\in\mathbb{N}$ is trivial, and reduces to the elementary Binomial expansion. In what follows, we may assume that $p\neq 0,1,2,\cdots$.

To prove 1), Taylor's Theorem is not needed in fact, and the Identity Theorem does the job.

Proof of part 1). Let us apply the Identity Theorem to $f(x) = (1+x)^p$ and its Taylor expansion $P(x)$ on the interval $(-1,1)$. Both are differentiable on $(-1,1)$, and, by chain rule,

$$f'(x) = \frac{d}{dx}\exp(p\ln(1+x)) = p(1+x)^p\frac{1}{1+x} = \frac{p}{1+x}f(x)$$

for $x > -1$, so that f satisfies the *differential equation*:

$$(1+x)f'(x) = pf(x)$$

where $-1 < x < 1$. One may expect that its Taylor expansion $P(x)$ should satisfies the same differential equation. In fact, we may write

$$P(x) = 1 + \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}x^n$$

which is a power series with convergence radius $R = 1$, so that $P(x)$ is differentiable on $(-1,1)$ and its derivative can be evaluated by differentiating it term by term:

$$P'(x) = \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{(n-1)!}x^{n-1}.$$

Hence

$$\begin{aligned}
(1+x)P'(x) &= \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{(n-1)!}(1+x)x^{n-1} \\
&= \sum_{n=0}^{\infty}\frac{p(p-1)\cdots(p-n)}{n!}x^n + \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}nx^n \\
&= p + \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}((p-n)+n)x^n \\
&= p + p\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}x^n \\
&= pP(x).
\end{aligned}$$

We apply the Identity Theorem to $h(x) = P(x)/f(x)$ on $(-1,1)$, which is differentiable as well as $f(x) \neq 0$ for $x \in (-1,1)$. Now

$$\begin{aligned}
h' &= \frac{P'f - Pf'}{f^2} \\
&= \frac{(1+x)P'f - (1+x)f'P}{(1+x)f^2} \\
&= \frac{pPf - pfP}{(1+x)f^2} = 0
\end{aligned}$$

so that, according to Identity Theorem, $P(x)/f(x)$ is constant in $(-1,1)$, and therefore

$$\frac{P(x)}{f(x)} = \frac{P(0)}{f(0)} = 1 \quad \text{for all } x \in (-1,1).$$

Hence

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n \quad \text{for } x \in (-1,1)\ .$$

Proof of 2). By 1) we only need to show that $f(1) = P(1)$ if $p > 0$. In fact, if $p > 0$, we prove that $f(x) = P(x)$ for $x \in [0,1]$ via Taylor's Theorem.

We may assume that $p \in (0,1)$. Let us apply Taylor's Theorem to $f(x) = (1+x)^p$ which has derivatives of any order on $(-1,\infty)$. Hence, for any $x > -1$, there is a number ξ_n between 0 and x such that

$$(1+x)^p = 1 + \sum_{k=1}^{n-1} \frac{p(p-1)\cdots(p-(k-1))}{k!} x^k + E_n(x)$$

where

$$E_n(x) = \frac{f^{(n)}(\xi_n)}{n!} x^n$$

for some $\xi_n \in (-1,1)$, where

$$\frac{f^{(n)}(x)}{n!} = \frac{p(p-1)\cdots(p-(n-1))}{n!}(1+x)^{p-n}.$$

Hence

$$E_n(x) = \frac{p(p-1)\cdots(p-(n-1))}{n!}(1+\xi_n)^p \left(\frac{x}{1+\xi_n}\right)^n.$$

If $x \in [0,1]$, then $\xi_n \in (0,1)$ so that

$$\left|(1+\xi_n)^p \left(\frac{x}{1+\xi_n}\right)^n\right| \le 2^p$$

and therefore

$$\begin{aligned}
|E_n(x)| &\le 2^p \left|\frac{p(p-1)\cdots(p-(n-1))}{n!}\right| \\
&= 2^p p \frac{(1-p)(2-p)\cdots(n-1-p)}{n!} \\
&= 2^p p \frac{1-p}{1}\frac{2-p}{2}\cdots\frac{n-1-p}{n-1}\frac{1}{n} \\
&\le \frac{2^p p}{n} \to 0
\end{aligned}$$

so that, by the Sandwich lemma, E_n converges to zero uniformly on $[0,1]$. It follows that $(1+x)^p = P(x)$ for $x \in [0,1]$. Together with the first part 1), 2) now follows. ■

For $p > 0$, we can show that $(1+x)^p = P(x)$ for every $x \in [-1,1]$, which will be the context of the following theorem. Before doing this, we observe that, for $\alpha > 0$

$$\lim_{x>0, x\to 0} x^\alpha = \lim_{x \downarrow 0} \exp(\alpha \ln x) = 0,$$

so we naturally define $0^\alpha = 0$ for $\alpha > 0$. Hence the power function x^α is continuous on $[0,\infty)$ if the power $\alpha > 0$.

Theorem 2.4.8 *Let p be a real number, and $P(x)$ denote the Taylor expansion of $(1+x)^p$ at 0, that is*

$$P(x) = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n. \tag{2.4.7}$$

1) If $p > -1$ then $(1+x)^p = P(x)$ for all $x \in (-1,1]$.

2) If $p > 0$, then $(1+x)^p = P(x)$ for all $x \in [-1,1]$, and the convergence of the power series $P(x)$ is uniform on $[-1,1]$.

Proof. Assume that $p \neq 0,1,2,\cdots$. According to the Taylor Theorem, for every $x > -1$ and $n \in \mathbb{N}$, there is ξ_n between 0 and x such that

$$(1+x)^p = 1 + \sum_{n=1}^{n-1} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n + E_n(x)$$

where the error term is given by, as we have seen in the theorem,

$$\begin{aligned} E_n(x) &= \frac{p(p-1)\cdots(p-(n-1))}{n!}(1+\xi_n)^{p-n}x^n \\ &= \frac{p(p-1)\cdots(p-(n-1))}{n!}(1+\xi_n)^p \left(\frac{x}{1+\xi_n}\right)^n. \end{aligned}$$

Step 1. If $x \in [0,1]$, then $\left|\frac{x}{1+\xi_n}\right| < 1$ so that

$$|E_n(x)| \leq 2^p \frac{|p(p-1)\cdots(p-(n-1))|}{n!} = 2^p\,|a(p)_n|$$

where

$$\begin{aligned} a(p)_n &= \frac{p(p-1)\cdots(p-(n-1))}{n!} \\ &= (-1)^n \frac{(-p)(1-p)\cdots((n-1)-p)}{n!}. \end{aligned}$$

If $p \in (0,1)$ then

$$a(p)_n = (-1)^{n-1}\frac{p}{n}\left(1-\frac{p}{1}\right)\left(1-\frac{p}{2}\right)\cdots\left(1-\frac{p}{n-1}\right)$$

so that

$$|a(p)_n| \leq \frac{p}{n} \to 0$$

which implies that $E_n \to 0$ uniformly on $[0,1]$ for this case that $p > 0$.

If $p \in (-1,0)$ then $1+p \in (0,1)$ and we may rewrite

$$\begin{aligned} a(p)_n &= (-1)^n \frac{(1-(p+1))(2-(p+1))\cdots(n-(1+p))}{n!} \\ &= (-1)^n \left(1-\frac{p+1}{1}\right)\left(1-\frac{p+1}{2}\right)\cdots\left(1-\frac{p+1}{n}\right). \end{aligned}$$

Let us prove the elementary inequality

$$1 - t \leq e^{-t} \quad \text{for } t \geq 0. \tag{2.4.8}$$

Let $g(t) = 1 - t - e^{-t}$. Then $g(0) = 0$ and $g'(t) = -1 + e^{-t} \leq 0$ for $t \geq 0$. Hence g is decreasing on $[0, \infty)$ and therefore $g(t) \leq 0$ for all $t \geq 0$.

By using this inequality we obtain, as $0 < 1 + p < 1$,

$$|a(p)_n| \leq \exp\left(-(1+p)\sum_{k=1}^{n}\frac{1}{k}\right) \to 0$$

as $1 + p > 0$ and $\sum_{k=1}^{n}\frac{1}{k} \to \infty$. Therefore $E_n \to 0$ as $n \to \infty$ uniformly on $[0, 1]$ and $p > -1$, so that, together with Theorem 2.4.7, we thus have

$$(1+x)^p = 1 + \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}x^n \quad \text{for } x \in (-1, 1],$$

and the convergence is uniform on $[-1+\delta, 1]$ for any $0 < \delta < 1$. This proves 1) and part of 2).

Step 2. Now we prove 2), so that we assume that $p > 0$. Without losing generality, let us assume that $p \in (0, 1)$. We want to show that $(1+x)^p = P(x)$ for all $x \in [-1, 1]$ and the convergence is uniform on $[-1, 1]$. Note that

$$P(x) = 1 + px + \sum_{n=2}^{\infty} a(p)_n x^n \qquad \forall x \in [-1, 1],$$

where

$$a(p)_n = \frac{p(p-1)\cdots(p-(n-1))}{n!}.$$

Of course we only need to show that $P(x)$ is convergent at -1. According to Abel's theorem, we only need to prove that the power series is convergent at $x = -1$, that is,

$$1 - p + \sum_{n=2}^{\infty}(-1)^n a(p)_n$$

is convergent. As we have mentioned, we may rewrite

$$a(p)_n = (-1)^{n-1}\frac{p}{n}\left(1-\frac{p}{1}\right)\left(1-\frac{p}{2}\right)\cdots\left(1-\frac{p}{n-1}\right)$$

so that

$$(-1)^n a(p)_n = -\frac{p}{n}\left(1-\frac{p}{1}\right)\left(1-\frac{p}{2}\right)\cdots\left(1-\frac{p}{n-1}\right)$$

for $n \geq 2$, which has a definite sign (always negative) for $p \in (0, 1)$. Using the elementary inequality (2.4.8) one obtains that

$$\begin{aligned}
0 &\leq -(-1)^n a(p)_n \\
&\leq \frac{p}{n}\exp\left\{-p\sum_{k=1}^{n-1}\frac{1}{k}\right\} \\
&= \frac{p}{n}\exp\{-p\gamma_{n-1} - p\ln(n-1)\} \\
&= \frac{p}{n}\frac{1}{(n-1)^p}e^{-p\gamma_{n-1}}
\end{aligned}$$

where

$$\gamma_{n-1} = \sum_{k=1}^{n-1}\frac{1}{k} - \ln(n-1) \to \gamma$$

the Euler constant. Hence $e^{-p\gamma_{n-1}} \to e^{-p\gamma}$ as $n \to \infty$, and therefore sequence $e^{-p\gamma_{n-1}}$ is bounded by some constant C. Therefore

$$0 \le -(-1)^n a(p)_n < pC\frac{1}{n(n-1)^p}$$

for any $n \ge 2$. Since $p > 0$, $\sum \frac{1}{n(n-1)^p}$ is convergent, so that, by the comparison test for series,

$$\sum_{n=2}^{\infty} (-1)^{n-1} a(p)_n$$

is convergent. Since

$$\left| \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n \right| \le (-1)^{n-1} a(p)_n < pC\frac{1}{n(n-1)^p}$$

for every $x \in [-1,1]$ and for every $n \ge 1$, by M-test for uniform convergence, together with Abel's theorem, for $p > 0$, the power series

$$\sum_{n=2}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n$$

converges uniformly to $(1+x)^p - 1 - px$ on $[-1,1]$, which proves 2). ∎

For example

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-(n-1))}{n!} x^n \qquad \forall x \in [-1,1]$$

and the convergence of the Taylor expansion on $[-1,1]$ is uniform, and

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-(n-1))}{n!} x^n \qquad \forall x \in (-1,1].$$