3. Asymptotic Approximations

3.1 Definitions

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges at fixed z if $\forall \varepsilon > 0, \exists N_0(z,\varepsilon) \in \mathbb{N}$ such that
$$\left| \sum_{n=m}^{N} f_n(z) \right| < \epsilon \quad \forall N \geq m > N_0$$

- A series $\sum_{n=0}^{\infty} f_n(z)$ converges to $f(z)$ at fixed z if $\forall \varepsilon > 0, \exists N_0(z,\varepsilon) \in \mathbb{N}$ such that
$$\left| f(z) - \sum_{n=0}^{N} f_n(z) \right| < \varepsilon \quad \forall N \geq N_0$$

- A series converges if its terms decay sufficiently rapidly as $n \to \infty$

- Less useful in practice than might be believed.

Example $$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \qquad z \in \mathbb{C}.$$

e^{-t^2} is a holomorphic function of $t \in \mathbb{C}$.

Thus it has a convergent power series with infinite radius of convergence

$$\sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}$$

Integrate term by term

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\, n!} = \frac{2}{\sqrt{\pi}} \left(z - z^3/3 + z^5/10 \cdots\cdots \right)$$

Has infinite radius of convergence.

For accuracy of 10^{-5}, 16 terms needed for $z = 2$
31 terms needed for $z = 3$
75 terms needed for $z = 5$

Cancellation required between large powers... need lot of terms for good approximation

Alternative approach to approximating erf(z).

Rewrite $\quad \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt$

Parts

$$\int_z^\infty e^{-t^2} dt = \int_z^\infty \left(-\frac{1}{2t}\right)\left(-2te^{-t^2}\right) dt$$

$-\tfrac{1}{2t} \qquad -2te^{-t^2}$

$\tfrac{1}{2t^2} \qquad e^{-t^2}$

$$= \left[-\tfrac{1}{2t}\, e^{-t^2}\right]_z^\infty - \int_z^\infty \frac{1}{2t^2}\, e^{-t^2} dt$$

$$= \frac{1}{2z} e^{-z^2} - \int_z^\infty \frac{e^{-t^2}}{2t^2}\, dt$$

Continuing the integration by parts

$$\operatorname{erf}(z) = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{1.3.}{(2z^2)^3} - \frac{1.3.5}{(2z^2)^5} + \cdots\right)$$

This series diverges $\forall z \in \mathbb{C}$, but truncated series very useful.

- For accuracy of 10^{-5} only two terms are needed for $z = 3$.
- Importantly The leading term is almost correct and each additional term gets us closer to the answer, with each additional correction of decreasing size until eventually they start increasing.

- This is an asymptotic series

Asymptoticness

- A sequence $\{f_n(\epsilon)\}_{n \in \mathbb{N}_0}$ is asymptotic if $\forall n \geq 1$

$$\frac{f_n(\varepsilon)}{f_{n-1}(\varepsilon)} \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

- A series $\sum_{n=0}^{\infty} f_n(\epsilon)$ is an <u>asymptotic expansion</u> of a function $f(\varepsilon)$ as $\varepsilon \to 0$ if $\forall N \in \mathbb{N}_0$ $\dfrac{f(\varepsilon) - \sum_{n=0}^{N} f_n(\epsilon)}{f_N(\epsilon)} \to 0$ as $\varepsilon \to 0$.

- In other words, the remainder is smaller than the last term included once ε is sufficiently small.

- We write $f(\varepsilon) \sim \sum_{n=0}^{\infty} f_n(\epsilon)$ as $\varepsilon \to 0$

- Usually first few terms are sufficient for a good approximation

- Often $f_n(\varepsilon) = a_n \varepsilon^n$ with a_n real, in which case

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \varepsilon^n \quad \text{as } \varepsilon \to 0$$

$\left\{ \begin{array}{l} f_n = a_n \delta_n(\varepsilon) \\ \text{with } \{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0} \\ \text{asymptotic also common} \end{array} \right\}$.

is called an <u>asymptotic power series</u>.

Order Notation

- $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means

  $\exists K, \delta > 0$ s.t $|f(\varepsilon)| < K|g(\varepsilon)| \quad \forall\, |\varepsilon - \varepsilon_0| < \delta$

- $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to 0$ as $\varepsilon \to \varepsilon_0$

- $f(\varepsilon) = \mathrm{ord}(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ means

  $\exists K \in \mathbb{R} \setminus \{0\}$ s.t. $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to K$ as $\varepsilon \to \varepsilon_0$

Examples

$\sin(x) = O(1),\ o(1),\ O(x),\ \mathrm{ord}(x)$ as $x \to 0$

$\sin(x) = O(1)$ as $x \to \infty$

$\log(x) = o(x^{-\delta})$ as $x \to 0$ for any $\delta > 0$.

3.2 Uniqueness and manipulation of an asymptotic series

- If a function $f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon)$ as $\varepsilon \to 0$ then induction implies that

$\{a_n\}_{n \in \mathbb{N}_0}$ is uniquely determined by $a_k = \lim_{\varepsilon \to 0} \left[\frac{f(\varepsilon) - \sum_{n=0}^{k-1} a_n \delta_n(\varepsilon)}{\delta_k(\varepsilon)} \right]$

- Uniqueness is for a given sequence $\{\delta_n(\varepsilon)\}_{n \in \mathbb{N}_0}$

- The sequence need not be unique e.g.

$$\tan \varepsilon \sim \varepsilon + \varepsilon^3/3 + 2\varepsilon^5/15 + \dots \qquad \text{as } \varepsilon \to 0$$

$$\tan \varepsilon \sim \sin \varepsilon + \tfrac{1}{2}(\sin \varepsilon)^3 + 3/8\,(\sin \varepsilon)^5 + \dots \qquad \text{as } \varepsilon \to 0$$

- Uniqueness for a given function... two functions may share the same asymptotic expansion e.g.

$$e^{\varepsilon} \sim \sum_{n=0}^{\infty} \varepsilon^n/n! \qquad \text{as } \varepsilon \to 0$$

$$e^{\varepsilon} + e^{-1/\varepsilon^2} \sim \sum_{n=0}^{\infty} \varepsilon^n/n! \qquad \text{as } \varepsilon \to 0$$

- Two distinct functions with the same asymptotic power series can only differ by a function that is not holomorphic as two holomorphic functions with the same power series are identical.

- Asymptotic expansions can be naively added, subtracted, divided, multiplied and divided (though the sequence eg. the $\{\delta_n(\epsilon)\}_{n\in\mathbb{N}_0}$ may be larger).

- This underlies expansion method for algebraic equations.

- One series can be substituted into another, but take care with exponentials .... always expand exponents to ord(1).

<u>Example</u> $f(z) = e^{z^2}$ $z = 1/\varepsilon + \varepsilon$ Naively $f(z) \sim e^{1/\varepsilon^2}$ at leading order $\times$

$$f(z) = \exp\left(\left(\frac{1}{\varepsilon}+\varepsilon\right)^2\right) = \exp\left(\frac{1}{\varepsilon^2}+2+\varepsilon^2\right) = e^{1/\varepsilon^2}\cdot e^2\cdot\left(1+\varepsilon^2+\frac{(\varepsilon^2)^2}{2!}+\frac{(\varepsilon^2)^3}{3!}+\dots\right)$$

- Sine and Cosine and complex exponentials require analogous care in this context.

- Asymptotic expansions can be integrated term by term with respect to ε resulting in the correct asymptotic expansion for the integral.

- In general asymptotic expansions <u>cannot</u> be differentiated safely

<u>Example</u>

$$f(\varepsilon) = \varepsilon\cos(1/\varepsilon) = O(\varepsilon) \text{ as } \varepsilon \to 0$$

$$f'(\varepsilon) = \frac{1}{\varepsilon}\sin(1/\varepsilon) + \cos(1/\varepsilon) = O(1/\varepsilon) \text{ as } \varepsilon \to 0$$

Differentiating the asymptotic expansion with the $O(\varepsilon)$ start would naively give $O(1)$ ... but the derivative is $O(1/\varepsilon)$ as $\varepsilon \to 0$.

- Terms move down an asymptotic expansion with differentiation (eg. $\frac{d}{dx}x^n = nx^{n-1}$) and thus terms at higher orders may cause problems on differentiation.

- Often, first few terms sufficient. If higher accuracy required...

  <u>optimal truncation</u> : truncate asymptotic series at smallest term

3.4 Parametric Expansions

- Integrals, differential equations and partial differential equations involve functions with one, or more, variables $f(x, \varepsilon)$ with ε a small parameter.

- There is an obvious generalisation of the definition of an asymptotic expansion by allowing the coefficients to depend on x. For fixed x

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} a_n(x)\,\delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0 \quad \text{if and only if}$$

$$\frac{1}{\delta_N(\varepsilon)}\left[f(x;\varepsilon) - \sum_{n=0}^{N} a_n(x)\,\delta_n(\varepsilon)\right] \to 0 \qquad \text{as } \varepsilon \to 0$$