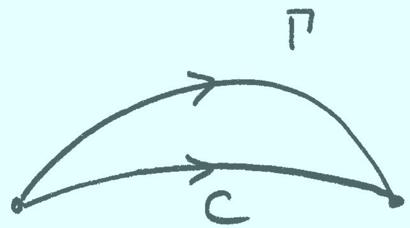
4.7 Method of Steepest Descents

- Generalises Laplace's method to consider
$$I(x) = \int_C f(t)\, e^{x\varphi(t)}\, dt \quad \text{as } x \to \infty,\ x \text{ real},$$
where $f(t), \varphi(t)$ are holomorphic (and thus analytic), with C a contour in the complex t plane.

- <u>Key idea</u> $I(x)$ unchanged upon deforming C to a new contour Γ, with the same start and end points.



$$I(x) = \int_\Gamma f(t)\, e^{x\varphi(t)}\, dt$$

- If we find a contour Γ on which $\operatorname{Im}(\varphi(t))$ is piecewise constant, i.e. Γ_j, v_j such that $\Gamma = \bigcup_j \Gamma_j$ with $\operatorname{Im}\varphi(t) = v_j = \text{const}$ on Γ_j then

$$I(x) = \sum_j e^{ixv_j} \int_{\Gamma_j} f(t)\, e^{x \operatorname{Re} \varphi(t)}\, dt$$

and each integral can be analysed as $x \to \infty$ using Laplace's method.

- Let $\varphi(t) = u(\xi, \eta) + iv(\xi, \eta)$ with $t = \xi + i\eta$.
- As φ is holomorphic, we have the Cauchy Riemann Equations (CRE):

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}, \quad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi}.$$

Hence $\nabla u \cdot \nabla v = u_\xi v_\xi + u_\eta v_\eta = 0 \quad \therefore \quad \nabla u \perp \nabla v$

Also $\nabla v \perp$ contours with v const $\quad \therefore \quad$ Contours with v const $\parallel \nabla u$.

∇u points in direction u increases at fastest rate

$-\nabla u$ points in direction u decreases at fastest rate

$\therefore$ Contour with v constant is a path of steepest ascent/descent of u.

Landscape of $u(\xi,\eta)$

- CRE. $\quad u_{\xi\xi}+u_{\eta\eta} = (v_\eta)_\xi + (-v_\xi)_\eta = 0$

- Hence u cannot have a maximum or a minimum (unless we are also considering a point where u is singular or a branch point, where u is not holomorphic).

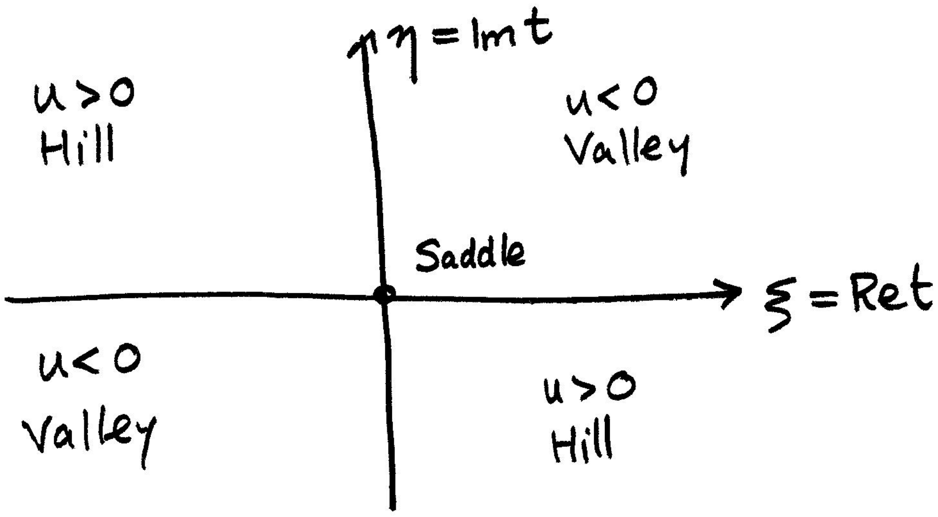
- At a stationary point, where $u_\xi = u_\eta = 0$, we have a SADDLE.

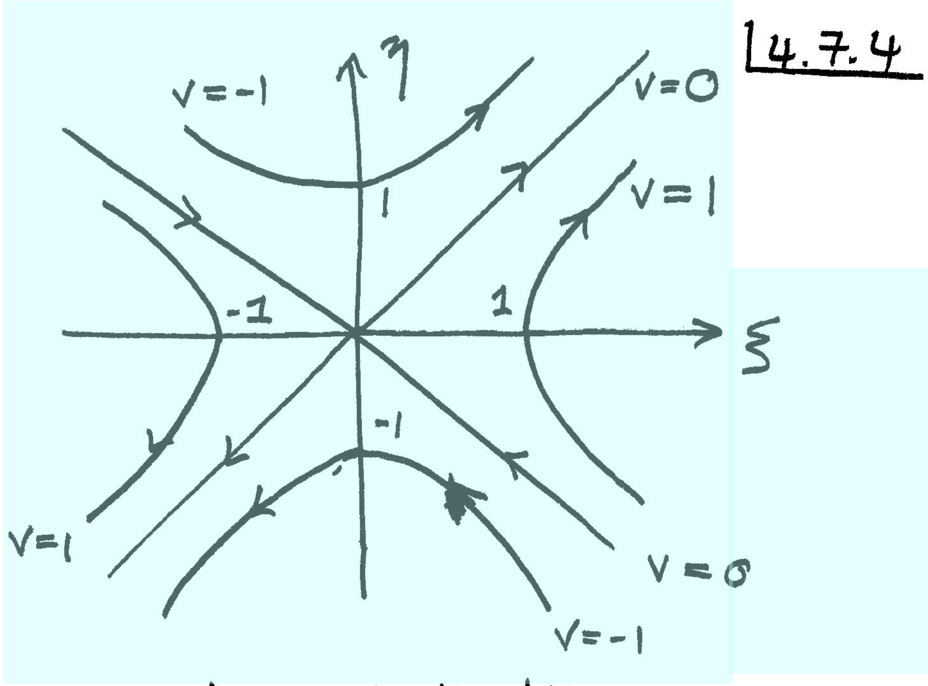
- Landscape of u has hills $(u>0)$, valleys $(u<0)$ at infinity with saddle points in the interior of the complex plane.

Example

$$\varphi(t) = it^2 = i(\xi+i\eta)^2 = -2\xi\eta + i(\xi^2-\eta^2) \quad \therefore\ u=-2\xi\eta,\ v=\xi^2-\eta^2$$

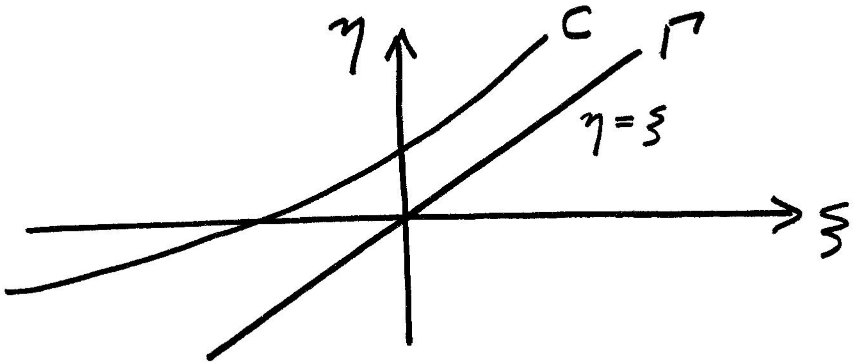
$$\nabla u = -2(\eta,\xi) \quad \therefore \text{ Saddle point at } \xi=\eta=0$$





Arrows in direction of decreasing u with STEEPEST DESCENT

- Contour C infinite, with endpoints in different valleys.
  - If endpoints not in valleys, integral $I(x)$ not well defined.



Deform C into Γ.

Integrals at infinity subleading

Hence method known as "Method of steepest descents" or saddle point method

To use the method ...

* Deform contour to union of steepest descent (v const) contours through the endpoints and any relevant saddle points

* Evaluate local contributions from saddle and end points using Laplace's method.