Example **Semi-Classical Quantum. Turning Points.**

The non-dimensional steady state Schrödinger equation for the even wave-functions of the simple harmonic oscillator is given by

$$\psi'' - x^2\psi = -E\psi$$
$$\psi \to 0 \text{ as } x \to \infty, \quad \psi'(0) = 0.$$

Find the large, $E \gg 1$, energy eigenvalues.

Let $y = \psi$. $x = \bar{x}/\sqrt{\varepsilon}$ with $\varepsilon = 1/E$. Then, dropping bars,

$$\varepsilon^2 y'' + (1 - x^2) y = 0$$
$$y(\infty) = 0, \; y'(0) = 0, \quad 0 < \varepsilon \ll 1.$$

Let $y = e^{i\varphi/\varepsilon} A(x, \varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$

WKB $\quad O(\varepsilon^0) \quad \varphi' = \pm\sqrt{1 - x^2} \qquad O(\varepsilon^1) \quad A_0 = \dfrac{\text{const}}{(1 - x^2)^{1/4}}$

Hence

For $0<x<1$,
$$y \sim \frac{M_0}{(1-x^2)^{1/4}} e^{1/\varepsilon \int_0^x \sqrt{1-s^2}\,ds} + \frac{N_0}{(1-x^2)^{1/4}} e^{-1/\varepsilon \int_0^x \sqrt{1-s^2}\,ds}$$

using $y'(0)=0$

$$\sim \frac{P_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\,ds\right)$$

For $x>1$
$$y \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-1/\varepsilon \int_1^x \sqrt{s^2-1}\,ds}$$
using $y(\infty)=0$

However, these breakdown near $x \simeq 1$ as $\varphi'(1)=0$.

Resolve using matched asymptotics

Inner region around $x=1$

Let $x = 1+\delta_1(\varepsilon)X$ $\qquad Y(X) = \delta_2(\varepsilon)\, y(x)$

$$\frac{\varepsilon^2}{\delta_1^2}\frac{d^2Y}{dX^2} + \underbrace{\left(1-(1+2\delta_1 X + \delta_1^2 X^2)\right)Y}_{2\delta_1 XY + \delta_1^2 X^2 Y} = 0$$

Dominant balance when $2\delta_1^3 = \varepsilon^2$ $\therefore$ Let $\delta_1 = \dfrac{\varepsilon^{2/3}}{2^{1/3}}$.

δ_2 undetermined as yet

small "oh"

With $Y = Y_0(X) + o(1)$

$$\frac{d^2Y_0}{dX^2} - XY_0 = 0 \quad \therefore \quad Y_0 = \boxed{R_0\,Ai(X) + S_0\,Bi(X)}$$ where Ai, Bi are Airy functions.

<u>Airy Functions</u>

$$Ai(X) = \frac{1}{\pi}\int_0^\infty \cos\left(t^3/3 + Xt\right) dt \sim \frac{1}{2\sqrt{\pi}\,X^{1/4}} e^{-2/3 X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}} \sin\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty.$$

$$Bi(X) = \frac{1}{\pi}\int_0^\infty \exp\left(-t^3/3 + Xt\right) dt \sim \frac{1}{\sqrt{\pi}\,X^{1/4}} e^{2/3 X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}} \cos\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty$$

<u>Matching</u> Inner ($X \to \infty$) with RH outer ($x \to 1^+$)

$S_0 = 0$ else Y_0 blows up as $X \to \infty$.

On matching everything scales with $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ whether using Van Dyke or intermediate region. Naively one gets simply $0=0$. Thus, on matching, insist the coefficients in front of $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ match.

<u>Matching</u> (intermediate variable)

Let $x-1 = \delta_1^{\beta}\hat{x} = \delta_1 X \qquad (0<\beta<1)$ with $\hat{x} = \mathrm{ord}(1)$, $x \to 1$, $X \to \infty$, $\hat{x} > 0$.

$$Y_0 = R_0 \mathrm{Ai}\left(\frac{\hat{x}}{\delta_1^{1-\beta}}\right) \sim \frac{R_0}{2\sqrt{\pi}} \frac{(\delta_1^{1-\beta})^{1/4}}{\hat{x}^{1/4}} \exp\left[-\frac{2}{3} \cdot \frac{1}{(\delta_1^{1-\beta})^{3/2}} \hat{x}^{3/2}\right]$$

$$y \sim \frac{Q_0}{[(x-1)(x+1)]^{1/4}} \exp\left[-\frac{1}{\varepsilon}\int_1^x \sqrt{s^2-1}\, ds\right]$$

$s^2-1 = (s-1)(s+1)$, $s = 1+\eta$

$$\int_1^x \sqrt{s^2-1}\, ds = \int_0^{x-1} \eta^{1/2} 2^{1/2} \sqrt{1+\eta/2}\, d\eta$$

$$= \sqrt{2} \cdot \frac{2}{3}(x-1)^{3/2} + \dots$$

$$= \frac{2\sqrt{2}}{3} \delta_1^{3\beta/2} \hat{x}^{3/2} + \dots$$

$$\therefore \frac{1}{\varepsilon}\int_1^x \sqrt{s^2-1}\, ds = \frac{1}{(2^{1/3}\delta_1)^{3/2}} \frac{2\sqrt{2}}{3} \delta_1^{3\beta/2} \hat{x}^{3/2} + \dots$$

$$\therefore\ y \sim \frac{Q_0}{2^{1/4}\delta_1^{\beta/4}\,\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\frac{1}{(\delta_1^{1-\beta})^{3/2}}\hat{x}^{3/2}\right] + \cdots$$

$$\therefore\ Y = \delta_2 y \sim \frac{Q_0\,\delta_2(\varepsilon)}{2^{1/4}(\delta_1)^{\beta/4}\hat{x}^{1/4}} \exp\left[-\tfrac{2}{3}\frac{1}{(\delta_1^{1-\beta})^{3/2}}\hat{x}^{3/2}\right] + \cdots \sim \frac{R_0\,\delta_1^{1/4}}{2\sqrt{\pi}}\frac{1}{\delta_1^{\beta/4}}\frac{1}{\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\frac{\hat{x}^{3/2}}{(\delta_1^{1-\beta})^{3/2}}\right]$$

$$\therefore\quad \delta_2 = \delta_1^{1/4} = \left(\varepsilon^{2/3}/2^{1/3}\right)^{1/4} = \frac{1}{2^{1/12}}\varepsilon^{1/6} \quad \text{and} \quad Q_0 = \frac{1}{2^{3/4}\sqrt{\pi}}R_0$$

<u>Matching</u> inner ($X \to -\infty$) with LH outer ($x \to 1^-$).

Let $x - 1 = \delta_1^{\gamma}\hat{x} = \delta_1 X \quad (0<\gamma<1)$ with $\hat{x} = \text{ord}(1)$, $x \to 1$, $X \to -\infty$, $\hat{x} < 0$.

$$Y_0 = R_0\,Ai\left(\frac{\hat{x}}{\delta_1^{1-\gamma}}\right) \sim \frac{R_0(\delta_1)^{\frac{1-\gamma}{4}}}{\sqrt{\pi}(-\hat{x})^{1/4}} \sin\left(\frac{2}{3}(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}} + \frac{\pi}{4}\right)$$

$$y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\,\delta_1^{\gamma/4}} \cos\left(\pi/4\varepsilon - \frac{1}{\varepsilon}\int_x^1 \sqrt{1-s^2}\,ds\right) \quad \text{using } \int_0^1\sqrt{1-s^2}\,ds = \pi/4$$

$$\therefore \quad y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{1}{\varepsilon}\cdot\frac{2\sqrt{2}}{3}(1-x)^{3/2} + \ldots\right)$$

← Substituting $s = 1-\eta$ in integral and using $\sqrt{1-s^2} = \eta^{1/2}(2^{1/2} + O(\eta))$

$$\sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\pi/4\varepsilon - \underbrace{\frac{2\sqrt{2}}{3\varepsilon}\delta_1^{3\gamma/2}}_{\frac{2}{3}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}}(-\hat{x})^{3/2} + \ldots\right)$$

$$\sim \delta_2^{-1} Y_0 = \frac{R_0}{\sqrt{\pi}\,(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \sin\left(\pi/4 + 2/3(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}\right)$$

With $w = 2/3(-\hat{x})^{3/2}\dfrac{1}{(\delta_1^{1-\gamma})^{3/2}}$,

$$\frac{P_0}{2^{1/4}}\cos\left(\pi/4\varepsilon - w\right) \sim \frac{R_0}{\sqrt{\pi}}\sin\left(\pi/4 + w\right)$$

$$\therefore \quad \frac{P_0}{2^{1/4}}\left[\cos\frac{\pi}{4\varepsilon}\cos w + \sin(\pi/4\varepsilon)\sin w\right] \sim \frac{R_0}{\sqrt{\pi}}\left[\sin\pi/4\cos w + \cos\pi/4\sin w\right]$$

$$\therefore \quad \frac{P_0}{2^{1/4}}\cos\frac{\pi}{4\varepsilon} \sim \frac{R_0\sin\pi/4}{\sqrt{\pi}}, \qquad \frac{P_0\sin(\pi/4\varepsilon)}{2^{1/4}} \sim \frac{R_0}{\sqrt{\pi}}\cos\pi/4$$

For $P_0, R_0 \neq 0$ $\qquad \tan\left(\frac{\pi}{4\varepsilon}\right) \sim \cot(\pi/4) = 1$ as $\varepsilon \to 0$

$$\therefore \quad \frac{\pi}{4\varepsilon} \sim \frac{\pi}{4} + n\pi \quad \text{as } n \to \infty, \text{ with } n \in \mathbb{N}$$

$$\therefore \quad E_n = \frac{1}{\varepsilon_n} = 1 + 4n \quad \text{as } n \to \infty, \text{ for the energy levels.}$$

Once this holds $\qquad \cos\left(\frac{\pi}{4\varepsilon}\right) \sim \cos(\pi/4 + n\pi) = \frac{1}{\sqrt{2}}(-1)^n$ $\qquad \therefore P_0 = \frac{2^{1/4}}{\sqrt{\pi}}(-1)^n R_0 = 2\,(-1)^n Q_0$ Connection formula

$$y_n \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-1/\varepsilon_n \int_1^x \sqrt{s^2-1}\, ds} \qquad x > 1,\ x \not\approx 1$$

$$\sim \frac{2^{1/12}}{\varepsilon^{1/6}} \cdot 2^{3/4} \cdot \sqrt{\pi}\, Q_0 \operatorname{Ai}\left(\frac{2^{1/3}(x-1)}{\varepsilon_n^{2/3}}\right) \qquad x \simeq 1$$

$$\sim \frac{2(-1)^n Q_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon_n}\int_0^x \sqrt{1-s^2}\, ds\right) \quad x<1,\ x \not\approx 1, \quad \varepsilon_n = \frac{1}{1+4n}, \quad n \gg 1.$$