Differential Equations II

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Chapter 1

Second-order linear boundary value problems - Part 1

These lecture notes are based on material written by Derek Moulton and Peter Howell. *Please send any corrections or comments to* Renaud Lambiotte.

1.1 Basic notation and concepts

In this section, we will develop various techniques to analyse and solve ordinary differential equations (ODEs), in particular *inhomogeneous linear boundary value problems* (BVPs). We start by briefly explaining what is meant by each piece of this expression. Although everything to follow can in principle be generalised to ODEs of arbitrary order, we restrict our attention to second order ODEs for the moment.

A second-order *linear* ODE is an equation of the form

$$\mathfrak{L}y(x) = f(x), \tag{1.1}$$

where f is a given *forcing function* and $\mathfrak{L}$ is a *linear differential operator*, that is,

$$\mathfrak{L}y(x) \equiv P_2(x)y''(x) + P_1(x)y'(x) + P_0(x)y(x) \tag{1.2a}$$

$$\equiv P_2(x)\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + P_1(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + P_0(x)y(x), \tag{1.2b}$$

for some given coefficients $P_0(x)$, $P_1(x)$, $P_2(x)$. The operator $\mathfrak{L}$ is linear in the sense that

$$\mathfrak{L}\big[\alpha_1 y_1(x) + \alpha_2 y_2(x)\big] \equiv \alpha_1 \mathfrak{L}y_1(x) + \alpha_2 \mathfrak{L}y_2(x), \tag{1.3}$$

for any constants α_i and functions $y_i(x)$. Here, and henceforth unless explicitly stated otherwise, we assume that y is sufficiently smooth for all the required derivatives to exist and be continuous. We will also assume that the coefficents P_i are at least continuous and (for reasons that will become clear) that P_2 is nonzero in the range of x of interest.

The linear ODE (1.1) is said to be *homogeneous* if the right-hand side f is identically zero, and if not then the equation is *inhomogeneous*. We will refer frequently to the homogeneous and inhomogeneous (or "Non-homogeneous") versions of (1.1), which we label as follows:

$$\textbf{homogeneous:} \qquad \mathfrak{L}y = 0, \tag{H}$$

$$\textbf{inhomogeneous:} \qquad \mathfrak{L}y = f \not\equiv 0. \tag{N}$$

Generally, we expect to need to supplement a second-order ODE of the form (1.1) with *two* boundary conditions to get a unique solution for $y(x)$, and the term *boundary value problem* refers to the way in which those boundary conditions are imposed. Much of the Differential Equations I course concerns the solution of *initial value problems* (IVPs), where the "initial values" of y and y' are given at a single point $x = a$, say. In a BVP, the ODE (1.1) is posed on an interval, say $a < x < b$, and the boundary conditions involve the values of y and y' at both ends of the domain $x = a$ and $x = b$. Provided the coefficients $P_i(x)$ and the forcing function $f(x)$ are sufficiently well behaved (and $P_2(x) \neq 0$), Picard's Theorem guarantees that an IVP for a linear ODE of the form (1.1) has a unique solution in a neighbourhood of the initial point $x = a$, but we will see that the same cannot be said of a linear BVP.

Example 1.1. Second order IVP and BVP

The simple 2nd order linear inhomogeneous ODE

$$y'' + y = 1 \tag{1.4}$$

has the general solution $y(x) = 1 + c_1 \cos x + c_2 \sin x$, *where* c_1 *and* c_2 *are arbitrary* integration constants. *A typical IVP would involve solving* (1.4) *in* $x > 0$ *subject to the initial conditions* $y(0) = 1$ *and* $y'(0) = 2$. *By imposing the two initial conditions, we can easily solve for the integration constants and thus obtain the solution* $y(x) = 1 + 2\sin x$.

A typical BVP would be to solve (1.4) *on an interval, say* $0 < x < \pi$, *subject to the boundary conditions* $y(0) = 1$ *and* $y'(\pi) = 2$. *Again, we can solve for the arbitrary constants, and this time we obtain the solution* $y(x) = 1 - 2\sin x$.

Suppose we replace the right-hand side of (1.4) *with a more complicated forcing function, for example*

$$y''(x) + y(x) = \tan x. \tag{1.5}$$

In principle, this ODE is solvable, subject to suitable boundary conditions, but now it is not at all obvious how to "spot" the particular integral!

Finally, suppose we slightly alter the boundary conditions to $y(0) = 1$ *and* $y(\pi) = 2$. *One can easily confirm that the ODE* (1.4) *has* no solution *subject to the modified boundary conditions.*

In the remainder of this section, we will derive general methods to solve ODEs of the form (1.1), as well as addressing the following general questions.

1. How can we construct a particular integral for the ODE (1.1) for arbitrary forcing function f?
2. Given suitable boundary conditions, when does a solution exist? When is it unique?

1.2 Space of solutions

If we ignore boundary conditions for the moment, then the following properties of solutions of (H) and (N) are easily established.

(i) The solutions of (H) form a vector space since, if $\mathfrak{L}y_1 = 0 = \mathfrak{L}y_2$, then $\mathfrak{L}[\alpha y_1 + \beta y_2] = 0$.

(ii) If y and Y satisfy (N), then $y - Y$ satisfies (H).

(iii) It follows that the general solution of (N) may be written in the form

$$y(x) = \underbrace{y_{\mathrm{PI}}(x)}_{\text{any solution of (N)}} + \underbrace{y_{\mathrm{CF}}(x)}_{\text{general solution of (H)}} \tag{1.6}$$

where y_{PI} is called the *particular integral* and y_{CF} the *complementary function*.

(iv) For a second-order ODE, the vector space of solutions to (H) has dimension two (see below). The complementary function therefore takes the form

$$y_{\mathrm{CF}}(x) = c_1 y_1(x) + c_2 y_2(x), \tag{1.7}$$

where c_1, c_2 are arbitrary constants, and y_1, y_2 are any two *linearly independent* solutions to (H).

1.3 Linear independence; the Wronskian

A pair of functions $y_1(x)$, $y_2(x)$ is *linearly independent* if there is no non-trivial linear combination that vanishes identically; in other words if

$$c_1 y_1(x) + c_2 y_2(x) \equiv 0 \quad \Leftrightarrow \quad c_1 = c_2 = 0. \tag{1.8}$$

They are *linearly dependent* if c_i, not both zero, can be found such that $c_1 y_1(x) + c_2 y_2(x)$ is identically zero. Provided y_1, y_2 are differentiable, this would also entail $c_1 y_1'(x) + c_2 y_2'(x) \equiv 0$. Therefore

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \equiv \mathbf{0}, \tag{1.9}$$

and non-trivial solutions can exist for (c_1, c_2) if and only if the determinant of the matrix is zero.

We define the *Wronskian* of a pair of functions to be this determinant:

$$W(x) = W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1(x) y_2'(x) - y_2(x) y_1'(x). \tag{1.10}$$

From what we have just seen, we conclude the following.

Proposition 1.1. *If two functions are linearly dependent then their Wronskian vanishes.*

The converse to this statement is not necessarily true, however. For example, the following (once) differentiable functions:

$$y_1(x) = \begin{cases} 0 & x < 0, \\ x^2 & x \geq 0, \end{cases} \qquad y_2(x) = \begin{cases} x^2 & x < 0, \\ 0 & x \geq 0, \end{cases} \tag{1.11}$$

are easily shown to be linearly independent, but have Wronskian equal to zero [**exercise**]. We will now show that there *is* a partial converse to Proposition 1.1 for the case where y_1 and y_2 are solutions to (H).

Suppose that y_1 and y_2 are two solutions to (H), i.e.

$$P_2 y_1'' + P_1 y_1' + P_0 y_1 = 0, \tag{1.12a}$$
$$P_2 y_2'' + P_1 y_2' + P_0 y_2 = 0. \tag{1.12b}$$

We can eliminate the P_0 term between these two equations by substracting $y_2\times$(1.12a) from $y_1\times$(1.12b) to get

$$P_2 \left(y_1 y_2'' - y_2 y_1'' \right) + P_1 \left(y_1 y_2' - y_2 y_1' \right) = 0. \tag{1.13}$$

The term multiplying P_1 in this equation is clearly the Wronskian $W[y_1, y_2]$, and the term multiplying P_2 is the derivative of W with respect to x, i.e.

$$P_2 \frac{\mathrm{d}W}{\mathrm{d}x} + P_1 W = 0. \tag{1.14}$$

Now, *provided P_2 is nowhere zero*, we can solve for W to get

$$W(x) = \text{const} \times \exp\left(-\int \frac{P_1(x)}{P_2(x)}\,\mathrm{d}x\right). \tag{1.15}$$

Since the exponential can't vanish, it follows that *if $W = 0$ at one point, then $W \equiv 0$ everywhere* and, conversely, *if $W \neq 0$ at one point, then $W \neq 0$ everywhere.* Now we can use this result to obtain a partial converse to Proposition 1.1.

Proposition 1.2. *Two solutions of a given homogeneous second-order ODE* (H) *are linearly dependent if and only if their Wronskian is zero.*

Proof. Suppose y_1 and y_2 are two solutions of (H); if they are linearly dependent then we know already that their Wronskian is zero so now suppose for the converse that their Wronskian is zero (everywhere, by (1.15)). If y_1 is the zero function then y_1 and y_2 are certainly linearly dependent and we are done. Suppose that there is at least one value of x, say $x = a$, with $y_1(a) \neq 0$, and pick μ so that $y_2(a) = \mu y_1(a)$. Then

$$0 = W(a) = y_1(a)y_2'(a) - y_2(a)y_1'(a) = y_1(a)\left(y_2'(a) - \mu y_1'(a)\right) \tag{1.16}$$

and, since $y_1(a) \neq 0$ by assumption, we conclude that $y_2'(a) = \mu y_1'(a)$.

Now define $y(x) = y_2(x) - \mu y_1(x)$; then $y(x)$ is a solution of (H) by linearity, and satisfies the initial conditions $y(a) = 0 = y'(a)$. Thus by uniqueness of solution of (H) (Picard's Theorem: again assuming that $P_2 \neq 0$) we conclude that $y(x) \equiv 0$ and therefore y_1 and y_2 are linearly dependent. □

1.4 A basis of solutions to (H)

We can choose two particular solutions y_1 and y_2 of (H) satisfying the following initial conditions at some point $x = a$:

$$y_1(a) = 1, \quad y_1'(a) = 0, \qquad\qquad y_2(a) = 0, \quad y_2'(a) = 1. \tag{1.17}$$

By Picard's Theorem both $y_1(x)$ and $y_2(x)$ exist and are unique at least in a neighbourhood of $x = a$ provided $P_2(a) \neq 0$. Also their Wronskian has $W = 1$ at $x = a$ and so is nonzero in the same neighbourhood of $x = a$, and hence they are linearly independent.

In fact, y_1 and y_2 span the vector space of solutions. Suppose $y(x)$ is any other solution of (H) and set

$$Y(x) = y_1(x)y(a) + y_2(x)y'(a). \tag{1.18}$$

Then $Y(x)$ is also a solution of (H) and satisfies the initial conditions

$$Y(a) = y(a), \qquad\qquad Y'(a) = y'(a). \tag{1.19}$$

By uniqueness (Picard again) $Y(x) \equiv y(x)$ and thus $y(x)$ is a linear combination of y_1 and y_2. Hence they do span the vector space of solutions, i.e. they are a basis, and we conclude the following.

Proposition 1.3.

(i) *The dimension of the space of solutions of H is* 2.

(ii) *Any pair of solutions of H with $W \neq 0$ is a basis.*

Exercise: generalise everything done so far to n-th order linear ODEs.

1.5 Solution methods for homogeneous problem

There are very few general methods of solution for second-order linear ODEs of the form (H). We will discuss three well known special cases of (H) where the general solution can be found relatively easily. All three methods can be used for higher order problems with similar properties.

1.5.1 Constant coefficients

If P_2, P_1 and P_0 are constants, then (H) admits exponential solutions of the form $y(x) = \mathrm{e}^{mx}$, where m satisfies the quadratic equation $P_2 m^2 + P_1 m + P_0 = 0$, known as the *auxiliary equation*. The general solution can then easily be found as a linear combination of solutions with different values of m. Care must be taken for cases where the roots m are complex or are repeated.

1.5.2 Cauchy–Euler equation

In a Cauchy–Euler equation, the coefficients are of the form $P_2(x) = \alpha x^2$, $P_1(x) = \beta x$, $P_0(x) = \gamma$, with α, β, γ constants, so (H) takes the form

$$\alpha x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \beta x \frac{\mathrm{d}y}{\mathrm{d}x} + \gamma y = 0. \tag{1.20}$$

(Note that the "power of x" is the same in each term.) In this case, solutions can be found of the form $y(x) = x^m$, and m again satisfies a quadratic equation, $\alpha m(m-1) + \beta m + \gamma = 0$. Again, extra care is needed if the roots m are repeated or complex. An alternative approach is to make the substitution $x = \mathrm{e}^t$, which transforms (1.20) into the constant-coefficients equation

$$\alpha \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (\beta - \alpha) \frac{\mathrm{d}y}{\mathrm{d}t} + \gamma y = 0. \tag{1.21}$$

1.5.3 Reduction of order

If one solution $y_1(x)$ is known, then the general solution can be found by solving an ODE of reduced order. The method is to express the solution to the ODE (H) in the form

$$y(x) = v(x) y_1(x). \tag{1.22}$$

We know that the function $v(x) = \text{const}$ is a possible answer but we seek something more general. We substitute (1.22) into (H) and simplify, using the fact that y_1 is a solution of (H), to obtain

$$P_2 y_1 v'' + (2P_2 y_1' + P_1 y_1) v' = 0, \tag{1.23}$$

which is a separable first-order ODE for v' with solution

$$v'(x) = \frac{\text{const}}{y_1(x)^2} \exp\left(-\int \frac{P_1(x)}{P_2(x)}\,\mathrm{d}x\right). \tag{1.24}$$

One further integration then gives v and thus the general solution $y(x) = v(x)y_1(x)$.

This method of constructing the general solution from a single known solution may also be derived from the expression (1.15) for the Wronskian, i.e.

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = y_1(x)^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_2(x)}{y_1(x)}\right) = \text{const} \times \exp\left(-\int \frac{P_1(x)}{P_2(x)}\,\mathrm{d}x\right), \tag{1.25}$$

from which we can construct $y_2(x)$ given $y_1(x)$.

1.6 Variation of parameters

We now know a good deal about the solutions of the homogeneous ODE (H). The general solution to the inhomogeneous version (N) given by (1.6) seems to rely on us spotting a particular integral $y_{\mathrm{PI}}(x)$. The method of variation of parameters allows us to construct a solution to (N) for any forcing function f without any guesswork, provided we already know the general solution to the homogeneous equation (H).

Suppose that (H) is solved by $y(x) = c_1y_1(x) + c_2y_2(x)$ with linearly independent y_1, y_2. We seek a solution to (N) of the form

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x), \tag{1.26}$$

i.e. we "vary the parameters" c_1 and c_2. First, differentiate (1.26) to find

$$y' = c_1y_1' + c_2y_2' + c_1'y_1 + c_2'y_2. \tag{1.27}$$

Now to eliminate the highest derivatives of c_i, we impose the condition

$$c_1'y_1 + c_2'y_2 = 0 \tag{1.28}$$

on c_1 and c_2. Note, since we are using two functions c_1 and c_2 to define one function y, we should have enough freedom to satisfy the additional constraint (1.28). Under the assumption (1.28), the expression (1.27) for y' simplifies to

$$y' = c_1y_1' + c_2y_2'. \tag{1.29}$$

We differentiate once more and substitute into (1.2) to get

$$\begin{aligned}\mathfrak{L}y &= P_2\left(c_1y_1'' + c_2y_2'' + c_1'y_1' + c_2'y_2'\right) + P_1\left(c_1y_1' + c_2y_2'\right) + P_0(c_1y_1 + c_2y_2)\\ &= c_1\mathfrak{L}y_1 + c_2\mathfrak{L}y_2 + P_2\left(c_1'y_1' + c_2'y_2'\right).\end{aligned} \tag{1.30}$$

But, since the y_i satisfy (H), the inhomogeneous ODE (N) becomes

$$\mathfrak{L}y = P_2\left(c_1'y_1' + c_2'y_2'\right) = f. \tag{1.31}$$

Together, (1.28) and (1.31) give two simultaneous linear equations for c_1' and c_2', namely

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f/P_2. \end{pmatrix} \tag{1.32}$$

Note that the determinant of the matrix on the left-hand side is the Wronskian W, which we know is nonzero by the assumed linear independence of y_1 and y_2. We can therefore invert (1.32) to get

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f/P_2. \end{pmatrix} = \frac{f}{P_2 W} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}. \tag{1.33}$$

We can thus integrate to obtain

$$c_1(x) = -\int^x \frac{f(\xi)y_2(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi, \qquad c_2(x) = \int^x \frac{f(\xi)y_1(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi, \tag{1.34}$$

and, by substituting into (1.26)

$$y(x) = -\int^x \frac{f(\xi)y_2(\xi)y_1(x)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi + \int^x \frac{f(\xi)y_1(\xi)y_2(x)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi. \tag{1.35}$$

In principle, (1.35) allows us to construct a particular solution to (N) for any right-hand side f. There is some freedom in the construction (1.35): firstly in the choice of two linearly independent solutions (y_1, y_2) of (H); and secondly in setting the lower limits in the integrals. We will show below how to use this freedom to fit boundary conditions, after doing an example.

Example 1.2. *Consider the equation*

$$y''(x) + y(x) = \tan x \quad \textit{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}. \tag{1.36}$$

The corresponding homogeneous equation is $y'' + y = 0$*, for which we may choose two linearly-independent solutions as*

$$y_1(x) = \cos x, \qquad y_2(x) = \sin x. \tag{1.37}$$

The Wronskian turns out to be

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = \cos^2 x + \sin^2 x = 1, \tag{1.38}$$

and so by (1.34) we have

$$c_1(x) = -\int \tan x \sin x \,\mathrm{d}x = \sin(x) - \log(\sec x + \tan x), \tag{1.39a}$$

$$c_2(x) = \int \tan x \cos x \,\mathrm{d}x = -\cos x. \tag{1.39b}$$

Thus a particular integral of the inhomogeneous ODE (1.36) *is given by*

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = -\cos(x)\log(\sec x + \tan x). \tag{1.40}$$

It would have been very difficult to "spot" this from (1.36)*!*

Chapter 2

Second-order linear boundary value problems - Part 2

These lecture notes are based on material written by Derek Moulton and Peter Howell. *Please send any corrections or comments to* Renaud Lambiotte.

2.1 Fitting boundary conditions

We now develop a general method to solve the inhomogeneous ODE (N) with homogeneous boundary conditions. We consider the BVP

$$P_2(x)y''(x) + P_1(x)y'(x) + P_0(x)y(x) = f(x) \quad a < x < b, \tag{2.1a}$$

with boundary data

$$y(a) = 0 = y(b). \tag{2.1b}$$

We will see later on how generalised boundary conditions more complicated than (2.1b) may be handled. We follow the Variation of Parameters recipe (1.26), but now making specific choices of the two basis solutions y_1 and y_2 such that $y_1(a) = 0$ and $y_2(b) = 0$. We assume for the moment that such y_1 and y_2 exist and are linearly independent so that $W[y_1, y_2] \neq 0$, and it follows that $y_1(b) \neq 0$ and $y_2(a) \neq 0$.

So the solution takes the form $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, with the c_i as in (1.34), and the boundary conditions (2.1b) lead to

$$y(a) = c_1(a)y_1(a) + c_2(a)y_2(a) = c_2(a)y_2(a) = 0, \tag{2.2a}$$

$$y(b) = c_1(b)y_1(b) + c_2(b)y_2(b) = c_1(b)y_1(b) = 0 \tag{2.2b}$$

with the choices made for y_i. This requires that we take $c_2(a) = 0 = c_1(b)$ and, by imposing these conditions on (1.34), we obtain explicit unique forms for c_1 and c_2, namely

$$c_1(x) = \int_x^b \frac{f(\xi)y_2(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi, \qquad c_2(x) = \int_a^x \frac{f(\xi)y_1(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi \tag{2.3}$$

(note the switching of the limits in the integral for c_1).

The solution to the BVP (2.1) can thus be written as

$$y(x) = \int_a^x \frac{f(\xi)y_1(\xi)y_2(x)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi + \int_x^b \frac{f(\xi)y_2(\xi)y_1(x)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi, \tag{2.4}$$

which we can write concisely as

$$y(x) = \int_a^b g(x,\xi) f(\xi)\,\mathrm{d}\xi, \tag{2.5}$$

where

$$g(x,\xi) = \begin{cases} \dfrac{y_1(\xi) y_2(x)}{P_2(\xi) W(\xi)} & a < \xi < x < b, \\ \dfrac{y_2(\xi) y_1(x)}{P_2(\xi) W(\xi)} & a < x < \xi < b, \end{cases} \tag{2.6}$$

is called the *Green's function.* We will return to study the properties of g in more detail in Section 2.

Example 2.3. *We illustrate the construction of g for the BVP*

$$y''(x) + y(x) = f(x) \quad \textit{for} \quad 0 < x < \frac{\pi}{2}, \tag{2.7a}$$

with boundary conditions

$$y(0) = 0 = y\left(\frac{\pi}{2}\right). \tag{2.7b}$$

1. *Identify* (H) *as $y'' + y = 0$.*
2. *Choose solutions y_1 and y_2 such that $y_1(0) = 0$ and $y_2(\pi/2) = 0$: $y_1(x) = \sin x$ and $y_2(x) = \cos x$ will do.*
3. *Note $P_2 = 1$ and calculate $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -1$.*

Therefore (2.6) *gives the Green's function as*

$$g(x,\xi) = \begin{cases} -\sin\xi\cos x & 0 < \xi < x < \frac{\pi}{2}, \\ -\cos\xi\sin x & 0 < x < \xi < \frac{\pi}{2}. \end{cases} \tag{2.8}$$

By (2.5), *the solution of the BVP* (2.7) *is then given by*

$$y(x) = \int_0^{\frac{\pi}{2}} g(x,\xi) f(\xi)\,\mathrm{d}\xi. \tag{2.9}$$

Example 2.4.: Nonexistence/nonuniqueness of solution

Here we consider the same ODE as in Example 2.3 *but with modified boundary conditions, namely*

$$y''(x) + y(x) = f(x) \quad \textit{for} \quad 0 < x < \frac{\pi}{2}, \tag{2.10a}$$

subject to

$$y(0) = 0 = y'\left(\frac{\pi}{2}\right). \tag{2.10b}$$

The problem here is that $y_1(x) = \sin(x)$ satisfies both *boundary conditions* (2.10b), *and it is impossible to find* linearly independent *y_1 and y_2 satisfying one boundary condition each. The construction that led to* (2.4) *therefore fails.*

However, from the discussion in §1.4, *we know that* any *solution of* (2.10a) *can be written in the form "particular integral + complementary function", that is,*

$$y(x) = \underbrace{c_1(x) y_1(x) + c_2(x) y_2(x)}_{PI} + \underbrace{\alpha y_1(x) + \beta y_2(x)}_{CF}, \tag{2.11}$$

where, as before,

$$c_1(x) = -\int_x^{\pi/2} f(\xi) y_2(\xi)\,\mathrm{d}\xi, \qquad c_2(x) = -\int_0^x f(\xi) y_1(\xi)\,\mathrm{d}\xi, \tag{2.12}$$

and α, β are arbitrary constants. Here we use variation of parameters just to find the particular integral: we have not yet attempted to apply the boundary conditions. Given the condition (1.28) *satisfied by c_1 and c_2, we can easily calculate*

$$y'(x) = \left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] + \left[\alpha y_1'(x) + \beta y_2'(x)\right]. \tag{2.13}$$

Now we impose the boundary conditions (2.10b). *Using the particular forms $y_1(x) = \sin x$ and $y_2(x) = \cos x$ and the conditions $c_2(0) = 0 = c_1(\pi/2)$, we calculate*

$$y(0) = \beta \qquad \text{and} \qquad y'(\pi/2) = -\beta - c_2(\pi/2), \tag{2.14}$$

and substitution into (2.10b) *gives $\beta = 0$ and $c_2(\pi/2) = 0$, i.e.*

$$\int_0^{\pi/2} f(\xi)\sin(\xi)\,\mathrm{d}\xi = 0. \tag{2.15}$$

The BVP (2.10) *has* no solution *unless f satisfies the* solvability condition (2.15). *If* (2.15) is *satisfied, then the solution of* (2.10) *exists but is not unique, since the value of α in* (2.13) *remains arbitrary.*

2.2 Analogy with linear algebra

The difficulty encountered in Example 2.4 is reminiscent of a difficulty that can occur in the solution of systems of linear equations. Consider the homogeneous and inhomogeneous problems

$$A\mathbf{x} = \mathbf{0}, \tag{$\mathcal{H}$}$$

$$A\mathbf{x} = \mathbf{b}, \tag{$\mathcal{N}$}$$

where $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x}$, $\mathbf{b} \in \mathbb{R}^n$. If A is invertible (i.e. has nonzero determinant), then ($\mathcal{H}$) has only the trivial solution $\mathbf{x} = \mathbf{0}$. In this case, ($\mathcal{N}$) has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

However, if ($\mathcal{H}$) has a solution $\mathbf{x} = \mathbf{x}_1 \neq \mathbf{0}$, then A must be singular and, for general $\mathbf{b}$, the solution of ($\mathcal{N}$) does not exist. If for some particular choice of $\mathbf{b}$ a solution of ($\mathcal{N}$) for $\mathbf{x}$ *does* exist, then it is non-unique, since any vector of the form $\mathbf{x} + \alpha\mathbf{x}_1$ is also a solution. In summary, if the homogeneous problem admits non-trivial solutions, then the inhomogeneous problem has either no solution or an infinite number of solutions, but how can we determine which it is?

One option is to note that (since the row and column ranks of A are equal) A^* is singular if and only if A is, where A^* here denotes the transpose of A. Thinking of A as a linear transformation on $\mathbb{R}^n$, we can also identify A^* as the corresponding *adjoint* transformation, in the sense that

$$\langle A\mathbf{x}, \mathbf{w}\rangle \equiv \langle \mathbf{x}, A^*\mathbf{w}\rangle\,, \tag{2.16}$$

where $\langle \mathbf{x}, \mathbf{w}\rangle \equiv \mathbf{x}\cdot\mathbf{w}$ denotes the usual Cartesian inner product.

If ($\mathcal{H}$) admits non-trivial solutions for $\mathbf{x}$, then the corresponding *adjoint problem*

$$A^*\mathbf{w} = \mathbf{0}, \tag{$\mathcal{H}^*$}$$

also admits non-trivial solutions for $\mathbf{w}$. By taking the inner product of $(\mathcal{N})$ with $\mathbf{w}$ and using (2.16), we deduce that a necessary condition for $(\mathcal{N})$ to be solvable is that

$$\langle \mathbf{b}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \text{ satisfying } (\mathcal{H}^*). \tag{2.17}$$

It can be shown that the solvability condition (2.17) is also sufficient, and hence that $(\mathcal{N})$ is solvable for $\mathbf{x}$ if and only if $\mathbf{b}$ is orthogonal to every vector in the kernel of A^*. Indeed, this is really just a re-phrasing of the standard result for finite-dimensional inner product spaces $\mathrm{im}(A) = \ker(A^*)^\perp$: "the image of A is the orthogonal complement of the kernel of A^*".

Collecting all the above together, we see that there are three alternative outcomes for the inhomogeneous problem $(\mathcal{N})$: there is either a unique solution, no solution, or an infinite number of solutions. These can be summarised as follows in the so-called Fredholm Alternative Theorem (FAT).

Theorem 2.1. ***Fredholm Alternative ($\mathbb{R}^n$ version)***

Exactly one of the following possibilities occurs.

1. *The homogeneous equation* $(\mathcal{H})$ $A\mathbf{x} = \mathbf{0}$ *has only the zero solution. In this case the solution of* $(\mathcal{N})$ $A\mathbf{x} = \mathbf{b}$ *is unique.*
2. *The homogeneous equation* $(\mathcal{H})$ $A\mathbf{x} = \mathbf{0}$ *admits non-trivial solutions, and so does* $(\mathcal{H}^*)$ $A^*\mathbf{w} = \mathbf{0}$. *In this case there are two sub-possibilities:*

   2(a) if $\langle \mathbf{b}, \mathbf{w} \rangle = 0$ *for all* $\mathbf{w}$ *satisfying* $(\mathcal{H}^*)$*, then* $(\mathcal{N})$ *has a non-unique solution;*

   2(b) otherwise, $(\mathcal{N})$ *has no solution.*

Now let us see how Theorem 2.1 relates to Examples 2.3 and 2.4.

Example 2.3 corresponds to alternative 1 of Theorem 2.1. The homogeneous problem $\mathfrak{L}y = y'' + y = 0$, subject to the boundary conditions $y(0) = y(\pi/2) = 0$ has no non-trivial solutions. In this case, we are able to find two linearly independent solutions satisfying $y_1(0) = 0 = y_2(\pi/2)$, and the construction in §2.1 provides a unique solution to the inhomogeneous problem $\mathfrak{L}y = f$ for arbitrary f.

In Example 2.4, the homogeneous problem $\mathfrak{L}y = y'' + y$, subject to the new boundary conditions $y(0) = y'(\pi/2) = 0$ does admit a non-trivial solution $y_1(x) = \sin x$. In this case, it is impossible to find two linearly independent solutions satisfying $y_1(0) = 0 = y_2'(\pi/2)$, and the construction of the Green's function given in §2.1 fails. This corresponds to alternative 2 of Theorem 2.1: the inhomogeneous problem $\mathfrak{L}y = f$ either has (2a) a non-unique solution, if f satisfies the solvability condition (2.15); or (2b) no solution, if (2.15) is not satisfied. However, to understand how (2.15) relates to (2.17), we need to define the adjoint of a differential operator.

2.3 Adjoint operator and boundary conditions

We define the *inner product* between two (suitably smooth) functions defined on an interval $[a, b]$ by

$$\langle u, v \rangle := \int_a^b u(x)\overline{v(x)}\,\mathrm{d}x, \tag{2.18}$$

where the overbar denotes complex conjugate. Where it is clear that we are dealing with real-valued functions, we will generally drop the overbar for simplicity.

In general, for a given linear operator $\mathfrak{L}$, the corresponding *adjoint operator* $\mathfrak{L}^*$ is defined by the inner product relation

$$\langle \mathfrak{L}y, w\rangle = \langle y, \mathfrak{L}^* w\rangle \tag{2.19}$$

for all y, w in a suitable inner product space. To determine the adjoint of a linear differential operator, one needs (i) to move the derivatives of the operator from y to w, using integration by parts, and (ii) to set the boundary conditions to ensure that all boundary terms vanish.

Example 2.5. *Let*

$$\mathfrak{L}y = y'' \tag{2.20}$$

for $a \le x \le b$. We use integration by parts to calculate

$$\begin{aligned}\langle \mathfrak{L}y, w\rangle = \int_a^b y''(x)w(x)\,\mathrm{d}x &= -\int_a^b y'(x)w'(x)\,\mathrm{d}x + \left[y'(x)w(x)\right]_a^b \\ &= \int_a^b y(x)w''(x)\,\mathrm{d}x + \left[y'(x)w(x) - y(x)w'(x)\right]_a^b \equiv \langle y, \mathfrak{L}^* w\rangle\,. \end{aligned} \tag{2.21}$$

To enforce this identity, we identify the integrand in (2.21) *with $\mathfrak{L}^*w$, i.e.*

$$\mathfrak{L}^* w = w''. \tag{2.22}$$

We note in this case that $\mathfrak{L} \equiv \mathfrak{L}^*$: the operator is self-adjoint.

We must also ensure that the boundary terms in (2.21) *vanish. Thus, the boundary conditions imposed on y imply corresponding* adjoint boundary conditions *to be imposed on w.*

As a first illustration, suppose that y satisfies the boundary conditions

$$\mathfrak{B}_1 y = y(a) = 0, \qquad \mathfrak{B}_2 y = y(b) = 0. \tag{BC1}$$

Then the boundary terms in (2.21) *reduce to*

$$y'(b)w(b) - y'(a)w(a) - y(b)w'(b) + y(a)w'(a) = y'(b)w(b) - y'(a)w(a) \tag{2.23}$$

and, to ensure that this vanishes for all $y'(a)$ and $y'(b)$, we deduce the adjoint boundary conditions

$$\mathfrak{B}_1^* w = w(a) = 0, \qquad \mathfrak{B}_2^* w = w(b) = 0. \tag{BC1*}$$

Alternatively, if we impose the more complicated boundary condtions

$$\mathfrak{B}_1 y = y'(a) = 0, \qquad \mathfrak{B}_2 y = 3y(a) - y(b) = 0 \tag{BC2}$$

on y, then the boundary terms in (2.21) *may be expressed in the form*

$$y'(b)w(b) - y'(a)w(a) - y(b)w'(b) + y(a)w'(a) = y(a)w'(a) - 3y(a)w'(b) + y'(b)w(b). \tag{2.24}$$

To ensure that this expression vanishes for all $y(a)$ and $y'(b)$, we deduce the adjoint boundary conditions

$$\mathfrak{B}_1^* w = w'(a) - 3w'(b) = 0, \qquad \mathfrak{B}_2^* w = w(b) = 0. \tag{BC2*}$$

Example 2.5 illustrates the following general points about the adjoint of a linear differential operator.

(i) We can calculate the adjoint $\mathfrak{L}^*$ of an operator $\mathfrak{L}$ without worrying about the boundary conditions.

(ii) If $\mathfrak{L}^* = \mathfrak{L}$, then the operator $\mathfrak{L}$ is *self-adjoint.*

(iii) When $\mathfrak{L}$ is supplemented with homogeneous boundary conditions to give a problem of the form $(\mathfrak{L} + \text{BC})$, then corresponding *adjoint boundary conditions* are generated to gave an *adjoint problem* $(\mathfrak{L}^* + \text{BC}^*)$.

(iv) If $\mathfrak{L} = \mathfrak{L}^*$ and $\text{BC} = \text{BC}^*$ then the problem is said to be *fully self-adjoint* (as in the case (BC1) above).

(v) As illustrated by (BC2) and (BC2*), it is possible for the *operator* to be self-adjoint but the boundary conditions not to be (sometimes this case is called "formally self-adjoint").

By following through the integration by parts procedure, one can find a general form for the adjoint operator:

$$\mathfrak{L}y = P_2 y'' + P_1 y' + P_0 y \tag{2.25a}$$
$$\Leftrightarrow \mathfrak{L}^* w = (P_2 w)'' - (P_1 w)' + P_0 w. \tag{2.25b}$$

One can easily check that an analagous procedure works for higher-order operators: to find the adjoint, move all the coefficients inside the derivatives, and switch the sign of any odd-ordered derivatives. Using (2.25), we calculate

$$\begin{aligned} w\mathfrak{L}y - y\mathfrak{L}^* w &= w\left[P_2 y'' + P_1 y' + P_0 y\right] - y\left[(P_2 w)'' - (P_1 w)' + P_0 w)\right] \\ &= \left[P_2 w y' - (P_2 w)' y + P_1 w y\right]' \end{aligned} \tag{2.26}$$

and therefore

$$\langle \mathfrak{L}y, w\rangle - \langle y, \mathfrak{L}^* w\rangle = \left[P_2 w y' - (P_2 w)' y + P_1 w y\right]_a^b. \tag{2.27}$$

Given appropriate homogeneous boundary conditions for y, we can deduce the corresponding adjoint boundary conditions for w by setting the final integrated term in (2.27) equal to zero. This integrated term must then be expressible in the form

$$\begin{aligned} \langle \mathfrak{L}y, w\rangle - \langle y, \mathfrak{L}^* w\rangle &= \left[P_2 w y' - (P_2 w)' y + P_1 w y\right]_a^b \\ &= (K_1^* w)(\mathfrak{B}_1 y) + (K_2^* w)(\mathfrak{B}_2 y) + (K_1 y)(\mathfrak{B}_1^* w) + (K_2 y)(\mathfrak{B}_2^* w), \end{aligned} \tag{2.28}$$

where $K_1 y$ and $K_2 y$ are linearly independent of $\mathfrak{B}_1 y$ and $\mathfrak{B}_2 y$, and likewise $K_1^* w$ and $K_2^* w$ are linearly independent of $\mathfrak{B}_1^* w$ and $\mathfrak{B}_2^* w$. For example, in the case of (BC2) from Example 2.5, we can write

$$\left[y'w - yw'\right]_a^b = \underbrace{-w(a)}_{K_1^* w}\underbrace{y'(a)}_{\mathfrak{B}_1 y} + \underbrace{w'(b)}_{K_2^* w}\underbrace{\big(3y(a) - y(b)\big)}_{\mathfrak{B}_2 y} + \underbrace{y(a)}_{K_1 y}\underbrace{\big(w'(a) - 3w'(b)\big)}_{\mathfrak{B}_1^* w} + \underbrace{y'(b)}_{K_2 y}\underbrace{w(b)}_{\mathfrak{B}_2^* w}. \tag{2.29}$$

We then see how the given boundary conditions $\mathfrak{B}_1 y = \mathfrak{B}_2 y = 0$ enforce the corresponding ajoint conditions $\mathfrak{B}_1^* w = \mathfrak{B}_2^* w = 0$.

Expanding out $\mathfrak{L}^*$ in (2.25), we find

$$\mathfrak{L}^* w = P_2 w'' + \left(2P_2' - P_1\right) w' + \left(P_2'' - P_1' + P_0\right) w, \tag{2.30}$$

and, by comparing with $\mathfrak{L}$, we deduce that $\mathfrak{L}$ is self-adjoint if and only if $P_1 = P_2'$. If so then, setting $P_2 = -p$, $P_1 = -p'$ and $P_0 = q$, we can write $\mathfrak{L}$ as

$$\mathfrak{L}y = -\left(py'\right)' + qy, \tag{2.31}$$

which is the most general formally self-adjoint second-order differential operator.

Finally, we are ready for a statement (without proof!) of the Fredholm Alternative Theorem (FAT) for linear differential operators of the form (2.25a).

Theorem 2.2. ***Fredholm Alternative (linear ODE version)***
We consider the linear homogeneous and inhomogeneous ODEs

$$\mathfrak{L}y = 0, \tag{H}$$
$$\mathfrak{L}y = f \not\equiv 0, \tag{N}$$

for $0 < x < a$*, supplemented by linear homogeneous boundary conditions of the form*

$$\left.\begin{aligned} \mathfrak{B}_1 y = \alpha_1 y(a) + \alpha_2 y'(a) + \beta_1 y(b) + \beta_2 y'(b) = 0, \\ \mathfrak{B}_2 y = \alpha_3 y(a) + \alpha_4 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = 0, \end{aligned}\right\} \tag{BC}$$

(with $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ *and* $(\alpha_3, \alpha_4, \beta_3, \beta_4)$ *linearly independent). We also define the homogeneous adjoint equation*

$$\mathfrak{L}^* w = 0, \tag{H*}$$

and corresponding adjoint boundary conditions (BC*)*, computed as described above.*

Exactly one of the following possibilities occurs.

1. *The homogeneous problem* (H+BC) *has only the zero solution. In this case the solution of* (N+BC) *is unique.*

2. *The homogeneous problem* (H+BC) *admits non-trivial solutions, and so does* (H*+BC*)*. In this case there are two sub-possibilities:*

   2(a) if $\langle f, w\rangle = 0$ *for all* w *satisfying* (H*+BC*)*, then* (N+BC) *has a non-unique solution;*

   2(b) otherwise, (N+BC) *has no solution.*

Exercise: Demonstrate that Examples 2.3 and 2.4 are consistent with FAT.

2.4 Inhomogeneous boundary conditions and FAT

Our statement of the Fredholm Alternative in Theorem 2.2 concerns ODEs subject to homogeneous boundary conditions. A little more work is required to apply the results to problems with inhomogeneous boundary conditions. Suppose that we replace the boundary conditions (BC) with

$$\left.\begin{aligned} \mathfrak{B}_1 y = \alpha_1 y(a) + \alpha_2 y'(a) + \beta_1 y(b) + \beta_2 y'(b) = \gamma_1, \\ \mathfrak{B}_2 y = \alpha_3 y(a) + \alpha_4 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = \gamma_2, \end{aligned}\right\} \tag{NBC}$$

for some constants γ_1 and γ_2. First we note that the condition for a unique solution of the modified problem (N+NBC) is exactly the same as case 1 in Theorem 2.2. To see this, let $v(x)$ be any twice differentiable function that satisfies the conditions (NBC): it need not be a solution of the ODE (H). We can then make the boundary conditions homogeneous by subtracting off $v(x)$, i.e. defining $\tilde{y}(x) = y(x) - v(x)$, so that $\tilde{y}$ satisfies the problem

$$\mathfrak{L}\tilde{y} = f - \mathfrak{L}v = \tilde{f}, \tag{2.32}$$

say, with homogeneous boundary conditions $\mathfrak{B}_1\tilde{y} = 0 = \mathfrak{B}_2\tilde{y}$. We can now apply FAT to deduce that there is a unique solution for $\tilde{y}$, and therefore also for y, if and only if the homogeneous problem (H+BC) has no non-trivial solutions.

If (H+BC) *does* admit non-trivial solutions, then we can apply Case 2 of FAT to deduce that there is no solution unless $\langle \tilde{f}, w\rangle = 0$ for all w in the kernel of (H*+BC*), in which case the solution is non-unique. The solvability condition in this case may be expressed as

$$\begin{aligned} 0 &= \langle \tilde{f}, w\rangle = \langle f, w\rangle - \langle \mathfrak{L}v, w\rangle \\ &= \langle f, w\rangle - \langle v, \mathfrak{L}^* w\rangle - (K_1^* w)(\mathfrak{B}_1 v) - (K_2^* w)(\mathfrak{B}_2 v) - (K_1 v)(\mathfrak{B}_1^* w) - (K_2 v)(\mathfrak{B}_2^* w), \end{aligned} \tag{2.33}$$

when we apply the decomposition (2.28). Since w satisfies the homogeneous adjoint problem (H*+BC*), the right-hand side of (2.33) only involves functions of v that are known by the given boundary conditions $\mathfrak{B}_1 v = \gamma_1$ and $\mathfrak{B}_2 v = \gamma_2$, and we thus deduce the solvability condition

$$\langle f, w\rangle = \gamma_1 K_1^* w + \gamma_2 K_2^* w. \tag{2.34}$$

We note that (2.34) does not involve the function v that was introduced to make the boundary conditions homogeneous, and indeed one can obtain (2.34) directly without first simplifying the boundary conditions. As above, let w be any solution of the homogeneous adjoint problem (H*+BC*), and take the inner product of (N) with w to get

$$\langle f, w\rangle = (K_1^* w)(\mathfrak{B}_1 y) + (K_2^* w)(\mathfrak{B}_2 y) + (K_1 y)(\mathfrak{B}_1^* w) + (K_2 y)(\mathfrak{B}_2^* w). \tag{2.35}$$

Application of the relevant boundary conditions then immediately produces (2.34).

In summary, when the boundary conditions are inhomogeneous, we have shown the following.

- The condition for a unique solution to exist (Case 1 of FAT) is unaffected.
- For cases where there is not a unique solution, the solvability condition is still obtained by taking the inner product with a non-trivial solution w of the homogeneous adjoint problem. Now the boundary terms produced by integration by parts do not disappear identically but do only involve quantites that are in principle known from the specified boundary conditions.

Example 2.6. Solve $y''(x) = f(x)$ on $0 < x < 1$ with $y(0) = 0$ and $y'(1) = 7$.

Here $\mathfrak{L}$ is self-adjoint, and the homogeneous adjoint problem is $L^* w = w'' = 0$ with $w(0) = w'(1) = 0$. This only has the trivial solution $w \equiv 0$, so original BVP has a unique solution for any $f(x)$.

For this simple ODE, we can construct the solution straightforwardly as follows. First let's make the boundary conditions homogeneous by subtracting off a suitable solution of the homogeneous problem, namely $u(x) = 7x$. Thus $\tilde{y} = y - u$ satisfies

$$\tilde{y}''(x) = f(x) \quad \text{on } 0 < x < 1, \qquad \tilde{y}(0) = 0 = \tilde{y}'(1). \tag{2.36}$$

We can easily integrate this simple ODE directly; alternatively, the Green's function for this problem is easily found to be given by

$$g(x,\xi) = \begin{cases} -x & 0 < x < \xi < 1, \\ -\xi & 0 < \xi < x < 1, \end{cases} \tag{2.37}$$

and the solution of the BVP is then

$$y(x) = 7x + \int_0^1 g(x,\xi) f(\xi)\, \mathrm{d}\xi. \tag{2.38}$$

Example 2.7. Solve the same ODE $y'' = 3$ with boundary conditions $y'(0) = 0$ and $y'(1) = \beta$.

The problem is again self-adjoint. The homogeneous adjoint problem $w'' = 0$, $w'(0) = 0 = w'(1)$ *has the non-trivial solution* $w = 1$ *(or any multiple thereof). Now calculate*

$$\begin{aligned} \langle y'', w\rangle &= \langle f, w\rangle \\ \Rightarrow \quad \int_0^1 y''(x)\,\mathrm{d}x &= \int_0^1 3\,\mathrm{d}x = 3 \\ \Rightarrow \quad \big[y'\big]_0^1 &= \beta = 3 \end{aligned} \tag{2.39}$$

Thus if $\beta \neq 3$*, we have a contradiction and no solution exists, while if* $\beta = 3$*, we have a non-unique solution.*

Example 2.8. When is the BVP

$$y''(x) + y(x) = f(x) \text{ for } 0 < x < \frac{\pi}{2}, \qquad y(0) = 1, \qquad y'\left(\frac{\pi}{2}\right) = 0 \tag{2.40}$$

solvable for y?

This is a very slightly altered version of Example 2.4*. The problem is again self-adjoint, and we know that* $w(x) = \sin x$ *satisfies the homogeneous problem. So take the inner product with* $\sin x$ *and integrate by parts to get*

$$\int_0^{\pi/2} \big(y''(x) + y(x)\big)\sin x\,\mathrm{d}x \equiv [y'(x)\sin x - y(x)\cos x]_0^{\pi/2} = 1, \tag{2.41}$$

when we evaluate the right-hand side using the given boundary conditions. The solvability condition in this case is therefore

$$\int_0^{\pi/2} f(x)\sin x\,\mathrm{d}x = 1. \tag{2.42}$$

Chapter 3

The Green's function

These lecture notes are based on material written by Derek Moulton and Peter Howell. *Please send any corrections or comments to* Renaud Lambiotte.

3.1 Properties of the Green's function

We recall from §2.1 that the solution of the second-order inhomogeneous ODE

$$\mathfrak{L}y = P_2 y'' + P_1 y' + P_0 y = f \quad a < x < b, \tag{3.1}$$

subject to the simple boundary conditions

$$y(a) = 0 = y(b), \tag{3.2}$$

may be written as

$$y(x) = \int_a^b g(x, \xi) f(\xi) \, \mathrm{d}\xi, \tag{3.3}$$

where the *Green's function* is given by

$$g(x, \xi) = \begin{cases} \dfrac{y_1(\xi) y_2(x)}{P_2(\xi) W(\xi)} & a < \xi < x < b, \\ \dfrac{y_2(\xi) y_1(x)}{P_2(\xi) W(\xi)} & a < x < \xi < b. \end{cases} \tag{3.4}$$

Here y_1 and y_2 are linearly independent solutions of the homogeneous ODE $\mathfrak{L}y = 0$ satisfying one boundary condition each, i.e. $y_1(a) = 0 = y_2(b)$.

We note that the construction of g depends only on the solution of the homogeneous ODE (3.1) and the imposed boundary conditions: it does not depend at all on f. If we are given the linear operator $\mathfrak{L}$ and suitable boundary conditions, in principle we can solve for g "once and for all", and then use (3.3) to give us the solution for arbitrary right-hand side f.Thus the Green's function provides a kind of *inverse* to the differential operator $\mathfrak{L}$ in the sense that $\mathfrak{L}y = f$ (plus suitable boundary conditions) is equivalent to $y = \mathfrak{L}^{-1} f$, with $\mathfrak{L}^{-1}$ defined by (3.3).

It is easily verified that the Green's function defined by (3.4) has the following properties.

(i) $g(x,\xi)$ (viewed as a function of x) satisfies the homogeneous ODE (H) everywhere other than the special point $x=\xi$, i.e.

$$\mathfrak{L}_x g = P_2(x)g_{xx} + P_1(x)g_x + P_0(x)g = 0 \tag{3.5}$$

in $a<x<\xi<b$ and in $a<\xi<x<b$. (Note here for clarity the subscript x indicates that the derivatives are with respect to x rather than ξ.)

(ii) $g(x,\xi)$ (again viewed as a function of x) satisfies the same boundary conditions as y, i.e. $g(a,\xi)=g(b,\xi)=0$.

(iii) $g(x,\xi)$ is continuous at $x=\xi$, i.e.

$$\lim_{x\to\xi+} g(x,\xi) = \lim_{x\to\xi-} g(x,\xi). \tag{3.6}$$

However, the first derivative of g is discontinuous, with a jump given by

$$\lim_{x\to\xi+} g_x(x,\xi) - \lim_{x\to\xi-} g_x(x,\xi) = \frac{1}{P_2(\xi)}\,. \tag{3.7}$$

3.2 Reverse-engineering g

Suppose we start from the form of the solution (3.3), and try to work out what properties g must have to make (3.3) satisfy the given BVP. Considering first the boundary conditions (3.2), we get

$$\int_a^b g(a,\xi)f(\xi)\,\mathrm{d}\xi = 0 = \int_a^b g(b,\xi)f(\xi)\,\mathrm{d}\xi \quad \text{for all functions } f(\xi), \tag{3.8}$$

which indeed leads us to property (ii) above.

Second, let us substitute (3.3) into the ODE (3.1), assuming (a risky assumption as we will see) that the x-derivatives may be passed through the integral sign so that

$$\mathfrak{L}\int_a^b g(x,\xi)f(\xi)\,\mathrm{d}\xi = \int_a^b \mathfrak{L}_x g(x,\xi)f(\xi)\,\mathrm{d}\xi = f(x). \tag{3.9}$$

To make this work, we apparently need g to satisfy

$$\mathfrak{L}_x g(x,\xi) = \delta(x-\xi), \tag{3.10}$$

where δ is a function (if one exists) with the property that

$$\int_a^b \delta(x-\xi)\phi(\xi)\,\mathrm{d}\xi \equiv \phi(x), \tag{3.11}$$

for any (suitably smooth) function ϕ. The property (3.11) is known as the *sifting property* — δ is somehow supposed to pick out the value of the test function ϕ at a specific point. Luckily, a function with the property (3.11) does exist (though it isn't really a function) and is called the Dirac delta function.

3.3 The delta function

3.3.1 Definition

The delta function may be thought of as describing a point source, and may be characterised by the properties

$$\delta(x) = 0 \quad \text{for all } x \neq 0, \tag{3.12a}$$

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1. \tag{3.12b}$$

The first property (3.12a) captures the notion of a point function. The second property (3.12a) constrains the area under the graph (which you might think of as infinitely thin and infinitely high). An idealized unit point source at $x = 0$ is described by $\delta(x)$; a point source at some other point $x = \xi$ would be given by $\delta(x - \xi)$.

If a δ existed satisfying (3.12), then it would also have the desired sifting property (3.11). By property (3.12), for any $x \in (a, b)$ we can write

$$\int_a^b \delta(x-\xi)\phi(\xi)\,\mathrm{d}\xi = \int_{x-\epsilon}^{x+\epsilon} \delta(x-\xi)\phi(\xi)\,\mathrm{d}\xi, \tag{3.13}$$

where ϵ is an arbitrarily small positive parameter. For sufficiently smooth ϕ, we can thus approximate

$$\int_a^b \delta(x-\xi)\phi(\xi)\,\mathrm{d}\xi \sim \big[\phi(x) + O(\epsilon)\big] \int_{x-\epsilon}^{x+\epsilon} \delta(x-\xi)\,\mathrm{d}\xi, \tag{3.14}$$

and by letting $\epsilon \to 0$, we find that the right-hand side is equal to $\phi(x)$ as required.

3.3.2 Approximating the delta function

The problem is that no classical function satisfies both properties (3.12) (any function that is non-zero only at a point either is not integrable or integrates to zero). One way around this difficulty is to replace δ by an approximating sequence of increasingly narrow functions with normalised area, i.e. $\delta_n(x)$ where

$$\int_{-\infty}^{\infty} \delta_n(x)\,\mathrm{d}x = 1 \quad \text{for all } n = 1,\,2,\,\ldots, \tag{3.15a}$$

$$\lim_{n\to\infty} \delta_n(x) = 0 \quad \text{for all } x \neq 0. \tag{3.15b}$$

One possibility is "hat" functions of the form

$$\delta_n(x) = \begin{cases} 0 & \text{for } |x| > 1/n, \\ n/2 & \text{for } |x| \leq 1/n. \end{cases} \tag{3.16}$$

It is easily verified that the sequence of functions $\delta_n(x)$ defined by (3.16) has the desired properties (3.15). As illustrated in figure 3.1, as n increases, $\delta_n(x)$ approaches a "spike", equal to zero everywhere apart from a neighbourhood of the origin but nevertheless with unit area under the graph.

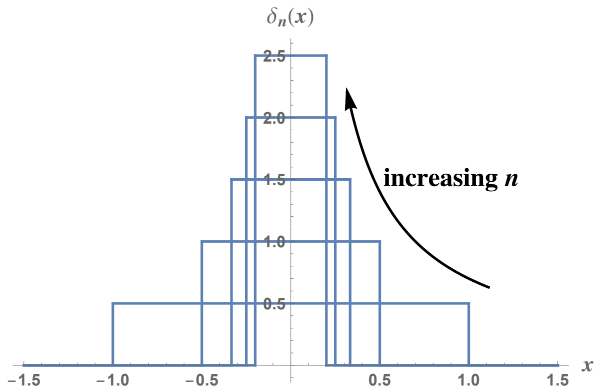


Figure 3.1: Hat functions defined by equation (3.16).

3.3.3 Properties of delta function

Approximating sequences like (3.16) can be used to establish various properties of the delta function.

Sifting property Let $\phi(x)$ be a smooth function, and $\Phi(x) = \int \phi(x)\,\mathrm{d}x$ its antiderivative. If we use the particular approximating sequence (3.16), then

$$\int_{-\infty}^{\infty} \delta_n(x-a)\phi(x)\,\mathrm{d}x = \int_{a-1/n}^{a+1/n} (n/2)\phi(x)\,\mathrm{d}x = \frac{n}{2}\left[\Phi(a+1/n) - \Phi(a-1/n)\right]. \tag{3.17}$$

Now letting $n \to \infty$ we get

$$\int_{-\infty}^{\infty} \delta_n(x-a)\phi(x)\,\mathrm{d}x \to \Phi'(a) = \phi(a). \tag{3.18}$$

Therefore δ does have the desired sifting property

$$\int_{-\infty}^{\infty} \delta(x-a)\phi(x)\,\mathrm{d}x \equiv \phi(a) \tag{3.19}$$

(for suitably smooth test functions ϕ) *if* we make the identification that

$$\int_{-\infty}^{\infty} \delta(x-a)\phi(x)\,\mathrm{d}x \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x-a)\phi(x)\,\mathrm{d}x. \tag{3.20}$$

This final identification (3.20) is *not valid* in the space of classical functions (the convergence of δ_n to δ is non-uniform) but it *does* hold for so-called *distributions*. Rather than trying to approximate δ with a classical function, instead, one defines it as a linear functional on the space of "test functions" $\mathcal{T}$:

$$\delta : \mathcal{T} \to \mathbb{R}, \tag{3.21a}$$

$$\delta : \phi(x) \mapsto \phi(0). \tag{3.21b}$$

See ASO Integral Transforms for more details about this more systematic approach.

Antiderivative of δ The antiderivative of the delta function is the so-called *Heaviside function*:

$$\int_{-\infty}^{x} \delta(s)\,\mathrm{d}s = H(x) := \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases} \tag{3.22}$$

(The value of $H(x)$ *at* $x = 0$ is indeterminate: it is sometimes taken to be 1 and sometimes taken to be 1/2.)

Note that (3.22) may be obtained by integrating the sequence (3.16) of approximating functions and showing that the limit is the Heaviside function, that is [**Exercise**]

$$\lim_{n\to\infty} \int_{-\infty}^{x} \delta_n(s)ds = H(x) \tag{3.23}$$

(with the same caveat as above about the validity of taking the limit through the integral). Alternatively, one can convince oneself that $H'(x) = 0$ for $x \neq 0$ but

$$\int_{-\infty}^{\infty} H'(x)\,\mathrm{d}x = \int_{-\epsilon}^{\epsilon} H'(x)\,\mathrm{d}x = \big[H\big]_{-\epsilon}^{\epsilon} = 1, \tag{3.24}$$

for any $\epsilon > 0$. Thus H' has the defining properties (3.12) of δ. Again, all of these arguments can be made more watertight using the theory of distributions.

3.4 Green's function via delta function

Now let us return to the problem of finding a Green's function $g(x,\xi)$ satisfying (3.10). We start by doing a very simple case with $\mathfrak{L}y = y''$ and $y(0) = 0 = y(1)$.

Example 3.9. Find $g(x,\xi)$ satisfying

$$g_{xx}(x,\xi) = \delta(x-\xi) \quad \text{for } 0 < x,\, \xi < 1, \tag{3.25a}$$

$$g(0,\xi) = 0 = g(1,\xi). \tag{3.25b}$$

Since its right-hand side is zero for $x \neq \xi$, we can easily integrate (3.25a) *to obtain the solution in $x < \xi$ and in $x > \xi$. By applying the boundary conditions* (3.25b) *we deduce that*

$$g(x,\xi) = \begin{cases} A(\xi)x & 0 < x < \xi < 1, \\ B(\xi)(1-x) & 0 < \xi < x < 1, \end{cases} \tag{3.26}$$

where A and B are two arbitrary functions of integration. To evaluate A and B we need to decide how to join the two solutions together across $x = \xi$. To do this, we integrate (3.25a) *across the singularity at $x = \xi$, that is,*

$$\int_{\xi-}^{\xi+} g_{xx}(x,\xi)\,\mathrm{d}x = \int_{\xi-}^{\xi+} \delta(x-\xi)\,\mathrm{d}x$$

$$\Rightarrow \quad \big[g_x(x,\xi)\big]_{x=\xi_-}^{x=\xi_+} = 1, \tag{3.27}$$

where ξ_- and ξ_+ refer to the limits as x approaches ξ from below and from above, respectively. From (3.27) *we deduce that* there must be a unit jump in the derivative of g across the point $x = \xi$. *The second condition to determine A and B is that g* itself must be continuous *(more about this below).*

So we impose the jump conditions

$$\big[g(x,\xi)\big]_{x=\xi_-}^{x=\xi_+} = 0, \qquad \big[g_x(x,\xi)\big]_{x=\xi_-}^{x=\xi_+} = 1, \tag{3.28}$$

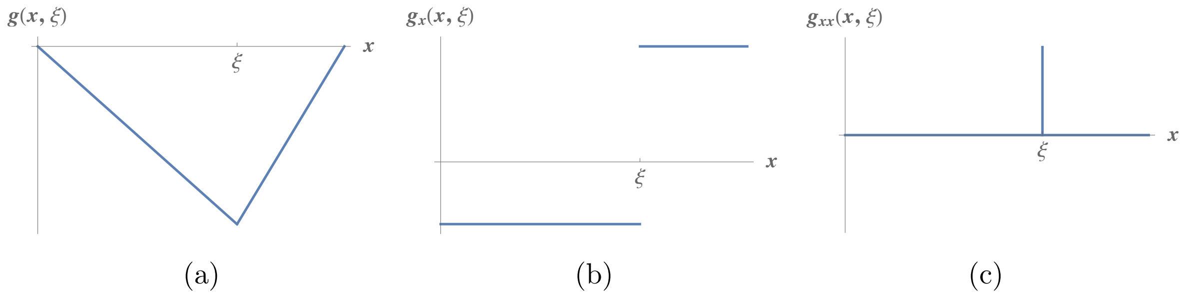


Figure 3.2: The Green's function defined by (3.30) and its first two derivatives (with $\xi = 0.65$).

to obtain

$$B(1-\xi) - A\xi = 0, \tag{3.29a}$$

$$-B - A = 1, \tag{3.29b}$$

and hence $A(\xi) = -(1-\xi)$, $B(\xi) = -\xi$, *and the Green's function in this case is given by*

$$g(x,\xi) = \begin{cases} -x(1-\xi) & 0 < x < \xi < 1, \\ -(1-x)\xi & 0 < \xi < x < 1. \end{cases} \tag{3.30}$$

The Green's function given by (3.30) is sketched in Figure 3.2(a). As we imposed, g satisfies the boundary conditions $g = 0$ at $x = 0$ and $x = 1$, and is continuous everywhere. The first x-derivative of g undergoes a unit jump across $x = \xi$, as shown in Figure 3.2(b), and in fact resembles a Heaviside function. It follows that the second derivative has a delta function at $x = \xi$, as illustrated in Figure 3.2(c).

Now: what would have happened if we didn't impose continuity of g across $x = \xi$? In that case g_x would have a delta function at $x = \xi$ and g_{xx} would have an even worse singularity ($\delta'(x)$, the derivative of the delta function, which is a well-defined distribution). So, continuity of g ensures that we only have a delta-function singularity at $x = \xi$ and nothing stronger.

We illustrate the approach more generally with a less trivial example.

Example 3.10. Find the Green's function for the problem

$$y''(x) + y(x) = f(x) \quad \text{for } 0 < x < \frac{\pi}{2}, \qquad y(0) = 0 = y\left(\frac{\pi}{2}\right). \tag{3.31}$$

So we have to solve

$$g_{xx} + g = \delta(x-\xi) \quad \textit{for } 0 < x,\, \xi < \frac{\pi}{2}, \qquad g(0,\xi) = 0 = g\left(\pi/2, \xi\right). \tag{3.32}$$

Since the right-hand side is zero for $x \neq \xi$, *we can find the solution on either side of the singularity and thus, applying the boundary conditions, we get*

$$g(x,\xi) = \begin{cases} A(\xi)\sin x & 0 < x < \xi < 1, \\ B(\xi)\cos x & 0 < \xi < x < 1. \end{cases} \tag{3.33}$$

To derive the appropriate jump conditions, we again integrate (3.32) *across* $x = \xi$, *as follows:*

$$\int_{\xi-}^{\xi+} g_{xx}(x,\xi) + g(x,\xi)\,\mathrm{d}x = \int_{\xi-}^{\xi+} \delta(x-\xi)\,\mathrm{d}x$$

$$\Rightarrow \quad \left[g_x(x,\xi)\right]_{x=\xi_-}^{x=\xi_+} = 1, \tag{3.34}$$

since the integral of g over the infinitesimal interval $[\xi_-, \xi_+]$ is zero. Again, we impose continuity of g itself (to eliminate any stronger singularity than δ) and thus we have exactly the same jump conditions (3.28) *as above.*

We can then easily solve for A and B and hence obtain

$$g(x,\xi) = \begin{cases} -\cos\xi \sin x & 0 < x < \xi < \frac{\pi}{2}, \\ -\sin\xi \cos x & 0 < \xi < x < \frac{\pi}{2}, \end{cases} \tag{3.35}$$

which agrees exactly with the solution found in Example 2.3 *using variation of parameters.*

We can generalise the above arguments to obtain the appropriate jump conditions for a general second-order linear operator of the form

$$\mathfrak{L}_x g(x,\xi) = P_2(x) g_{xx}(x,\xi) + P_1(x) g_x(x,\xi) + P_0(x) g(x,\xi) = \delta(x-\xi), \tag{3.36}$$

namely

$$\big[g(x,\xi)\big]_{x=\xi_-}^{x=\xi_+} = 0, \qquad \big[g_x(x,\xi)\big]_{x=\xi_-}^{x=\xi_+} = \frac{1}{P_2(\xi)}. \tag{3.37}$$

These conditions reproduce property (iii) of g noted in §3.4. Note once again the importance of P_2 being nonzero on the interval of interest.

Exercise: (i) derive (3.37); (ii) hence obtain the general formula (3.4) for g.

3.5 Generalisation

We now show how to generalise the concepts developed above to linear ODEs of arbitrary order and with more complicated (but still linear) boundary conditions. A general linear differential operator of order $n \in \mathbb{N}$ may be written as

$$\mathfrak{L}y(x) \equiv P_n(x) y^{(n)}(x) + P_{n-1}(x) y^{(n-1)}(x) + \cdots + P_1(x) y'(x) + P_0(x) y(x) \tag{3.38a}$$

$$\equiv P_n(x) \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + P_{n-1}(x) \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + P_1(x) \frac{\mathrm{d}y}{\mathrm{d}x} + P_0(x) y(x), \tag{3.38b}$$

for some given coefficients $P_0, \ldots, P_n$; (3.38) is equivalent to (1.2) when $n = 2$. As in Section 1, we assume that all P_i are at least continuous and that the coefficient P_n of the highest derivative is nonzero.

In terms of $\mathfrak{L}$, we define homogeneous and inhomogeneous linear ODEs of order n by

$$\mathfrak{L}y = 0, \tag{H}$$

$$\mathfrak{L}y = f \not\equiv 0. \tag{N}$$

In a general nth-order linear BVP, the ODE (N) is supplemented by n boundary conditions, each of which consists of a linear combination of y and its derivatives up to order $n-1$, evaluated at the boundary points $x = a$ and $x = b$. We will write these generically as

$$\mathfrak{B}_i y\Big|_{x=a,b} = \gamma_i, \quad i = 1, 2, \ldots, n, \tag{BCN}$$

where γ_i are constants and each $\mathfrak{B}_i$ is of the form

$$\mathfrak{B}_i y = \sum_{j=1}^{n} \left(\alpha_{ij} y^{(j-1)}(a) + \beta_{ij} y^{(j-1)}(b) \right), \tag{3.39}$$

for some constants α_{ij}, β_{ij} (which must be such that (BCN) comprises n independent equations). For instance, for a 2nd order system, with $n = 2$ the most general linear boundary conditions would have the form

$$\mathfrak{B}_1 y = \gamma_1, \qquad \mathfrak{B}_2 y = \gamma_2, \tag{3.40}$$

where

$$\mathfrak{B}_1 y = \alpha_{11} y(a) + \alpha_{12} y'(a) + \beta_{11} y(b) + \beta_{12} y'(b), \tag{3.41a}$$

$$\mathfrak{B}_2 y = \alpha_{21} y(a) + \alpha_{22} y'(a) + \beta_{21} y(b) + \beta_{22} y'(b), \tag{3.41b}$$

which is equivalent to (BC) in the homogeneous case where $\gamma_1 = \gamma_2 = 0$.

The boundary conditions (BCN) are *homogeneous* if $\gamma_i = 0$ for all i, in which case we have

$$\mathfrak{B}_i y\Big|_{x=a,b} = 0, \quad i = 1, 2, \ldots, n, \tag{BCH}$$

We assume that

the homogeneous problem (H + BCH) *has no non-trivial solutions*, $(\star)$

and then by FAT (Theorem 2.2), we expect the inhomogeneous problem to have a unique solution.

We can reduce the full problem (N+BCN) to one with homogeneous boundary conditions by subtracting off a suitable solution of the homogeneous problem (H). Let u be the solution of the problem (H + BCN), i.e.

$$\mathfrak{L}u(x) = 0, \quad a < x < b, \tag{3.42a}$$

$$\mathfrak{B}_i u\Big|_{x=a,b} = \gamma_i, \quad i = 1, 2, \ldots, n. \tag{3.42b}$$

It may be shown that, under the assumption $(\star)$, a unique solution for u exists. Then defining $\tilde{y} = y - u$, we see that $\tilde{y}$ satisfies the inhomogeneous ODE (N) but with the homogeneous boundary conditions (BCH). We may therefore focus on the problem (N+BCH).

3.6 Green's function for a general BVP

As above, we assume that the boundary conditions have been made homogeneous so we can consider a general nth-order BVP of the form (N+BCH), i.e.

$$\mathfrak{L}y(x) = \sum_{j=1}^{n} P_i(x) y^{(j-1)}(x) = f(x) \quad a < x < b, \tag{3.43a}$$

subject to n linearly independent homogeneous boundary conditions

$$\mathfrak{B}_i y\big|_{x=a,b} = \sum_{j=1}^{n} \left(\alpha_{ij} y^{(j-1)}(a) + \beta_{ij} y^{(j-1)}(b) \right) = 0, \quad i = 1, 2, \ldots, n. \tag{3.43b}$$

The corresponding problem for g is

$$\mathfrak{L}_x g(x,\xi) = \delta(x-\xi) \quad a < x,\, \xi < b, \tag{3.44a}$$

with boundary conditions

$$\mathfrak{B}_i g(x,\xi)\big|_{x=a,b} = 0, \quad i = 1,\, 2,\, \ldots,\, n. \tag{3.44b}$$

Since $\delta(x-\xi)$ is zero for $x \neq \xi$, we can in principle solve (3.44a) to get two distinct solutions in each of the sub-intervals $a < x < \xi$ and $a < \xi < x < b$. Given that $\mathfrak{L}$ is of order n, we will then have $2n$ degrees of freedom, i.e. n arbitrary integration constants. After applying the n independent boundary conditions (3.44b) we will have n remaining constants (actually functions of ξ) to determine. We therefore need n jump conditions at $x = \xi$, which come as above by integrating across $x = \xi$:

$$\int_{\xi-}^{\xi+} \left[P_n(x) \frac{\partial^n}{\partial x^n} g(x,\xi) + \cdots + P_0(x) g(x,\xi) \right] \mathrm{d}\xi = \int_{\xi-}^{\xi+} \delta(x-\xi)\, \mathrm{d}\xi = 1. \tag{3.45}$$

By integrating the first term on the left-hand side by parts, we obtain

$$\int_{\xi-}^{\xi+} \left[\big(P_{n-1}(x) - P_n'(x)\big) \frac{\partial^{n-1}}{\partial x^{n-1}} g(x,\xi) + \cdots + P_0(x) g(x,\xi) \right] \mathrm{d}\xi + \left[P_n(x) \frac{\partial^{n-1}}{\partial x^{n-1}} g(x,\xi) \right]_{x=\xi-}^{x=\xi+} = 1. \tag{3.46}$$

This equation is balanced by setting a jump condition on the $(n-1)$th derivative:

$$\left[\frac{\partial^{n-1}}{\partial x^{n-1}} g(x,\xi) \right]_{x=\xi-}^{x=\xi+} = 1/P_n(\xi), \tag{3.47}$$

and taking all lower derivatives to be continuous across $x = \xi$:

$$\big[g(x,\xi)\big]_{x=\xi-}^{x=\xi+} = \big[g_x(x,\xi)\big]_{x=\xi-}^{x=\xi+} = \cdots = \left[\frac{\partial^{n-2}}{\partial x^{n-2}} g(x,\xi) \right]_{x=\xi-}^{x=\xi+} = 0. \tag{3.48}$$

Once the Green's function is determined, following the above procedure, the solution to the BVP (3.43) is given by

$$y(x) = \int_a^b g(x,\xi) f(\xi)\, \mathrm{d}\xi. \tag{3.49}$$

It can be verified by direct substitution that (3.49) satisfies (3.44), provided (i) g satisfies (3.44) and (ii) it is legitimate to pass the differential operator $\mathfrak{L}$ through the integral in (3.49).

3.7 Green's function in terms of adjoint

There is an alternative way to construct the Green's function that eliminates the need for any dicey differentiating through integrals. Start from the ODE (3.43) and take an inner product with an *a priori* unknown function $G(x,\xi)$ on both sides of the equation to obtain

$$\langle \mathfrak{L}y, G \rangle = \langle G(x,\xi), f(x) \rangle = \int_a^b G(x,\xi) f(x)\, \mathrm{d}x. \tag{3.50}$$

(Note here the integration is with respect to x). Now, using the adjoint, we can write

$$\langle \mathfrak{L}y, G\rangle = \langle y, \mathfrak{L}^* G\rangle = \int_a^b y(x)\mathfrak{L}_x^* G(x,\xi)\,\mathrm{d}x, \tag{3.51}$$

provided G satisfies the *adjoint* boundary conditions corresponding to the boundary conditions (3.43b) imposed on y.

The idea now is to isolate y. This can be accomplished if G satisfies

$$\mathfrak{L}_x^* G(x,\xi) = \delta(x-\xi) \tag{3.52}$$

(as well as the corresponding adjoint boundary conditions), in which case the right-hand side of (3.51) leaves just $y(\xi)$, and we have the solution

$$y(\xi) = \int_a^b G(x,\xi) f(x)\,\mathrm{d}x. \tag{3.53}$$

To make comparison with our previous construction easier, we can switch the roles of x and ξ to write (3.53) in the equivalent form

$$y(x) = \int_a^b G(\xi,x) f(\xi)\,\mathrm{d}\xi = \int_a^b g(x,\xi) f(\xi)\,\mathrm{d}\xi. \tag{3.54}$$

We deduce that $G(\xi,x) \equiv g(x,\xi)$: we might say that G is the *transpose* of g (*cf* §2.2). In summary, if

- $g(x,\xi)$ satisfies $\mathfrak{L}_x g(x,\xi) = \delta(x-\xi)$ with homogeneous boundary conditions (BC),

then

- $g(\xi,x)$ satisfies the corresponding adjoint equation $\mathfrak{L}_x^* g(\xi,x) = \delta(x-\xi)$ and boundary conditions (BC*).

In particular,

- if ($\mathfrak{L}$ + BC) is fully self-adjoint, then g is *symmetric*, i.e. $g(x,\xi) \equiv g(\xi,x)$ (and *vice versa*).

3.8 FAT and Green's function

As we have seen, the Green's function approach apparently gives the explicit constructive solution to $\mathfrak{L}y = f$ with homogeneous boundary conditions (BC). So, if the Green's function approach works, i.e. if we can find g, then we have both existence and uniqueness of the solution y. But we know from FAT (Theorem 2.2) that, when there is a non-trivial solution of the homogeneous problem ($\mathfrak{L}y = 0$+BC), the solution of the inhomogeneous problem should either not exist or not be unique. Clearly, in such cases something must go wrong with the construction of the Green's function. So, let us suppose that ($\mathfrak{L}y = 0$+BC) admits non-zero solutions and thus, similarly, the adjoint problem ($\mathfrak{L}^* w = 0$+BC*) admits a non-zero solution w. Then, starting from the delta function formulation

$$\mathfrak{L}_x g(x,\xi) = \delta(x-\xi), \tag{3.55}$$

and taking the inner product with w, we get the solvability condition

$$0 = \langle G(x,\xi), \mathfrak{L}^* w(x) \rangle = \langle \mathfrak{L}_x G(x,\xi), w(x) \rangle = \langle \delta(x-\xi), w(x) \rangle = w(\xi) \tag{3.56}$$

which clearly does not hold since $w \not\equiv 0$ (by assumption).

Thus, in situations where ($\mathfrak{L}$+BC) has a non-trivial kernel, we can't construct the Green's function. (One can instead construct a so-called *modified Green's function*, and thus obtain the non-unique form of the solution in case 2(a) of FAT, but we won't go into details here).

Chapter 4

Eigenfunction expansions

These lecture notes are based on material written by Derek Moulton and Peter Howell. *Please send any corrections or comments to* Renaud Lambiotte.

4.1 Introduction

Our next approach to solving linear inhomogeneous BVP's is through an eigenfunction expansion. The idea is to exploit the linearity of the operator by constructing a solution as a *superposition* of a set of basis functions. In an n-dimensional vector space, we know that any set of n linearly independent vectors will serve as a basis, and then any other vector can always be expressed as a linear combination of the basis vectors. Here we are dealing with a space of functions, for example the set of twice continuously differentiable functions, which is an infinite dimensional vector space. So we expect to need an *infinite* set of linearly independent basis functions $\{y_n(x)\}_{n=1,2,\ldots}$ such that any "reasonable" function $\phi(x)$ can be written as a linear combination of these functions

$$\phi(x) = \sum_{n=0}^{\infty} c_n y_n(x), \tag{4.1}$$

for some choice of the coefficients c_n.

Many subtleties are involved in passing from finite to infinite dimensions. For example, we could ask ourselves whether the series (4.1) is guaranteed to converge, either pointwise or uniformly. Another key issue is the *completeness* of the set $\{y_i\}$, i.e. whether *every* "reasonable" function can be expressed as such a linear combination: indeed, what does "reasonable" mean? We are not going to get into such questions in this course (see Part B Functional Analysis if you're interested).

You have already seen in Fourier series the idea that any (periodic, piecewise continuous) function can be written as an infinite sum of sines and cosines; (4.1) generalises this idea to an alternative set of basis functions. Still, the issues that can be encountered in Fourier series (e.g. nonuniform convergence, Gibbs phenomenon) give some hints of the subtleties alluded to above.

4.2 Eigenfunctions of linear BVP

We will perform the decomposition (4.1) using a particular set of basis functions, namely solutions of the BVP

$$\mathfrak{L}y = \lambda y, \tag{E}$$

along with homogeneous boundary conditions. We observe that the trivial solution $y \equiv 0$ always satisfies (E). We are interested in particular values of the parameter λ, known as *eigenvalues*, for which (E) admits non-trivial solutions. For the linear BVPs considered in this course, it may be shown that the eigenvalues form a discrete countable set, say $\{\lambda_i\}_{i=1,,2,\ldots}$, and the corresponding non-trivial solutions $\{y_i(x)\}_{i=1,,2,\ldots}$ are known as *eigenfunctions*.

This approach is analogous to the linear algebra eigenproblem

$$A\mathbf{x} = \lambda\mathbf{x} \tag{4.2}$$

where $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$. The trivial solution $\mathbf{x} = \mathbf{0}$ always satisfies (4.2). Non-trivial solutions exist if and only if λ is an eigenvalue of A, and then $\mathbf{x}$ is the corresponding eigenvector. Moreover (provided A is diagnonalisable), the eigenvectors form a linearly independent set of n vectors, which can be used as a basis of $\mathbb{R}^n$.

We will show how to construct the solution of an inhomogeneous BVP as a linear combination of eigenfunctions y_i satisfying (E), by exploiting the following two fundamental properties.

Proposition 4.4. *The adjoint problem has the same eigenvalues as the original problem.*

Here, the "original problem" refers to the ODE (E) plus homogeneous boundary conditions (BC); the adjoint problem is

$$\mathfrak{L}^* w = \overline{\lambda} w, \tag{E*}$$

subject to the corresponding adjoint boundary conditions (BC*), where the overbar denotes complex conjugate. For the moment we do not assume that the eigenvalues are real, although we will see below that it is usually a reasonable assumption. The proposition (which we will not prove here: see §4.7.5 though) states that (E+BC) has non-trivial solutions for y if and only if (E*+BC*) has non-trivial solutions for w.

Proposition 4.5. *The eigenfunction and adjoint eigenfunction corresponding to distinct eigenvalues are orthogonal.*

That is, if $\mathfrak{L}y_j = \lambda_j y_j$ (so $\mathfrak{L}^* w_j = \overline{\lambda}_j w_j$) and $\mathfrak{L}y_k = \lambda_k y_k$ (so $\mathfrak{L}^* w_k = \overline{\lambda}_k w_k$), then

$$\langle y_j, w_k \rangle = 0 \quad \text{whenever} \quad \lambda_j \neq \lambda_k. \tag{4.3}$$

Proof. The proof is exactly as for matrix problems.

$$\begin{aligned} 0 &= \langle \mathfrak{L}y_j, w_k \rangle - \langle y_j, \mathfrak{L}^* w_k \rangle \\ &= \langle \lambda_j y_j, w_k \rangle - \left\langle y_j, \overline{\lambda}_k w_k \right\rangle \\ &= (\lambda_j - \lambda_k) \langle y_j, w_k \rangle . \end{aligned} \tag{4.4}$$

But $\lambda_j \neq \lambda_k$ so $\langle y_j, w_k \rangle = 0$. □

4.3 Inhomogeneous solution process

We are now in a position to construct the solution of the inhomogeneous BVP

$$\mathfrak{L}y = f, \tag{N}$$

subject to linear homogeneous boundary conditions (BC).

Step 1: Solve the eigenvalue problem (E+BC) to obtain the eigenvalue-eigenfunction pairs $\{(\lambda_j, y_j)\}_{j=1,2,\ldots}$.

Step 2: Solve the adjoint eigenvalue problem (E*+BC*) to obtain $\{(\lambda_j, w_j)\}_{j=1,2,\ldots}$ (easier since we already know λ_j).

Step 3: Assume a solution to the inhomogeneous problem (N+BC) of the form

$$y(x) = \sum_i c_i y_i(x). \tag{4.5}$$

To determine the coefficients c_i, start from (N) and take an inner product with w_k:

$$\begin{aligned}
\langle f, w_k \rangle &= \langle \mathfrak{L}y, w_k \rangle \\
&= \langle y, \mathfrak{L}^* w_k \rangle \\
&= \left\langle y, \overline{\lambda}_k w_k \right\rangle \\
&= \lambda_k \left\langle \sum_i c_i y_i, w_k \right\rangle \\
&= \lambda_k c_k \langle y_k, w_k \rangle . \qquad (4.6)
\end{aligned}$$

We can solve the last equality for the c_k, and we are done! Note that in the last step we have assumed that the inner product may be interchanged with the sum and used the orthogonality property (4.3).

4.4 Eigenfunction expansion and Green's function

Taking (4.6) one step further, we have

$$y(x) = \sum_{k=1}^{\infty} \frac{\langle f, w_k \rangle}{\lambda_k \langle y_k, w_k \rangle} y_k(x) \tag{4.7}$$

Let n_k denote the constant $\langle y_k, w_k \rangle$ (one could normalise y_k and w_k such that $n_k \equiv 1$), then:

$$\begin{aligned}
y(x) &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k n_k} \left(\int_a^b f(\xi) w_k(\xi) \, \mathrm{d}\xi \right) y_k(x) \\
&= \int_a^b \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k n_k} w_k(\xi) y_k(x) \right) f(\xi) \, \mathrm{d}\xi \\
&= \int_a^b g(x, \xi) f(\xi) \, \mathrm{d}\xi, \qquad (4.8)
\end{aligned}$$

where

$$g(x,\xi) = \sum_{k=1}^{\infty} \frac{w_k(\xi) y_k(x)}{\lambda_k n_k}. \tag{4.9}$$

Thus we have derived an alternative form for the Green's function. One can also derive (4.9) by solving the problem $\mathfrak{L}_x g(x,\xi) = \delta(x-\xi)$ (or indeed the problem $\mathfrak{L}^*_\xi g(x,\xi) = \delta(x-\xi)$) using an eigenfunction expansion. Note in particular that, if $\mathfrak{L}$ is self-adjoint, then $w_k \equiv y_k$ and therefore g is symmetric, as we already showed in §3.7.

4.5 Eigenfunction expansion and FAT

Note that there is a difficulty in calculating the coefficients c_k if one of the eigenvalues is zero. If $\lambda_k = 0$, then (4.6) cannot be solved for c_k. In this case either

(a) $\langle f, w_k \rangle = 0$, in which case c_k is arbitrary: the solution is non-unique; or

(b) $\langle f, w_k \rangle \neq 0$, in which case (4.6) is inconsistent: the solution does not exist.

We see that this behaviour is exactly in line with FAT. If $\lambda_k = 0$, then $\mathfrak{L}y = 0$ has a non-trivial solution y_k, and likewise $\mathfrak{L}^* w_k = 0$ has a non-trivial solution w_k. Then case (a) and (b) above correspond precisely to 2(a) and 2(b) of Theorem 2.2.

4.6 Inhomogeneous boundary conditions

In the construction in §4.3 we assumed homogeneous boundary conditions. In the general case of an inhomogeneous system with inhomogenous boundary conditions,

$$\mathfrak{L}y = f, \qquad \mathfrak{B}_i y = \gamma_i \quad (i = 1, 2, \ldots, n), \tag{4.10}$$

one possibility is to make the boundary conditions homogeneous by subtracting a suitable solution of the homogeneous ODE, as described in §3.5. That is, we write $\tilde{y} = y - u$, where u satisfies

$$\mathfrak{L}u = 0, \qquad \mathfrak{B}_i u = \gamma_i \quad (i = 1, 2, \ldots, n). \tag{4.11}$$

Then $\tilde{y}$ satisfies homogeneous boundary conditions and can thus be constructed exactly as described in §4.3.

For completeness it is worth noting that one *can* solve BVPs with inhomogeneous boundary conditions using an eigenfunction expansion and without needing to perform the above decomposition. The key steps are as follows.

1. The eigenfunctions are *always* determined using homogeneous boundary conditions. Thus, eigenfunctions won't change whether you "decompose" or not.

2. However, in going from Line 1 to Line 2 of (4.6), care must be taken in the integration by parts, as boundary terms will generally still be present. These extra boundary terms then carry through to the formula for the coefficients c_k.

Example 4.11. BCs incorporated solution route

Solve $y''(x) = f(x)$ on $0 < x < 1$ with $y(0) = \alpha$ and $y(1) = \beta$.

1. *To determine eigenvalues and eigenvectors, solve* $y'' = \lambda y$*, with* $y(0) = 0$ *and* $y(1) = 0$*. We get* $y_k(x) = \sin(k\pi x)$ *and* $\lambda_k = -k^2\pi^2$ *with* $k = 1, 2, 3, \ldots$*.*
2. *The problem is self-adjoint* [**Check!**]*, so* $w_k(x) = y_k(x) = \sin(k\pi x)$*.*
3. *Note that*

$$\langle \mathfrak{L}y, w_k \rangle - \langle y, \mathfrak{L}^* w_k \rangle = \int_0^1 y''(x)w_k(x) - y(x)w_k''(x)\,\mathrm{d}x = \left[y'w_k - yw_k'\right]_0^1 \tag{4.12}$$

   and, when we put in $w_k(x) = \sin(k\pi x)$ *and the boundary conditions for* y*,*

$$\langle \mathfrak{L}y, w_k \rangle - \langle y, \mathfrak{L}^* w_k \rangle = \left[y'w_k - yw_k'\right]_0^1 = w_k'(0)y(0) - w_k'(1)y(1) = k\pi\left(\alpha - (-1)^k\beta\right). \tag{4.13}$$

4. *So, following the approach of* (4.6)*, we get*

$$\begin{aligned} \langle f, w_k \rangle &= \langle \mathfrak{L}y, w_k \rangle \\ &= \langle y, \mathfrak{L}^* w_k \rangle + k\pi\left(\alpha - (-1)^k\beta\right) \\ &= \cdots = \lambda_k c_k \langle y_k, w_k \rangle + k\pi\left(\alpha - (-1)^k\beta\right). \end{aligned} \tag{4.14}$$

5. *Now substitute for* λ_k*,* y_k *and* w_k*; note that*

$$\langle y_k, w_k \rangle = ||y_k||^2 = \int_0^1 \sin^2(k\pi x)\,\mathrm{d}x = \frac{1}{2}. \tag{4.15}$$

   Thus obtain the solution for $y(x)$ *as a Fourier series:*

$$y(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x) \tag{4.16a}$$

$$\textit{with} \quad c_k = \frac{2}{k\pi}\left(\alpha - (-1)^k\beta\right) - \frac{2}{k^2\pi^2}\int_0^1 f(x)\sin(k\pi x)\,\mathrm{d}x. \tag{4.16b}$$

Example 4.12. Decomposed solution route

We solve the same problem, this time by first making the boundary conditions homogeneous.

1. *Solve the homogeous ODE with inhomgeneous BCs:*

$$u'' = 0, \qquad u(0) = \alpha, \qquad u(1) = \beta, \tag{4.17}$$

   to get $u(x) = \alpha(1 - x) + \beta x$*.*
2. *Solve for* $\tilde{y} = y - u$*. Since the boundary conditions are now homogeneous, we can jump straight to the formula* (4.6) *to determine the coefficients:*

$$c_k = \frac{\langle f, w_k \rangle}{\lambda_k \langle y_k, w_k \rangle} = -\frac{2}{k^2\pi^2}\int_0^1 f(x)\sin(k\pi x)\,\mathrm{d}x. \tag{4.18}$$

3. *The full solution is then*

$$y(x) = \alpha(1 - x) + \beta x + \sum_{k=1}^{\infty} c_k y_k(x). \tag{4.19}$$

Although they look different, the solutions (4.16) and (4.19) produced using the two approaches are the same. Either way, we see that self-adjoint problems are very convenient, since the adjoint eigenfunctions w_k are the same as y_k.

4.7 Sturm–Liouville theory

4.7.1 Homogeneous SL problem

Sturm–Liouville (SL) theory concerns self-adjoint linear ODEs of the form

$$\mathfrak{L}y(x) = \lambda r(x)y(x), \tag{4.20}$$

where $r(x) \geq 0$ is a *weighting function*, and the operator $\mathfrak{L}$ is of the form

$$\mathfrak{L}y(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right) + q(x)y(x) \quad a < x < b. \tag{4.21}$$

It is easy to check (and we already saw in §2.3) that the operator $\mathfrak{L}$ is formally self-adjoint. It is fully self-adjoint if the boundary conditions take the separated form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{4.22a}$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0. \tag{4.22b}$$

4.7.2 Properties of SL eigenfunctions and eigenvalues

Orthogonality Due to the presence of the weighting function, the orthogonality relation for SL eigenfunctions is

$$\int_a^b y_j(x)\overline{y}_k(x)r(x)\,\mathrm{d}x \equiv \langle y_j, ry_k\rangle = 0. \tag{4.23}$$

One can (though we won't) incorporate r into the definition of the inner product, i.e.

$$\langle y_j, y_k\rangle_r := \int_a^b y_j(x)\overline{y}_k(x)r(x)\,\mathrm{d}x. \tag{4.24}$$

This does indeed define an inner product provided $r > 0$ (almost everywhere) on $[a, b]$.

Eigenvalues The functions p, q, r are assumed to be real, so $\overline{\mathfrak{L}} = \mathfrak{L}$. Thus, when y_k is an eigenfunction of $\mathfrak{L}$ with eigenvalue λ_k, we have

$$\begin{aligned} 0 &= \langle \mathfrak{L}y_k, y_k\rangle - \langle y_k, \mathfrak{L}y_k\rangle \\ &= \langle \lambda_k r y_k, y_k\rangle - \langle y_k, \lambda_k r y_k\rangle \\ &= \lambda_k \langle r y_k, y_k\rangle - \overline{\lambda}_k \langle y_k, r y_k\rangle. \end{aligned} \tag{4.25}$$

Since

$$\langle r y_k, y_k\rangle = \langle y_k, r y_k\rangle = \int_a^b |y_k(x)|^2 r(x)\,\mathrm{d}x > 0 \tag{4.26}$$

(this is just $||y_k||_r^2 := \langle y_k, y_k\rangle_r$), we must have $\overline{\lambda}_k = \lambda_k$ and thus, *all eigenvalues are real.*

Moreover, if $a \leq x \leq b$ is a finite domain, then the eigenvalues λ are *discrete and countable.* They can be ordered such that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots$$

with $\lim_{k\to\infty} \lambda_k = \infty$.

"Regular" Sturm–Liouville problems

Proposition 4.6. *If a SL system satisfies the additional conditions*

(i) $p(x) > 0$ *and* $r(x) > 0$ *on* $a \le x \le b$*;*

(ii) $q(x) \ge 0$ *on* $a \le x \le b$*;*

(iii) the boundary conditions (4.22) *have* $\alpha_1\alpha_2 \le 0$ *and* $\beta_1\beta_2 \ge 0$*,*

then all eigenvalues $\lambda_k \ge 0$*.*

Proof. Using $\langle y_k, \mathfrak{L}y_k - \lambda_k r y_k\rangle = 0$, we have

$$
\begin{aligned}
& -\int_a^b y_k(x)\left(p(x)y_k'(x)\right)' \,\mathrm{d}x + \int_a^b q(x)y_k(x)^2\,\mathrm{d}x - \int_a^b \lambda_k r(x)y_k(x)^2\,dx = 0 \\
\Rightarrow \quad & -\left[pyy'\right]_a^b + \int_a^b p(x)y_k'(x)^2\,\mathrm{d}x + \int_a^b q(x)y_k(x)^2\,\mathrm{d}x - \lambda_k\int_a^b r(x)y_k(x)^2\,dx = 0 \qquad (4.27)
\end{aligned}
$$

and hence

$$
\lambda_k = \frac{\displaystyle\int_a^b p(x)y_k'(x)^2\,\mathrm{d}x + \int_a^b q(x)y_k(x)^2\,\mathrm{d}x - \left[pyy'\right]_a^b}{\displaystyle\int_a^b r(x)y(x)^2\,\mathrm{d}x} \ge 0. \qquad (4.28)
$$

□

4.7.3 Singular SL problems

Suppose that the function $p(x)$ in the SL operator (4.21) is zero at one of the end points of the interval, say $p(a) = 0$ but $p > 0$ on $(a, b]$. Then the ODE (4.20) is *singular* at $x = a$ — we will see in Section 4 how to analyse the behaviour of the solutions at such a singular point. Now when we use integration by parts to calculate the adjoint boundary conditions, we find

$$
\langle \mathfrak{L}y, w\rangle - \langle y, \mathfrak{L}w\rangle = \left[p\left(yw' - wy'\right)\right]_a^b = p(b)\left(y(b)w'(b) - w(b)y'(b)\right), \qquad (4.29)
$$

and the contribution from $x = a$ is zero *regardless of the boundary conditions imposed on* y *or* w. We only need to impose that $y(x)$, $y'(x)$, $w(x)$ and $w'(x)$ are *bounded* as $x \to a$ (along with suitable boundary conditions at $x = b$) to ensure that $\langle \mathfrak{L}y, w\rangle = \langle y, \mathfrak{L}w\rangle$. If p is zero at *both* end points, i.e. $p(a) = 0 = p(b)$ but $p(x) > 0$ for $x \in (a, b)$, then $\langle \mathfrak{L}y, w\rangle = \langle y, \mathfrak{L}w\rangle$ with no boundary conditions needing to be imposed on y or w (provided they are bounded). In such cases, $[a, b]$ is called the *natural interval* for the problem.

4.7.4 Inhomogeneous SL problems

Since a SL operator is self-adjoint, the eigenfunction expansion process is quite straightforward. Consider $\mathfrak{L}y = f$, with $\mathfrak{L}$ as in (4.21) and homogeneous boundary conditions (4.22).

The problem can be solved with an eigenfunction expansion in the same manner as in §4.3, exploiting the fact that $\mathfrak{L}^* \equiv \mathfrak{L}$ and $w_k \equiv y_k$. With $y = \sum c_k y_k$, we get

$$\begin{aligned} \langle f, y_k \rangle &= \langle \mathfrak{L}y, y_k \rangle \\ &= \langle y, \mathfrak{L}y_k \rangle \\ &= \langle y, \lambda_k r y_k \rangle \\ &= \lambda_k \left\langle \sum_i c_i y_i, r y_k \right\rangle \\ &= \lambda_k c_k \langle y_k, r y_k \rangle , \end{aligned} \tag{4.30}$$

where we have used the orthogonality property (4.23). Thus we obtain the coefficients in the eigenfunction expansion in the form

$$c_k = \frac{\langle f, y_k \rangle}{\lambda_k \langle y_k, r y_k \rangle}. \tag{4.31}$$

4.7.5 Transforming an operator to SL form

Any 2nd order linear operator

$$\mathfrak{L}y \equiv P_2 y'' + P_1 y' + P_0 y \tag{4.32}$$

with $P_2 \neq 0$ can be converted to a SL operator as follows. We multiply by an integrating factor function $r(x)$:

$$r\mathfrak{L}y = rP_2 y'' + rP_1 y' + rP_0 y, \tag{4.33}$$

and then choose r so that the right-hand side can be expressed in the form $r\mathfrak{L}y = -(py')' + qy$.

Exercise: show that

$$r(x) = -\frac{1}{P_2(x)} \exp\left(\int \frac{P_1(x)}{P_2(x)} \, \mathrm{d}x \right). \tag{4.34}$$

Suppose we are considering an eigenvalue problem

$$\mathfrak{L}y = \lambda y, \tag{4.35}$$

where $\mathfrak{L}$ is not self-adjoint. We could instead convert (4.35) into the equivalent SL form

$$\hat{\mathfrak{L}}y = r\mathfrak{L}y = -\left(py'\right)' + qy = \lambda r y, \tag{4.36}$$

and the transformed problem is fully self-adjoint provided the boundary conditions are of the self-adjoint form (4.22). The eigenvalues λ_k and eigenfunctions y_k of (4.35) and (4.36) must be identical (because they are essentially the same equation). Thus, although $\mathfrak{L}$ is not self-adjoint, nevertheless its eigenvalues must be real provided the boundary conditions are of self-adjoint form.

Because $\mathfrak{L}$ is not self-adjoint, its eigenfunctions y_k are not orthogonal; instead they satisfy the orthogonality relation

$$\langle y_j, w_k \rangle = 0 \quad \text{for } j \neq k, \tag{4.37}$$

with the eigenfunctions w_k of the corresponding adjoint problem. In contrast, the eigenfunctions of (4.36) (which are the same functions y_k) *are* orthogonal, albeit with respect to a modified inner product

$$\langle y_j, ry_k \rangle = 0 \quad \text{for } j \neq k. \tag{4.38}$$

It may be verified that the adjoint eigenfunctions are given by $w_k = ry_k$ so that (4.37) and (4.38) are consistent.

Exercise: Show that

$$\mathfrak{L}^*(ry) \equiv r\mathfrak{L}y, \tag{4.39}$$

where $\mathfrak{L}$ is given by (4.32) and r is as in (4.34).

Chapter 5

Power series solution of linear ODEs

These lecture notes are based on material written by Derek Moulton and Peter Howell. *Please send any corrections or comments to* Renaud Lambiotte.

5.1 Singular points of ODEs

5.1.1 Introduction

This section concerns nth order homogeneous linear ODEs of the form

$$\mathfrak{L}y(x) = y^{(n)}(x) + P_{n-1}(x)y^{(n-1)}(x) + \cdots + P_1(x)y^{(n)}(x) + P_0(x)y(x) = 0. \tag{5.1}$$

Note, in comparison with (3.38), we have divided through by $P_n(x)$ so that the coefficient of the highest-order derivative $y^{(n)}(x)$ is equal to 1. We will seek the solution to (5.1) in the form of a *power series expansion* in the neighbourhood of some point $x = x_0$. Both the procedure and the nature of the solution depend on how well-behaved the functions $P_j(x)$ are as $x \to x_0$.

5.1.2 Ordinary points

The point x_0 is an *ordinary point* of the ODE (5.1) if all $P_j(x)$ are *analytic* in a neighbourhood of $x = x_0$, i.e. they each have a convergent power series expansion of the form $\sum_{k=0}^{\infty} c_k(x-x_0)^k$. In this case, it may be shown that:

1. all n linearly independent solutions of (5.1) are also analytic in a neighbourhood of $x = x_0$, i.e. can be expressed in the form

$$y(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k; \tag{5.2}$$

2. the radius of convergence of the series solution (5.2) $\geq$ distance (in $\mathbb{C}$) to nearest singular point of the coefficent functions $P_j(x)$.

The procedure at an ordinary point is straightforward: just (i) plug the expansion (5.2) into the ODE (5.1), using the power series expansions of each of the P_j, then (ii) by equating the coefficient of each power of x to zero, obtain a sequence of equations for the coefficients a_k that can be solved recursively.

Example 5.13. Find the solution of

$$y'(x) + \frac{2x}{(1+x^2)}\, y(x) = 0 \tag{5.3}$$

as a power series expansion about $x = 0$.

Here $x_0 = 0$ is an ordinary point. The nearest singular points of $P_0(x) = 2x/\left(1+x^2\right)$ are at $x = \pm i$, distance 1 from 0, so the solution of (5.3) can be written as a regular power series expansion whose radius of convergence $R \geq 1$.

By substituting (5.2) into (5.3) and multiplying through by $\left(1+x^2\right)$, we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \left[\left(1+x^2\right) k a_k x^{k-1} + 2a_k x^{k+1}\right] \\ &= \sum_{k=0}^{\infty} \left[k a_k x^{k-1} + (k+2) a_k x^{k+1}\right]. \end{aligned} \tag{5.4}$$

Now we want to increase k by 2 in the first term in the sum so that the exponents of x agree: we have to take care of the cases $k = 0$ and $k = 1$ separately and so end up with

$$0 = 0 \times a_0 x^{-1} + 1 \times a_1 + \sum_{k=0}^{\infty} \left[(k+2) a_{k+2} x^{k+1} + (k+2) a_k x^{k+1}\right]. \tag{5.5}$$

The coefficient of x^{-1} is zero identically. By setting the coefficient of x^0 to zero, we deduce that a_1 must be zero. Then by setting to zero all the coefficients of x, x^2, x^3, ..., we get the recurrence relation

$$a_{k+2} = -a_k \quad (k = 0, 1, 2, \ldots). \tag{5.6}$$

Since $a_1 = 0$, it follows that the odd coefficients $a_3, a_5, \ldots$ are all equal to zero, and the even coefficients are given by $a_{2k} = (-1)^k a_0$. The solution of (5.3) is thus given by

$$y(x) = a_0 \sum_{k=0}^{\infty} (-1)^k x^{2x}. \tag{5.7}$$

One can easily verify that the radius of convergence of the series (5.7) is equal to 1. Indeed, it is easy to solve the simple ODE (5.3) exactly to get $y(x) = \text{const}/\left(1+x^2\right)$, of which (5.7) is just the Maclaurin expansion.

5.1.3 Singular points

The point x_0 is called a *singular point* of the ODE (5.1) if at least one of the coefficient functions $P_j(x)$ is not analytic there. In this case, the general solution $y(x)$ may not be analytic at x_0: $y(x)$ or its derivatives might "blow-up" as $x \to x_0$. The following simple example illustrates how solutions can behave near a singular point.

Example 5.14. *Consider the first-order ODE*

$$y'(x) - \lambda x^{-m} y(x) = 0, \tag{5.8}$$

where $\lambda \in \mathbb{R}$ and m is a non-negative integer. The general solution of (5.8) can easily be found via separation of variables, and the generic behaviour as $x \to 0$ depends on the value of m.

(i) *For $m = 0$, the point $x = 0$ is ordinary. The solution $y(x) = \text{const} \times e^{\lambda x}$ can be expanded as a power series about $x = 0$ which converges for all $x \in \mathbb{C}$.*

(ii) *For $m = 1$, the point $x = 0$ is singular. The solution in this case is $y(x) = \text{const} \times x^{\lambda}$, which is analytic if λ is a non-negative integer (despite 0 being a singular point). For any other λ, the solution is singular at $x = 0$, but with a relatively benign singularity: either a pole (if λ is a negative integer) or a branch point (otherwise).*

(iii) *For $m = 2$, the behaviour is much worse: the solution of (5.8) is $y(x) = \text{const} \times \exp(-\lambda/x)$, which has an* essential singularity *at $x = 0$. Similarly, there is an essential singularity at $x = 0$ for any value of $m \geq 2$.*

Example 5.14 suggests that the strength of the singularity in the solution at a singular point tends to increase the higher the order of the poles in the coefficients in front of the lower order terms of the ODE. Indeed, this is the key idea behind the classification of singular points.

5.1.4 Regular singular points

If the coefficients $P_j(x)$ are not all analytic at $x = x_0$, but the modified coefficients

$$p_j(x) \equiv P_j(x)(x - x_0)^{n-j} \text{ are all analytic at } x = x_0, \tag{5.9}$$

then $x = x_0$ is a *regular singular point* of the ODE (5.1). For example, Case (ii) of Example 5.14 has a regular singular point at $x = 0$. For the general second-order ODE

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \tag{5.10}$$

there is a regular singular point at $x = x_0$ if at least one of $P(x)$ and $Q(x)$ is not analytic at $x = x_0$ but both $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are.

Any singular point that does not satisfy the criterion (5.9) is an *irregular singular points.* At a regular singular point, the singularity in the solution is "not too bad", and a modification of the power series approach can be used. For irregular singular points, though, there is no general theory!

Example 5.15. Cauchy–Euler equation

The Cauchy–Euler equation

$$y''(x) + \frac{a}{x}\,y(x) + \frac{b}{x^2}\,y(x) = 0 \tag{5.11}$$

has a regular singular point at $x = 0$. The general solution can be found via the ansatz $y = x^{\alpha}$, where α satisfies the characteristic equation $\alpha(\alpha - 1) + a\alpha + b = 0$, and there are two cases to consider.

(i) *If the characteristic equation has two distinct roots α_1 and α_2, then, the general solution of* (5.11) *is given by*

$$y(x) = C_1 x^{\alpha_1} + C_2 x^{\alpha_2} \tag{5.12}$$

(where C_1 and C_2 are arbitrary constants).

(ii) *If the characteristic equation has a double root α, then the general solution is*

$$y(x) = C_1 x^{\alpha} + C_2 x^{\alpha} \log x. \tag{5.13}$$

Note that if the roots are two distinct non-negative integers, then the general solution in case (i) *is analytic (even though the ODE has a singular point). In general, however, the behaviour as $x \to 0$ could be a negative, fractional or even complex power of x, and the solution generically has a pole or a branch point at $x = 0$.*

The behaviour illustrated by Example 5.15 carries over to regular singular points in general, except that the functions x^{α_1} and x^{α_2} are each multiplied by an analytic function (i.e. a regular power series in x). The general theory for regular singular points will be explained below, but first we show how the point at infinity can be analysed.

5.1.5 The point at infinity

The point $x_0 = \infty$ can also be classified by changing the independent variable via the substitution

$$t = 1/x, \qquad u(t) = y(x), \tag{5.14}$$

and classifying the point $t = 0$ for the resulting ODE for $u(t)$.

Example 5.16. Find and classify the singular points of the ODEs

(i) $y'(x) - y(x) = 0$,

(ii) $y''(x) + \dfrac{1}{x^2}\,y(x) = 0$.

In case (i)*, the coefficient* $P_0(x) = -1$ *is analytic everywhere, and there don't appear to be any singular points. But if we make the change of variables* (5.14) *then, by the chain rule, we have* $\dot{u}(t) = -\left(1/t^2\right) y'(x)$*. The ODE* (i) *therefore becomes*

$$\dot{u}(t) + \frac{1}{t^2}\,u(t) = 0, \tag{5.15}$$

which has an irregular singular point *at* $t = 0$*, and it follows that* (i) *has an irregular singular point at* $x = \infty$*. Indeed, the solution* $y(x) = \mathrm{e}^x$ *has an essential singularity as* $x \to \infty$*.*

In case (ii)*, there is a regular singular point at* $x = 0$ *(since* $x^2 \times \left(1/x^2\right) = x$ *is analytic at* $x = 0$*). Again making the substitution* (5.14)*, we get* [**exercise**]

$$\ddot{u}(t) + \frac{2}{t}\,\dot{u}(t) + \frac{1}{t^2}\,u(t) = 0, \tag{5.16}$$

which likewise has a regular singular point at $t = 0$*. Therefore the ODE* (ii) *has regular singular points at* $x = 0$ *and at* $x = \infty$*.*

5.2 Frobenius method for 2nd order ODEs

5.2.1 The indicial equation

From now on, we restrict attention to regular singular points of 2nd order ODEs. If $x = x_0$ is a regular singular point, then we can write the ODE in the form

$$\mathfrak{L}y(x) = y''(x) + \frac{p(x)}{(x - x_0)}\,y'(x) + \frac{q(x)}{(x - x_0)^2}\,y(x) = 0, \tag{5.17}$$

where p and q are analytic, and can therefore be expanded as convergent power series:

$$p(x) = \sum_{k=0}^{\infty} p_k (x - x_0)^k, \qquad q(x) = \sum_{k=0}^{\infty} q_k (x - x_0)^k. \tag{5.18}$$

The idea is to seek a solution in the form of a *Frobenius series*

$$y(x) = (x - x_0)^{\alpha} \sum_{k=0}^{\infty} a_k (x - x_0)^k. \tag{5.19}$$

Note the similarity to the Cauchy–Euler Example 5.15: $y(x) \sim a_0(x-x_0)^\alpha$ as $x \to 0$, but now the power of x is multiplied by an *a priori* unknown analytic function $\sum_k a_k(x-x_0)^k$, with coefficients a_k to be determined. We may assume that $a_0 \neq 0$ by choosing α appropriately.

Now we plug (5.19) into the ODE (5.17), to get

$$\sum_{k=0}^{\infty}(\alpha+k)(\alpha+k-1)a_k(x-x_0)^{\alpha+k-2} + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(\alpha+k)p_j a_k(x-x_0)^{\alpha+k+j-2} + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} q_j a_k(x-x_0)^{\alpha+k+j-2} = 0, \quad (5.20)$$

and equate coefficients. At the lowest power, namely $(x-x_0)^{\alpha-2}$, we find

$$\big[\alpha(\alpha-1) + p_0\alpha + q_0\big]a_0 = 0. \quad (5.21)$$

Since a_0 is defined to be non-zero, the quadratic function in brackets must be zero. This polynomial plays an important role, and we will denote it by

$$F(\alpha) = \alpha(\alpha-1) + p_0\alpha + q_0. \quad (5.22)$$

The equation $F(\alpha) = 0$ is called the *indicial equation*, and it determines the possible indicial exponents α_1, α_2. Note that in general these exponents can be complex! In any case, we order them such that $\mathrm{Re}[\alpha_1] \geq \mathrm{Re}[\alpha_2]$.

5.2.2 The first series solution

Let us continue equating coefficients of powers of $(x-x_0)$. We find after some algebra that the coefficients of $(x-x_0)^{k+\alpha-2}$ satisfy

$$F(\alpha+k)a_k = -\sum_{j=0}^{k-1}[(\alpha+j)p_{k-j} + q_{k-j}]a_j \quad (5.23)$$

To generate the first series solution, we take $\alpha = \alpha_1$: the solution of the indicial equation with the larger real part. Since F is a quadratic function with roots at α_1 and α_2, with $\mathrm{Re}[\alpha_2] \leq \mathrm{Re}[\alpha_1]$, it follows that $F(\alpha_1 + k) \neq 0$ for any integer $k \geq 1$ We can therefore rearrange (5.23) to

$$a_k = -\frac{1}{F(\alpha_1+k)}\sum_{j=0}^{k-1}[(\alpha_1+j)p_{k-j} + q_{k-j}]a_j \quad (5.24)$$

and thus solve successively for all the coefficients a_1, a_2, ..., and we obtain one solution

$$y_1(x) = (x-x_0)^{\alpha_1}\sum_{k=0}^{\infty} a_k(x-x_0)^k. \quad (5.25)$$

Therefore at least one solution of (5.17) can always be expressed as a Frobenius series with indicial exponent $\alpha = \alpha_1$, and we call this the *first solution*.

5.2.3 The second solution Case I: $\alpha_1 - \alpha_2 \notin \mathbb{Z}$

For the *second solution*, we have to distinguish between several cases and sub-cases. The simplest case occurs when the indices α_1 and α_2 *do not differ by an integer* (so in particular they are not equal). In this case, $F(\alpha_2 + k) \neq 0$ for all $k \geq 1$, so we can solve (5.23) also with the second value of the exponent $\alpha = \alpha_2$. We call the coefficients the second solution b_n to distinguish from the previous coefficients a_k, and they satisfy the recurrence relations

$$b_k = -\frac{1}{F(\alpha_2 + k)} \sum_{j=0}^{k-1} [(\alpha_2 + j)p_{k-j} + q_{k-j}]b_j. \tag{5.26}$$

Thus, we obtain with no problems a second solution also as a Frobenius series, with indicial exponent α_2:

$$y_2(x) = (x - x_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k(x - x_0)^k. \tag{5.27}$$

5.2.4 Case II: $\alpha_1 = \alpha_2$

In the case of a double root we apparently only get one solution with the Frobenius method, and we have to multiply by logs to get a second solution (similar to the case of a double root in Cauchy–Euler). In particular, the second solution is of the form

$$y_2(x) = y_1(x) \log(x - x_0) + (x - x_0)^{\alpha_1} \sum_{k=0}^{\infty} b_k(x - x_0)^k, \tag{5.28}$$

where y_1 is the first solution (5.25).

The form of solution (5.28) can be derived using the so-called *derivative method*, which is outlined in §5.2.6. For the moment, we can at least verify that it works in principle by substituting (5.28) into (5.17). In doing so, note that, with $\mathfrak{L}$ defined by (5.17),

$$\mathfrak{L}\big[y_1(x)\log(x - x_0)\big] = \log(x - x_0)\mathfrak{L}y_1(x) + \frac{2}{(x - x_0)}\, y_1'(x) + \frac{p(x) - 1}{(x - x_0)^2}\, y_1(x) \tag{5.29}$$

and because $\mathfrak{L}y_1 = 0$, when (5.28) is substituted into (5.17), the logs vanish, and one can solve a sequence of recurrence relations for the coefficients b_k as above.

5.2.5 Case III: $\alpha_1 - \alpha_2$ a positive integer

If $\alpha_1 - \alpha_2 = N$, where $N > 0$ is an integer, then we will potentially run into trouble in equation (5.26) when $k = N$. In this case, there are two sub-possibilities.

Case III(a): For $k = N$, the right-hand side of (5.26) is non-zero.

Then we have a contradiction, and the standard Frobenius solution method doesn't work. To get a second solution, we use the same form as in Case II, i.e.

$$y_2(x) = y_1(x) \log(x - x_0) + (x - x_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k(x - x_0)^k. \tag{5.30}$$

Again, when we substitute (5.30) into the ODE (5.17), the logs vanish and one obtains a set of recurrence relations that determine the coefficients b_k. Note that the indicial exponent for the second series in (5.30) is α_2, whereas y_1 is given by the Frobenius series using the exponent α_1.

Case III(b): When $k = N$, the right-hand side of RHS of (5.23) is zero.

In this case, there is no contradiction, but any choice for b_N will satisfy (5.26), i.e. b_N remains undetermined. The second solution therefore has Frobenius form

$$y_2(x) = (x - x_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k, \tag{5.31}$$

where b_0 can be chosen to be $b_0 = 1$ (without loss of generality) and b_N is arbitrary. Since $\alpha_2 + N = \alpha_1$, changing b_N just corresponds to adding multiples of y_1 to (5.31).

Example 5.17. Find a series solution about the regular singular point $x = 0$ for the differential equation

$$4x^2 y''(x) + 4xy'(x) + (4x^2 - 1)y(x) = 0. \tag{5.32}$$

Step 1: *Assume a solution of form*

$$y(x) = x^{\alpha} \sum_{k=0}^{\infty} a_k x^k \tag{5.33}$$

with $a_0 \neq 0$. Compute the corresponding series for y', y'' by differentiating term by term.

Step 2: *Plug the series* (5.33) *into the ODE* (5.32) *and multiply everything out:*

$$\begin{aligned} 0 &= \underbrace{\sum_{k=0}^{\infty} 4(\alpha+k)(\alpha+k-1)a_k x^{\alpha+k}}_{4x^2y''} + \underbrace{\sum_{k=0}^{\infty} 4(\alpha+k)a_k x^{\alpha+k}}_{4xy'} - \underbrace{\sum_{k=0}^{\infty} a_k x^{\alpha+k}}_{y} + \underbrace{\sum_{k=0}^{\infty} 4a_k x^{\alpha+k+2}}_{4x^2y} \\ &= \sum_{k=0}^{\infty} \left(4(\alpha+k)^2 - 1\right) a_k x^{\alpha+k} + \sum_{k=0}^{\infty} 4a_k x^{\alpha+k+2}. \end{aligned} \tag{5.34}$$

Step 3: *The* indicial equation *comes from the balance at lowest order, in this case x^{α}:*

$$F(\alpha) = 4\alpha^2 - 1. \tag{5.35}$$

The indicial exponents are the roots of F, i.e.

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = -\frac{1}{2}. \tag{5.36}$$

Step 4: *Shift the terms in the series* (5.34) *so that* the exponents of x are the same *in each term. For this example, we need only shift the index in the last sum, so all the series have terms proportional to $x^{\alpha+k}$. Thus, by replacing k with $k-2$, we have*

$$\sum_{k=0}^{\infty} 4a_k x^{\alpha+k+2} \equiv \sum_{k=2}^{\infty} 4a_{k-2} x^{k+\alpha}, \tag{5.37}$$

and thus we obtain

$$0 = a_0 F(\alpha) x^{\alpha} + a_1 F(\alpha+1) x^{\alpha+1} + \sum_{k=2}^{\infty} \left[a_k F(\alpha+k) + 4a_{k-2} \right] x^{k+\alpha}. \tag{5.38}$$

We have chosen the α so that the equation balances at x^{α}, and hence a_0 is free. Balancing at all other orders will determine the coefficients a_k for $k \geq 1$.

Step 5: First series

Set $\alpha = \alpha_1 = 1/2$ in (5.38)*; note that*

$$F(\alpha_1 + k) = 4\left(\frac{1}{2} + k\right)^2 - 1 = 4k(k+1) \tag{5.39}$$

and thus we obtain

$$a_1 = 0, \qquad a_k = \frac{-1}{k(k-1)}\, a_{k-2} \quad k = 2,\, 3,\, \ldots. \tag{5.40}$$

Step 6: *Use the recursion relation* (5.40) *to determine a formula for a_k in terms of a_0. A good idea is to write out a few terms, and look for a pattern. Here, since $a_1 = 0$, we easily see that $a_3 = a_5 = \cdots = 0$, i.e. all the odd coeffients are zero, and we are left with*

$$\begin{aligned} a_2 &= \frac{-1}{2\cdot 3}\, a_0, \\ a_4 &= \frac{-1}{4\cdot 5}\, a_2 = \frac{1}{5\cdot 4\cdot 3\cdot 2}\, a_0, \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{2k} &= \frac{(-1)^k a_0}{(2k+1)!}. \end{aligned} \tag{5.41}$$

Step 7: *Input the formula* (5.41) *for the coefficients to obtain the first solution:*

$$y_1(x) = a_0 x^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k}. \tag{5.42}$$

Step 8: Second series

Repeat the process for the second root $\alpha_2 = -1/2$. In this case, $\alpha_1 - \alpha_2 = 1 = N$ is an integer, so we are in Case III.

The coefficients b_k in the second series satisfy

$$0 = b_0 F(\alpha_2) x^{\alpha_2} + b_1 F(\alpha_2+1) x^{\alpha_2+1} + \sum_{k=2}^{\infty} \left[b_k F(\alpha_2+k) + 4b_{k-2} \right] x^{k+\alpha_2}. \tag{5.43}$$

The coefficient of x^{α_2}, namely $F(\alpha_2)$, is zero by construction. At order $x^{\alpha_2+N} = x^{\alpha_2+1}$, we obtain $F(1/2)b_1 = 0 \times b_1 = 0$. There is no contradiction, and b_1 is arbitrary and can be set to zero: we are in CaseIII(b).

Step 9: *Following the recursion forward with $b_0 \neq 0$, analogous computations to the above yield*

$$y_2(x) = b_0 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k}. \tag{5.44}$$

Step 10: *The general solution is a linear combination of the two series solutions, i.e.*

$$y(x) = C_1 x^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} + C_2 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}. \tag{5.45}$$

In this example, we can recognise the series for sine and cosine and thus express the solution in closed form. In fact, the general solution to (5.32) *(which is called* Bessel's equation *of order* 1/2*) is*

$$y(x) = C_1 \frac{\sin x}{\sqrt{x}} + C_2 \frac{\cos x}{\sqrt{x}}. \tag{5.46}$$

5.2.6 Derivative method

Here we discuss Case II, where α_1 is a double root of $F(\alpha)$, and give a brief justification for the form (5.28) of the series solution. Without loss of generality, let $a_0 = 1$. Suppose we solve (5.23) for the coefficients $a_1, a_2, \ldots$ with *arbitrary* α, i.e. with $F(\alpha)$ not generally equal to zero. Thus, each coefficient a_k is a function of α, and we can think of α as a parameter in the series

$$y(x;\alpha) = (x-x_0)^\alpha + \sum_{k=1}^{\infty} a_k(\alpha)(x-x_0)^{k+\alpha}. \tag{5.47}$$

The recurrence relation (5.23) ensures that the coefficient of $(x-x_0)^{\alpha+k-2}$ in $\mathfrak{L}y$ is zero for all $k \geq 1$, and we are just left with

$$\mathfrak{L}y(x;\alpha) = (x-x_0)^{\alpha-2}F(\alpha). \tag{5.48}$$

Since $F(\alpha_1) = 0$, it follows that $\mathfrak{L}y(x;\alpha_1) = 0$ and thus

$$y_1(x) = y(x;\alpha_1) = \sum_{0}^{\infty} a_k(\alpha_1)(x-x_0)^{\alpha_1+k} \tag{5.49}$$

is a solution (as we already know). Now the idea is to differentiate (5.48) with respect to α, then set $\alpha = \alpha_1$. Since $\mathfrak{L}$ has no dependence on α,

$$\begin{aligned}\mathfrak{L}\left[\frac{\partial}{\partial\alpha} y(x;\alpha)\right] &= \frac{\partial}{\partial\alpha}\left[\mathfrak{L}y(x;\alpha)\right] \\ &= \frac{\partial}{\partial\alpha}\left[(x-x_0)^{\alpha-2}F(\alpha)\right] \\ &= (x-x_0)^{\alpha-2}\log(x-x_0)F(\alpha) + (x-x_0)^{\alpha-2}F'(\alpha).\end{aligned} \tag{5.50}$$

Since α_1 is a double root of F, the right-hand side of (5.50) is zero when $\alpha = \alpha_1$, and it follows that

$$y_2(x) = \left.\frac{\partial}{\partial\alpha} y(x;\alpha)\right|_{\alpha=\alpha_1} \tag{5.51}$$

satisfies $\mathfrak{L}y_2 = 0$. To get a more explicit form, calculate

$$\begin{aligned}\frac{\partial}{\partial\alpha} y(x;\alpha) &= \frac{\partial}{\partial\alpha}\left(\sum_{k=0}^{\infty} a_k(\alpha)(x-x_0)^{\alpha+k}\right) \\ &= \log(x-x_0)\sum_{k=0}^{\infty} a_k(\alpha)(x-x_0)^{\alpha+k} + \sum_{k=0}^{\infty} a_k'(\alpha)(x-x_0)^{\alpha+k}\end{aligned} \tag{5.52}$$

and set $\alpha = \alpha_1$ to get

$$y_2(x) = \log(x - x_0)y_1(x) + \sum_{k=0}^{\infty} b_k(x - x_0)^{\alpha_1+k}, \tag{5.53}$$

in agreement with (5.28), where $b_k = a_k'(\alpha_1)$.

In principle, the derivative method allows us to determine the coefficients b_k in the second series solution, as outlined above. However, to do so we require a closed form for $a_k(\alpha)$ for general α. In practice, it is usually easier just to substitute in the appropriate form (5.28) of the series and compare coefficients.

5.2.7 More examples

Example 5.18. Find a series solution about $x = 0$ for the differential equation

$$x(x - 1)y''(x) + 3xy'(x) + y(x) = 0. \tag{5.54}$$

First we divide through by $x(x - 1)$ to obtain the standard form

$$y''(x) + \frac{3}{x-1}\,y'(x) + \frac{1}{x(x-1)}\,y(x) = 0. \tag{5.55}$$

Since $p(x) = 3x/(x - 1)$ and $q(x) = x/(x - 1)$ are analytic at $x = 0$, it is a regular singular point. Thus we expect to find at least one solution in the form of a Frobenius series.

If we try for a solution with the local behaviour of the form $y(x) \sim x^\alpha$ as $x \to 0$, then (5.54) implies that

$$-\alpha(\alpha - 1)x^{\alpha-1} + \text{ higher order terms } = 0, \tag{5.56}$$

and we deduce that the indicial equation is

$$F(\alpha) = \alpha(\alpha - 1) = 0, \tag{5.57}$$

which has roots $\alpha_1 = 1$, $\alpha_2 = 0$.

More generally, by seeking the solution as a power series of the form

$$y(x) = x^\alpha \sum_{k=0}^{\infty} a_k x^k \tag{5.58}$$

we obtain

$$\underbrace{\sum_{k=0}^{\infty} -(k+\alpha)(k+\alpha-1)a_k x^{k+\alpha-1}}_{\text{series 1}} + \underbrace{\sum_{k=0}^{\infty} \left[(k+\alpha)(k+\alpha-1) + 3(k+\alpha) + 1\right] a_k x^{k+\alpha}}_{\text{series 2}} = 0. \tag{5.59}$$

Now, shift the index in series 2 so that the indices match series 1:

$$\text{series 2} = \sum_{k=1}^{\infty} \big[(k+\alpha-1)(k+\alpha-2) + 3(k+\alpha-1) + 1\big] a_{k-1} x^{k+\alpha-1}. \tag{5.60}$$

Now we can bring the two sums together and demand that the coefficients of $x^{k+\alpha-1}$ all vanish. The first term with $k = 0$ vaishes identically by the indicial equation (5.57). Simplifying the terms for $k > 0$, we obtain the recursion relation

$$(k+\alpha)(k+\alpha-1)a_k - (k+\alpha)^2 a_{k-1} = 0. \tag{5.61}$$

Note that the coefficient of a_k is just $F(k+\alpha)$, as expected.

On substituting $\alpha = \alpha_1 = 1$ into (5.61), we obtain

$$a_k = \frac{k+1}{k}\, a_{k-1}. \tag{5.62}$$

Without loss of generality setting $a_0 = 1$, we obtain the simple formula $a_k = k+1$, and thus one solution to (5.54) is given by the series

$$y_1(x) = \sum_{k=0}^{\infty}(k+1)x^{k+1} = \frac{x}{(1-x)^2}. \tag{5.63}$$

For a second solution, since $\alpha_1 - \alpha_2 = 1$ is an integer, we are in Case III*, and there may or may not be a Frobenius series solution. To find out, we seek a solution*

$$y_2 = x^{\alpha_2}\sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} b_k x^k. \tag{5.64}$$

Setting $\alpha = \alpha_2 = 0$ in (5.61), we have

$$(k-1)b_k = k b_{k-1}. \tag{5.65}$$

We immediately run into trouble, since we must take $b_0 \neq 0$, and thus with $k = 1$ we get the contradiction $0 \times b_1 = b_0 \neq 0$. Hence the second Frobenious solution does not work: we are in Case III(a)*, and the form of the second solution is*

$$y_2(x) = y_1(x)\log(x) + \sum_{k=0}^{\infty} b_k x^k. \tag{5.66}$$

Example 5.18 illustrates that the indicial equation can be found just by considering the leading-order terms, without bothering to substitute in an entire series. In Example 5.18, once we have obtained one series solution $y_1(x) = x/(1-x)^2$, we can construct the other using reduction of order. Setting $y(x) = y_1(x)v(x)$ in (5.54), we find that v satisfies the ODE

$$v''(x) + \frac{(2-x)}{x(1-x)}\, v'(x) = 0, \tag{5.67}$$

which is easily integrated to give

$$v(x) = C_1 + C_2\left(\log(x) + \frac{1}{x}\right). \tag{5.68}$$

A second solution to (5.54) is thus given by

$$y_2(x) = y_1(x)\left(\log(x) + \frac{1}{x}\right) = y_1(x)\log(x) + \frac{1}{(1-x)^2}, \tag{5.69}$$

which indeed has the form (5.66) when expanded about $x = 0$.

Example 5.19. Find the form of series solutions about $x = 0$ for the differential equation

$$\sin^2(x)y'' - \sin(x)\cos(x)y' + y = 0. \tag{5.70}$$

We consider the functions

$$p(x) = -x\frac{\sin(x)\cos(x)}{\sin^2(x)}, \qquad q(x) = x^2\frac{1}{\sin^2(x)}. \tag{5.71}$$

As both p and q are finite as $x \to 0$ (the singularities there are removable), $x = 0$ is a regular singular point. Note that

$$\lim_{x\to 0} p(x) = -1, \qquad \lim_{x\to 0} q(x) = 1, \tag{5.72}$$

as can be obtained with L'Hôpital's rule. This implies that the leading terms in the power series expansions of p and q are $p_0 = -1$ and $q_0 = 1$, and the indicial equation is

$$F(\alpha) = \alpha(\alpha - 1) + p_0\alpha + q_0 = (\alpha - 1)^2 = 0. \tag{5.73}$$

Hence $\alpha = 1$ is a repeated root.

We conclude that one solution is of the form

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+1} \tag{5.74a}$$

and a second solution is given by

$$y_2(x) = y_1(x)\log(x) + \sum_{k=0}^{\infty} b_k x^{k+1}. \tag{5.74b}$$

The coefficients $\{a_k,\, b_k\}$ can in principle be computed by inserting the solution forms (5.74) *into* (5.70) *and balancing coefficients, but we will not do so here.*

One can solve (5.70) exactly by spotting that $\sin x$ is a solution and then using reduction of order: this approach confirms that the local expansions (5.74) are indeed of the correct form.

Chapter 6

Special functions

These lecture notes are based on material written by Derek Moulton and Peter Howell. *Please send any corrections or comments to* Renaud Lambiotte.

6.1 Introduction

We have seen in the previous section a method to construct power series solutions to ODEs with non-constant coefficients and singular points. Except for a few examples, even if a closed form for the coefficients a_k can be found, the resulting power series cannot be expressed in terms of elementary functions, i.e. exponentials, sines, cosines, etc. Nevertheless, some particular ODEs occur frequently enough for their solutions to have been given special names and for their behaviour to be fully characterised. In this section, we explore some of these so-called *special functions.*

6.2 Bessel functions

6.2.1 Bessel's equation

Bessel's equation arises whenever one separates the variables in the Laplacian in cylindrical polar coordinates. For example, consider the vibrating membrane of a circular drum. It may be shown that the transverse displacement $w(x, y, t)$ of the membrane at time t and position (x, y) satisfies the two-dimensional *wave equation*

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = \nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \tag{6.1}$$

where c is a constant (representing the wave-speed and given by $c = \sqrt{T/\rho}$, where T and ρ are the membrane tension and density). If the membrane is pinned at the boundary of a disk of radius a, then we have to solve (6.1) in $x^2 + y^2 < a^2$, subject to the boundary condition

$$w = 0 \qquad \text{at } x^2 + y^2 = a^2. \tag{6.2}$$

We look for a *normal mode* in which the membrane oscillates with frequency ω, so that the displacement takes the form $w(x, y, t) = u(x, y)\cos(\omega t + \phi)$. By substituting into (6.1), we find that u satisfies the *Helmholtz equation*

$$\nabla^2 u + \lambda u = 0, \tag{6.3}$$

with $\lambda = \omega^2/c^2$.

Now let us switch to plane polar coordinates (r, θ) such that $(x, y) = r(\cos\theta, \sin\theta)$, and thus obtain the equation and boundary condition:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0 \qquad 0 \le r < a, \quad 0 \le \theta \le 2\pi, \tag{6.4a}$$

$$u = 0 \qquad r = a, \quad 0 \le \theta \le 2\pi, \tag{6.4b}$$

$$u \text{ } 2\pi\text{-periodic in } \theta. \tag{6.4c}$$

This is a PDE eigenvalue problem: $u \equiv 0$ always satisfies the problem (6.4), and our aim is to find values of λ such that there are non-trivial solutions $u(r, \theta)$.

Since $u(r, \theta)$ is periodic in θ we can expand u into a Fourier series of the form

$$u(r, \theta) = U_0(r) + \sum_{n=1}^{\infty} U_n(r)\cos n\theta + V_n(r)\sin n\theta. \tag{6.5}$$

Substitution of (6.5) into (6.4) gives

$$\frac{1}{r}\left(rU_n'(r)\right)' + \left(\lambda - \frac{n^2}{r^2}\right)U_n(r) = 0, \qquad \text{for } 0 \le r < a, \tag{6.6a}$$

$$U_n(r) = 0 \qquad \text{at } r = a. \tag{6.6b}$$

The same equation and boundary condition hold for $V_n(r)$. Now eliminate λ by the rescaling $U_n(r) = y(x)$ with $x = \lambda^{1/2} r$, resulting in

$$x^2 y''(x) + xy'(x) + \left(x^2 - n^2\right)y(x) = 0, \tag{6.7}$$

which is known as *Bessel's equation* of order n.

6.2.2 Bessel functions of first and second kind

Bessel's equation (6.7) has a regular singular point at $x = 0$, with indicial equation given by $F(\alpha) = \alpha^2 - n^2 = 0$, the solutions of which are $\alpha_1 = n$, $\alpha_2 = -n$, with a double root for $n = 0$. In general, the parameter n in (6.7) can be any complex number but, in the context described above where $u(r, \theta)$ is required to be 2π-periodic in θ, we need only consider n to be a non-negative integer. Similarly, since x is a scaled version of the radial coordinate r, we focus on non-negative values of x. A detailed analysis of the singular point at $x = 0$, as in §5.2, reveals that one solution of (6.7) is given by a Frobenius series about $x = 0$ with the exponent $\alpha_1 = n$, and the other solution is given by a Frobenius series with exponent $\alpha_2 = -n$ plus $\log(x)$ times the first solution (i.e. Case III(a) from §5.2.5).

The first Frobenius series, with a specific normalisation of the leading coefficient in the expansion, defines the *Bessel functions of first kind*

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k}, \tag{6.8}$$

for integer $n \ge 0$.

Similarly, a specifically normalised choice for the second series solution defines the *Bessel functions of second kind*

$$Y_n(x) = \frac{2}{\pi}\log\left(\frac{x}{2}\right)J_n(x) - \frac{1}{\pi}\left(\frac{2}{x}\right)^n \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x^2}{4}\right)^k \\ - \frac{1}{\pi}\left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty}\frac{\left[\psi(k+1)+\psi(n+k+1)\right]}{k!(n+k)!}\left(-\frac{x^2}{4}\right)^k, \quad (6.9)$$

where the *digamma function* $\psi(m)$ for integer $m \geq 1$ is given by $\psi(m) = -\gamma + \sum_{k=1}^{m-1} k^{-1}$, and $\gamma = 0.5772\cdots$ is the Euler–Mascheroni constant. More details regarding the expansions (6.8) and (6.9) are explored on Problem Sheet 3.

6.2.3 Properties of Bessel functions

The first few Bessel functions $J_n(x)$ and $Y_n(x)$ are plotted in Figure 6.1. Many properties of the Bessel functions are known — see for example the NIST Digital Library of Mathematical Functions (DLMF). — and we list here just a few.

(i) Since Bessel's equation (6.7) has only one singular point for finite x, the series (6.8) and (6.9) for J_n and in Y_n have infinite radius of convergence.

(ii) Also, J_n and Y_n are oscillating functions that decay slowly as $x \to \infty$. Each has an infinite set of discrete zeros in $x > 0$, which are quite useful and have therefore been tabulated, for example at mathworld. The first few zeros of J_n and Y_n (denoted $j_{n,1}$, $j_{n,2}$, $\ldots$ and $y_{n,1}$, $y_{n,2}$, $\ldots$) are listed in Table 6.1, and 6.2, respectively.

(iii) As $x \to 0$, the behaviours of the two kinds of Bessel function are quite different. For the first kind, we have $J_n(0) = 0$ if $n > 0$, and $J_0(0) = 1$, while the second kind Bessel functions are singular, with $Y_n(x) \to -\infty$ as $x \to 0$. (The singularity is logarithmic when $n = 0$, or has $Y_n(x) = O\left(x^{-n}\right)$ when $n > 0$.)

(iv) The following two recursion relations can be derived from the local expansion (6.8):

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x), \qquad J_{n+1}(x) = -2J_n'(x) + J_{n-1}(x). \quad (6.10)$$

The same relations also hold for the second-kind Bessel functions Y_n.

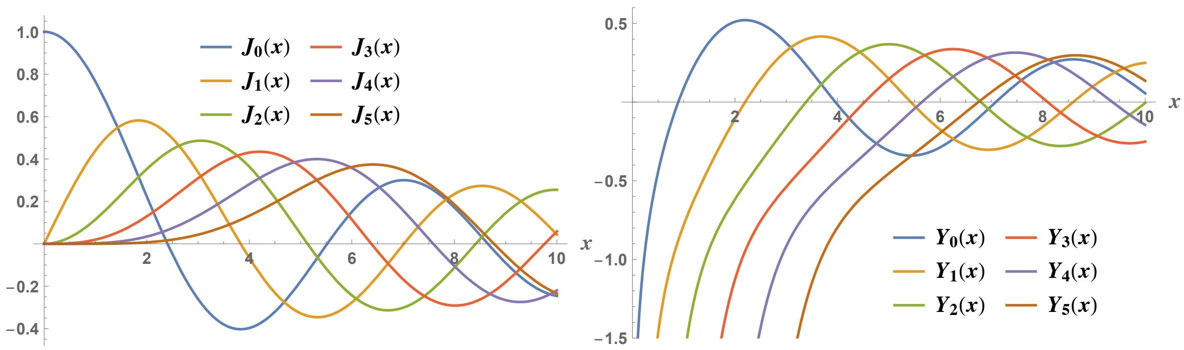
6.2.4 Normal modes of a circular drum

We can now express the general solution to (6.6a) in terms of Bessel functions as

$$U_n(r) = C_1 J_n\left(\lambda^{1/2} r\right) + C_2 Y_n\left(\lambda^{1/2} r\right), \quad (6.11)$$

for some arbitrary constants C_1 and C_2. We require the displacement to be bounded as $r \to 0$, and must therefore set $C_2 = 0$ to remove the singularity in Y_n. For a non-trivial solution we must have $C_1 \neq 0$, and the boundary condition (6.6b) at $r = a$ therefore leads to

$$J_n\left(\lambda^{1/2} a\right) = 0, \quad (6.12)$$



(a) (b)

Figure 6.1: (a) Bessel functions of the first kind $J_n(x)$. (b) Bessel functions of the second kind $Y_n(x)$.

m	$j_{0,m}$	$j_{1,m}$	$j_{2,m}$	$j_{3,m}$	$j_{4,m}$
1	2.40483	3.83171	5.13562	6.38016	7.58834
2	5.52008	7.01559	8.41724	9.76102	11.0647
3	8.65373	10.1735	11.6198	13.0152	14.3725
4	11.7915	13.3237	14.796	16.2235	17.616
5	14.9309	16.4706	17.9598	19.4094	20.8269

Table 6.1: The first five zeros of J_n with $n = 0, 1, 2, 3, 4$.

m	$y_{0,m}$	$y_{1,m}$	$y_{2,m}$	$y_{3,m}$	$y_{4,m}$
1	0.893577	2.19714	3.38424	4.52702	5.64515
2	3.95768	5.42968	6.79381	8.09755	9.36162
3	7.08605	8.59601	10.0235	11.3965	12.7301
4	10.2223	11.7492	13.21	14.6231	15.9996
5	13.3611	14.8974	16.379	17.8185	19.2244

Table 6.2: The first five zeros of Y_n with $n = 0, 1, 2, 3, 4$.

i.e. $\lambda^{1/2}a$ has to be one of the zeros $j_{n,m}$ of J_n. Thus the eigenvalues for (6.6) are given by

$$\lambda = \frac{j_{n,m}^2}{a^2}, \qquad n = 0,\, 1,\, \dots, \quad m = 1, 2, \dots \tag{6.13}$$

with corresponding eigenfunctions

$$U_{n,m}(r) = J_n\left(j_{n,m} r/a\right). \tag{6.14}$$

We can then read off the *normal frequencies* of the drum from the definition of λ, i.e.

$$\omega_{n,m} = j_{n,m}\,\frac{c}{a}. \tag{6.15}$$

6.2.5 Sturm–Liouville form

The differential equation (6.6a) can be written in Sturm-Liouville form by multiplying through by r. For convenience we also pose the problem on the unit interval (corresponding to the modes in a disk of unit radius $a = 1$, which may be obtained by rescaling r with a) to get the eigenvalue problem

$$\mathfrak{L}U(r) = -\left(rU'(r)\right)' + \frac{n^2}{r}\,U(r) = \lambda r U(r), \qquad \text{for } 0 < r < 1, \tag{6.16a}$$

$$U(r) = 0 \qquad \text{at } r = 1, \tag{6.16b}$$

$$U(r) \text{ bounded} \qquad \text{as } r \to 0. \tag{6.16c}$$

From above, we know that the eigenvalues and eigenfunctions for (6.16) are given by

$$\lambda_{n,m} = j_{n,m}^2, \qquad\qquad U_{n,m}(r) = J_n\left(j_{n,m} r\right). \tag{6.17}$$

We recognise (6.16a) as a singular Sturm–Liouville equation with weighting function r, and thus deduce the following orthogonality relation between eigenfunctions:

$$\int_0^1 J_n\left(j_{n,\ell} r\right) J_n\left(j_{n,m} r\right) r \,\mathrm{d}r = 0 \qquad \text{for } \ell \neq m. \tag{6.18}$$

A separate calculation for the case $\ell = m$ results in [*see Problem Sheet 3*]

$$\int_0^1 J_n^2\left(j_{n,m} r\right) r \,\mathrm{d}r = \frac{1}{2}\left(J_n'(j_{n,m})\right)^2. \tag{6.19}$$

6.3 Legendre functions

6.3.1 The Legendre equation

The Legendre equation arises when studying eigenvalue problems for the 3D Laplacian operator in spherical coordinates. Suppose again we are solving the Helmholtz equation (6.3) but now using spherical polars (r, θ, ϕ), so the Laplacian is given by

$$\nabla^2 u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = -k^2 u. \tag{6.20}$$

When we separate the variables by seeking a solution of the form

$$u(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi), \tag{6.21}$$

then (6.20) may be rearranged to

$$\frac{r\big(rR(r)\big)''}{R(r)} + k^2r^2 = -\frac{(\sin\theta\,\Theta'(\theta))'}{\sin\theta\,\Theta(\theta)} - \frac{\Phi''(\phi)}{\sin^2\theta\,\Phi(\phi)}. \tag{6.22}$$

By the usual argument, the left-hand side of (6.22) is a function only of r, while the right-hand side is independent of r, so they must both equal a constant, λ say. We then have

$$-\frac{\Phi''(\phi)}{\Phi(\phi)} = \frac{\sin\theta\,(\sin\theta\,\Theta'(\theta))'}{\Theta(\theta)} + \lambda\sin^2\theta, \tag{6.23}$$

which likewise must equal a constant. For Φ to be a 2π-periodic function, that constant must be of the form m^2, where $m \geq 0$ is an integer: this gives $\Phi = \text{const}$ if $m = 0$ or $\Phi(\phi) = \cos(m\phi + \alpha)$ if $m > 0$. We are then left with the following linear ODE for $\Theta(\theta)$:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{6.24}$$

Equation (6.24) is to be solved for $0 < \theta < \pi$. It may readily be verified that $\theta = 0$ and $\theta = \pi$ are both regular singular points of (6.24), and to get physically resonable solutions we must insist that $\Theta(\theta)$ is sufficiently well-behaved as $\theta \to 0,\ \pi$.

We can express (6.24) in a more helpful form by making the change of variable $\cos\theta = x$ and $\Theta(\theta) = y(x)$. Then $\mathrm{d}/\mathrm{d}\theta = -\sin\theta\,\mathrm{d}/\mathrm{d}x$, and (6.24) is transformed into the *associated Legendre equation* for $y(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0. \tag{6.25}$$

The parameters m and λ in (6.25) can in general take any complex values. We will focus on the case where m is a non-negative integer and (for reasons that will become clear below) $\lambda = \ell(\ell+1)$, where ℓ is also a non-negative integer. The solutions of the associated Legendre equation (6.25) are the *associated Legendre functions*; for $m = 0$, we drop the "associated" and speak of the Legendre equation and Legendre functions.

6.3.2 Properties of Legendre functions

Many properties and relations satisfied by solutions of (6.25) may be found, for example, at DLMF or mathworld. Here we list a few useful properties.

(i) The points $x = \pm 1$ and $x = \infty$ are regular singular points of the associated Legendre equation (6.25). The indicial exponents for $x = \pm 1$ are $-m/2$ and $m/2$. Thus, the local expansion yields one bounded and one unbounded solution as $x \to 1$, and similarly as $x \to -1$. (When $m = 0$, there is a repeated root of the indicial equation, and one solution is of order $\log(x \mp 1)$ as $x \to \pm 1$.)

(ii) If we consider bounded solutions of (6.25) on $-1 < x < 1$, we see that boundedness imposes *two* conditions, one at either end of the interval. This suggests that (6.25) can be posed as a singular Sturm–Liouville problem:

$$-\left(\left(1-x^2\right)y'(x)\right)' + \frac{m^2}{1-x^2}\,y(x) = \lambda y(x) \qquad \text{for } -1 < x < 1, \tag{6.26a}$$

$$y(x) \text{ bounded} \qquad \text{as } x \to \pm 1. \tag{6.26b}$$

(iii) The eigenvalues of (6.26) are given by $\lambda = \ell(\ell+1)$ with integer $\ell \geq m \geq 0$. The eigenfunctions are the corresponding associated Legendre functions, which are denoted by $y(x) = P_\ell^m(x)$. From Sturm–Liouville theory, we infer the orthogonality relation

$$\int_{-1}^{1} P_k^m(x) P_\ell^m(x)\,\mathrm{d}x = 0 \qquad \text{for } k \neq \ell. \tag{6.27}$$

The case $k = \ell$ requires explicit calculation: see Problem Sheet 3.

(iv) For $m = 0$ and integer $\ell \geq 0$, the Legendre functions (without "associated") are denoted by $P_\ell(x)$. It may be shown that P_ℓ is a polynomial of degree ℓ: if one seeks the solution of (6.25) as a power series expansion about $x = 0$,

$$y(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{6.28}$$

then the series *terminates*, with $a_k \equiv 0$ for $k > \ell$. The resulting *Legendre polynomials* are given explictly by the *Rodrigues' formula*:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left[(x^2-1)^\ell\right]. \tag{6.29}$$

(v) A second, linearly independent, solution of the Legendre equation (6.26a) with $m = 0$ is given by the Legendre function of second kind, denoted by $Q_\ell(x)$. These solutions are unbounded as $x \to \pm 1$. For the case $\ell = 0$, the solution Q_0 is found on Problem Sheet 2:

$$Q_0(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right). \tag{6.30}$$

(vi) For the general case of nonzero $m \leq \ell$, the associated Legendre functions of first and second kind are given by

$$P_\ell^m(x) = (-1)^m\left(1-x^2\right)^{m/2}\frac{\mathrm{d}^m P_\ell(x)}{\mathrm{d}x^m}, \tag{6.31a}$$

$$Q_\ell^m(x) = (-1)^m\left(1-x^2\right)^{m/2}\frac{\mathrm{d}^m Q_\ell(x)}{\mathrm{d}x^m}. \tag{6.31b}$$

The associated Legendre function P_ℓ^m is a polynomial if and only if m is even.

6.4 Generalisation: orthogonal polynomials

There are many other second order linear ODEs with families of orthogonal polynomials as solutions, satisfying orthogonality relations

$$\int_a^b p_m(x)p_n(x)r(x)\,\mathrm{d}x = 0 \qquad m \neq n \tag{6.32}$$

with a fixed weighting function $r(x)$ which can by inferred by formulating an appropriate Sturm–Liouville eigenvalue problem. One can in fact give a complete classification of all infinite families of orthogonal polynomials that can arise from second-order linear ODEs. The most important ones include the following.

1. The "Jacobi-like" polynomials, which include the Legendre, Chebyshev, and Gegenbauer polynomials, arise from ODEs of the type

   $$\left(1-x^2\right)y''(x) + (a+bx)y'(x) + \lambda y(x) = 0, \tag{6.33}$$

   posed on the interval $[-1,1]$, with constants a and b and an appropriate discrete set of values of λ.

2. The associated Laguerre polynomials satisfy Laguerre's equation:

   $$xy''(x) + (a+1-x)y'(x) + \lambda y(x) = 0, \tag{6.34}$$

   which admits a polynomial solution $y(x) = L_n^a(x)$ when λ is a non-negative integer n. They satisfy the orthogonality relation

   $$\int_0^\infty L_m^a(x)L_n^a(x)x^a\mathrm{e}^{-x}\,\mathrm{d}x = 0 \quad \text{for } m \neq n. \tag{6.35}$$

   The Laguerre polynomials (without "associated") correspond to $a = 0$ and are denoted by $L_n(x) \equiv L_n^0(x)$.

3. Hermite polynomials are solutions of the Hermite equation

   $$y''(x) - 2xy'(x) + \lambda y(x) = 0, \tag{6.36}$$

   which admits a polynomial solution $H_n(x)$ when $\lambda = 2n$ for integer $n \geq 0$. Hermite polynomials satisfy the orthogonality relation

   $$\int_{-\infty}^\infty H_m(x)H_n(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = 0 \quad \text{for } m \neq n. \tag{6.37}$$

Chapter 7

Asymptotic analysis - Part 1

These lecture notes are based on material written by Derek Moulton and Peter Howell. *Please send any corrections or comments to* Renaud Lambiotte.

7.1 Introduction

A complex mathematical problem often cannot be solved exactly, but it may contain parameters that represent physical constants or quantities in the problem. If some of these parameters are very small or very large, it may be possible to derive approximate solutions to the problem. Doing this in a systematic manner is the subject of *asymptotic analysis*. In this section a basic framework is presented for the use of this approach. Asymptotic methods can be put on a rigorous footing, but we will content ourselves with an informal approach.

Example 7.20. Consider a pendulum, initially hanging vertically and set in motion with velocity V.

The angle $\theta(t)$ made by the pendulum with the vertical at time t satisfies the equation

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0, \tag{7.1a}$$

where ℓ is the length of the pendulum and g is the acceleration due to gravity. The given initial state leads to the following initial conditions for θ:

$$\theta(0) = 0, \qquad \ell\dot{\theta}(0) = V. \tag{7.1b}$$

The problem (7.1) can *be solved exactly, but in a rather unpleasant form involving elliptic functions. Can we say anything about the how the solution depends on the sizes of the constants ℓ and V?*

The first step is to normalise the problem, i.e. to re-scale the variables to eliminate as many parameters as possible. The idea is that all of the variables and parameters in the normalised model should be dimensionless.

We can eliminate g/ℓ from (7.1a) *by defining a new time variable*

$$\tau = \left(\frac{g}{\ell}\right)^{1/2} t. \tag{7.2}$$

Note that g, ℓ and t have units of m^2/s, m *and* s, *respectively, so that τ is indeed* dimensionless. *The angle θ is already dimensionless, but nevertheless can be scaled to balance the left- and right-hand sides of* (7.1b), *i.e.*

$$\theta(t) = \alpha u(\tau), \tag{7.3}$$

where

$$\alpha = \frac{V}{\sqrt{\ell g}}. \tag{7.4}$$

Again, one can check that α is dimensionless.

The normalised version of the problem (7.1) *then reads*

$$\alpha\ddot{u}(\tau) + \sin\big(\alpha u(\tau)\big) = 0, \qquad u(0) = 0, \qquad \dot{u}(0) = 1. \tag{7.5}$$

Now we have collapsed all of the physical constants g, ℓ and V into the single dimensionless parameter α, and we can ask the question: how does the solution $u(\tau)$ of (7.5) *behave if α is very small or if α is very large?*

Example 7.20 illustrates how a process of *non-dimensionalisation* can produce a normalised mathematical problem containing a minimal number of dimensionless parameters that characterise the relative importance of the different physical effects in the problem. It then makes sense to ask what the approximate behaviour of solutions might be if a particular parameter is either very small or very large. More details on how to nondimensionalise a given physical problem can be found elsewhere and in Part B and C applied mathematical courses.

7.2 Asymptotic expansions

7.2.1 Order notation and twiddles

To start it is necessary to give a basic structure to describe approximations to a function when some parameter in the function becomes large or small. The following definitions allow the relative sizes of two different functions to be described. We consider two continuous real-valued functions $f(x)$ and $g(x)$, and compare their behaviours as x tends towards a particular value x_0 (often $x_0 = 0$ or ∞).

Definition 7.1. "Big O" notation
We write

$$f(x) = O\big(g(x)\big) \quad \textit{as } x \to x_0 \qquad \textit{if} \quad \exists A > 0 \textit{ such that } |f(x)| < A|g(x)| \tag{7.6}$$

for all x sufficiently close to x_0.

We say that "f is of order g" to capture the idea that $f(x)$ and $g(x)$ are "roughly the same size" in the limit as $x \to x_0$.

Example 7.21.

(i) $\sin(2x) = O(x) \quad \text{as } x \to 0;$

(ii) $3x + x^3 = O(x) \quad \text{as } x \to 0;$

(iii) $\log x = O(x-1) \quad \text{as } x \to 1;$

(iv) $5x^2 + x^{-3} - \mathrm{e}^{-x} = O\left(x^2\right) \quad \text{as } x \to \infty.$

Definition 7.2. "Twiddles" notation
We write

$$f(x) \sim g(x) \quad \textit{if} \quad \frac{f(x)}{g(x)} \to 1 \quad \textit{as } x \to x_0. \tag{7.7}$$

This notation could be read as "f is asymptotic to g" or "f looks like g" as $x \to x_0$, and captures the idea of two functions being approximately equal in some limit.

Example 7.22.

(i) $\sin(2x) \sim 2x \quad \text{as } x \to 0;$

(ii) $x + \mathrm{e}^{-x} \sim x \quad \text{as } x \to \infty.$

Definition 7.3. "Little o" notation
We write

$$f(x) = o\big(g(x)\big) \quad \text{as } x \to x_0 \quad \text{if} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{7.8}$$

This notation captures the idea that f is "much smaller than" g in the limit as $x \to x_0$, and can also be written as $f(x) \ll g(x)$ or indeed $g(x) \gg f(x)$ as $x \to x_0$.

Example 7.23.

(i) $9x^2 - 4x^5 = o(x) \quad \text{as } x \to 0;$

(ii) $\dfrac{3}{x^2} - 3\mathrm{e}^{-x} = o(1/x) \quad \text{as } x \to \infty.$

Whenever using the order or twiddles notation, one should include in the statement what value x is tending to (though it is often implicit).

Example 7.24. Taylor's Theorem
A smooth function $f(x)$ may be expanded in a Taylor series and thus one may make statements such as:

$$\begin{aligned}
f(x) &= f(x_0) + (x - x_0)f'(x_0) + O\left((x - x_0)^2\right) && \text{as } x \to x_0, && (7.9a)\\
f(x) &= f(x_0) + (x - x_0)f'(x_0) + o\big((x - x_0)\big) && \text{as } x \to x_0, && (7.9b)\\
f(x) &= f(x_0) + (x - x_0)f'(x_0) + o\big((x - x_0)^{3/2}\big) && \text{as } x \to x_0, && (7.9c)\\
f(x) &\sim f(x_0) && \text{as } x \to x_0, && (7.9d)\\
f(x) - f(x_0) &\sim (x - x_0)f'(x_0) && \text{as } x \to x_0. && (7.9e)
\end{aligned}$$

7.2.2 Asymptotic sequence and asymptotic expansion

In this course we are particularly interested in problems containing a small parameter, and we will therefore focus on the case $x_0 = 0$. We will follow convention by generally using the notation ϵ (rather than x) for the small parameter. Our aim then is to find the approximate behaviour of some function $f(\epsilon)$, say, in the limit as $\epsilon \to 0$.

Example 7.25.

(i) $\sin\big(\epsilon^{1/2}\big) \approx \epsilon^{1/2} - \dfrac{\epsilon^{3/2}}{6} + \cdots,$

(ii) $\tanh^{-1}(1 - \epsilon) \approx \dfrac{1}{2} \log\left(\dfrac{2}{\epsilon}\right) - \dfrac{\epsilon}{4} - \dfrac{\epsilon^2}{16} + \cdots,$

both in the limit as $\epsilon \to 0$.

If f is smooth, then one can express $f(\epsilon)$ as a Taylor expansion in powers of ϵ as $\epsilon \to 0$, as in Example 7.24. However, for a unbounded or non-smooth functions, integer powers of ϵ might not be appropriate to capture the local behaviour, as illustrated by Example 7.25. In general, we might want to write

$$f(\epsilon) \approx \sum_k a_k \phi_k(\epsilon), \tag{7.10}$$

where $\phi_k(\epsilon)$ are suitable *gauge functions*. For such a series to provide a useful approximation to the function f, we would expect the terms in the expansion to get successively smaller with increasing k, and this motivates the following definition.

Definition 7.4. *A set of functions* $\{\phi_k(\epsilon)\}_{k=0,1,2,\ldots}$ *is an* asymptotic sequence *as* $\epsilon \to 0$ *if* $\phi_{k+1}(\epsilon) = o\big(\phi_k(\epsilon)\big)$ *as* $\epsilon \to 0$*, i.e. each term in the sequence is of smaller magnitude than the previous term.*

Example 7.26. *Here are some examples of asymptotic sequences:*

(i) $\{1,\, \epsilon,\, \epsilon^2,\, \epsilon^3,\, \cdots\}$,

(ii) $\left\{1,\, \epsilon^{1/2},\, \epsilon,\, \epsilon^{3/2},\, \cdots\right\}$,

(iii) $\{1,\, \epsilon,\, \epsilon\log\epsilon,\, \epsilon^2,\, \epsilon^2\log\epsilon,\, \cdots\}$.

Definition 7.5. *A function* $f(\epsilon)$ *has an* asymptotic expansion *of the form*

$$f(\epsilon) \sim \sum_k a_k\phi_k(\epsilon) \quad \text{as } \epsilon \to 0 \tag{7.11}$$

if

(i) *the gauge functions* ϕ_k *for an asymptotic sequence, i.e.* $\phi_{k+1}(\epsilon) \ll \phi_k(\epsilon)$ *for all* k*;*

(ii) $f(\epsilon) - \sum_{k=0}^{N} a_k\phi_k(\epsilon) \ll \phi_N(\epsilon)$ *for all* $N = 0, 1, \ldots,$

as $\epsilon \to 0$*.*

Property (i) ensures that *the terms in the expansion get successively smaller*, and property (ii) ensures that *the approximation gets more accurate the more terms are included in the expansion.*

The definition of an asymptotic expansion differs crucially from that for a *convergent* series. For a convergent series of the form

$$f(\epsilon) = \sum_{k=0}^{\infty} a_k\phi_k(\epsilon), \tag{7.12}$$

we require that the partial sum

$$f_N(\epsilon) = \sum_{k=0}^{N} a_k\phi_k(\epsilon), \tag{7.13}$$

converges to $f(\epsilon)$ as $N \to \infty$, with ϵ held fixed. For an asymptotic expansion

$$f(\epsilon) \sim \sum_k a_k\phi_k(\epsilon), \tag{7.14}$$

we instead require that the partial sum (7.13) converges asymptotically to $f(\epsilon)$ as $\epsilon \to 0$, with N held fixed. In fact, an asymptotic expansion may well *diverge* as $N \to \infty$ (i.e. have radius of convergence equal to zero) but still be useful and perfectly well defined by Definition 7.5.

Elementary properties of asymptotic expansions include the following.

1. *Given* a particular choice of gauge functions $\{\phi_k\}$, the coefficients a_k are unique.

This can easily be proved by induction on N. Note that the gauge functions themselves are not unique, for example,

$$\begin{aligned}\tan\epsilon &\sim \epsilon + \frac{1}{3}\,\epsilon^3 + \frac{2}{15}\,\epsilon^5 + \cdots \\ &\sim \sin\epsilon + \frac{1}{2}\,\sin^3\epsilon + \frac{3}{8}\sin^5\epsilon + \cdots\,. \end{aligned} \tag{7.15}$$

Usually we use the simplest choice, namely powers of ϵ, or possibly exponentials or logs.

2. The function defines the expansion but not vice versa.

For example, if $\phi_k(\epsilon) = \epsilon^k$ for $k = 0,\, 1,\, 2,\, \ldots$, then

$$\frac{1}{1-\epsilon} \sim 1 + \epsilon + \epsilon^2 + \cdots \quad \text{as } \epsilon \to 0 \tag{7.16a}$$

but also

$$\frac{1}{1-\epsilon} + \mathrm{e}^{-1/\epsilon} \sim 1 + \epsilon + \epsilon^2 + \cdots \quad \text{as } \epsilon \to 0. \tag{7.16b}$$

In other words, we have two different functions with the same asymptotic expansion. This occurs because (for $0 < \epsilon \ll 1$)

$$\mathrm{e}^{-1/\epsilon} = o\left(\epsilon^k\right) \quad \text{for all } k, \tag{7.17}$$

and $\mathrm{e}^{-1/\epsilon}$ is said to be *exponentially small* or *transcendentally small.*

7.3 Approximate roots of algebraic equations

To start using asymptotic methods consider the problem of finding the roots of an algebraic equation containing a small parameter. To focus ideas, first we consider some simple cases where the exact roots can be easily found.

Example 7.27. Solve approximately the quadratic equation

$$x^2 + \epsilon x - 1 = 0 \tag{7.18}$$

in the limit as $\epsilon \to 0$.
Exact solution: *Here we can use the quadratic formula to get the exact solutions*

$$x = \frac{1}{2}\left(-\epsilon \pm \sqrt{4+\epsilon^2}\right). \tag{7.19}$$

A binomial expansion of the square root yields the following approximations for the two roots:

$$x^+ \sim 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + O\left(\epsilon^4\right), \tag{7.20a}$$

$$x^- \sim -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + O\left(\epsilon^4\right), \tag{7.20b}$$

both as $\epsilon \to 0$*. Now the question is, could we have derived the approximate solutions* (7.20) *directly from the equation* (7.18)*, without finding the exact solutions first?*

Asymptotic approach: *Since equation* (7.18) *contains only* ϵ*, and no other (e.g. fractional) powers of* ϵ*, we assume that the solution for* x *may be expressed as an asymptotic expansion of the form*

$$x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots \quad \text{as } \epsilon \to 0. \tag{7.21}$$

We substitute (7.21) *into* (7.18) *to obtain*

$$\begin{aligned} 0 &\sim \left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right)^2 + \epsilon\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right) - 1 \\ &\sim x_0^2 + 2x_0 x_1 \epsilon + \left(x_1^2 + 2x_0 x_2\right)\epsilon^2 + \left(2x_1 x_2 + 2x_0 x_3\right)\epsilon^3 + \cdots \epsilon\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) - 1. \end{aligned} \tag{7.22}$$

Since this must hold for all ϵ*, and we have assumed that* $x_0, x_1, \ldots$ are all independent of ϵ*, we conclude that equality must hold independently for each power of* ϵ*. Hence, we equate the coefficients of each power of* ϵ *to solve successively for* $x_0, x_1, \ldots$*.*

Considering the first few powers, we get:

$$O(1): \quad x_0^2 - 1 = 0, \quad \Rightarrow \quad x_0 = \pm 1, \tag{7.23a}$$

$$O(\epsilon): \quad 2x_0 x_1 + x_0 = 0, \quad \Rightarrow \quad x_1 = -\frac{1}{2}, \tag{7.23b}$$

$$O\left(\epsilon^2\right): \quad 2x_0 x_2 + x_1^2 + x_1 = 0, \quad \Rightarrow \quad x_2 = \frac{1}{8x_0} = \pm\frac{1}{8}, \tag{7.23c}$$

$$O\left(\epsilon^3\right): \quad 2x_0 x_3 + 2x_1 x_2 + x_2 = 0, \quad \Rightarrow \quad x_3 = 0, \tag{7.23d}$$

and so on. Thus we have obtained the first few terms in asymptotic expansions for each of the two roots of (7.18)*, namely*

$$x \sim \pm 1 - \frac{1}{2}\epsilon \pm \frac{1}{8}\epsilon^2 + O\left(\epsilon^4\right), \tag{7.24}$$

which clearly agrees with the exact solution (7.20)*.*

Example 7.28. Solve approximately the quadratic equation

$$\epsilon x^2 + x - 1 = 0 \tag{7.25}$$

in the limit as $\epsilon \to 0$.

Exact solution: *Again we can use the quadratic formula to get the exact solutions*

$$x = \frac{1}{2\epsilon}\left(-1 \pm \sqrt{1 + 4\epsilon}\right), \tag{7.26}$$

and expansion of the square root yields the following approximations for the two roots:

$$x^+ \sim 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + O\left(\epsilon^4\right), \tag{7.27a}$$

$$x^- \sim -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + O\left(\epsilon^3\right). \tag{7.27b}$$

Now we try to get the roots directly from equation (7.25)*.*

Asymptotic approach. First attempt: *It is reasonable to expect that the leading-order solution as* $\epsilon \to 0$ *could be found by just setting* $\epsilon = 0$ *in* (7.25)*. This approach gives* $x \sim 1$ *as a first approximation, which indeed agrees with the first root* (7.27a) *at lowest order in* ϵ*. We can then obtain an improved approximation by hypothesising that* x *can be expressed as an asymptotic expansion in powers of* ϵ*, i.e.*

$$x \sim 1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots \quad \text{as } \epsilon \to 0. \tag{7.28}$$

We substitute (7.28) *into the original equation* (7.25) *to get*

$$0 \sim \epsilon\left(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right)^2 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots. \tag{7.29}$$

As in Example 7.27*, we equate the coefficients of each power of ϵ to solve successively for x_1, x_2, $\ldots$.*

Considering the first few powers, we get:

$$O(\epsilon): \quad 1 + x_1 = 0, \quad \Rightarrow \quad x_1 = -1, \tag{7.30a}$$

$$O\left(\epsilon^2\right): \quad 2x_1 + x_2 = 0, \quad \Rightarrow \quad x_2 = 2, \tag{7.30b}$$

$$O\left(\epsilon^3\right): \quad x_1^2 + 2x_2 + x_3 = 0, \quad \Rightarrow \quad x_3 = -5, \tag{7.30c}$$

and so on. Hence we can systematically improve the approximation of the root near $x = 1$, and evidently we have managed to reproduce the expansion (7.27a)*.*

Second attempt: *The previous approach successfully produced an asymptotic expansion for the positive root x^+. But since* (7.25) *is a quadratic equation, we know that it has another root, which our method seems to have missed.*

Note that the root (7.28) *near $x = 1$ has been found by considering a* dominant balance *between two of the three terms in* (7.25)*, namely x and 1, while treating the third term ϵx^2 as a small correction, i.e.*

$$\underbrace{\epsilon x^2}_{\text{small}} + \underbrace{x - 1}_{\text{balance}} = 0. \tag{7.31}$$

To approximate the other root, we need to consider other possible balances between different terms in equation (7.25)*.*

Suppose we try to balance the terms $\epsilon^2 x$ and 1 in (7.25)*, which suggests that $x = O\left(\epsilon^{-1/2}\right)$. This choice would give the following sizes for the terms:*

$$\underbrace{\epsilon x^2}_{O(1)} + \underbrace{x}_{O\left(\epsilon^{-1/2}\right)} - \underbrace{1}_{O(1)} = 0. \tag{7.32}$$

Now we have a problem: the first and third terms balance, but the second term is much bigger than either of them. To get a dominant balance*, we need to ensure that the balanced terms are the dominant terms in the equation, and* (7.32) *fails this requirement.*

Third attempt: *The remaining possibility is to balance the terms ϵx^2 and x in* (7.25)*, i.e. to suppose that $x = O\left(\epsilon^{-1}\right)$. Then comparing the sizes of the terms in* (7.25)*, we get*

$$\underbrace{\epsilon x^2}_{O(\epsilon^{-1})} + \underbrace{x}_{O(\epsilon^{-1})} - \underbrace{1}_{O(1)} = 0. \tag{7.33}$$

This choice does give a dominant balance: when the first two terms are the same order, they are indeed much bigger than the third term.

Now we know this balance works, we use the scaling $x = \epsilon^{-1} y$, with $y = O(1)$, to reflect the anticipated size of x; then (7.25) *is transformed to*

$$\frac{y^2}{\epsilon} + \frac{y}{\epsilon} - 1 = 0 \quad \Leftrightarrow \quad y^2 + y - \epsilon = 0. \tag{7.34}$$

Now letting $\epsilon \to 0$ in (7.34)*, we get a sensible balance between the first two terms, but there seem to be two possible choices for y, namely $y \sim -1$ or $y \sim 0$. However, assuming that we have scaled the equation correctly, the desired root should have $y = O(1)$, so we ignore the second option (which in fact just reproduces the root x_+ that we have already found).*

We therefore seek the solution to (7.34) *as an asymptotic expansion of the form*

$$y \sim -1 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots \quad \text{as } \epsilon \to 0. \tag{7.35}$$

Substitution of (7.35) *into* (7.34) *leads to*

$$0 \sim \left(-1 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots\right)\left(\epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots\right) - \epsilon, \tag{7.36}$$

after some simplification by writing $y^2 + y = y(y+1)$. As above, this equation must be satisfied at every order in ϵ, and we can solve successively for the coefficients as follows:

$$O(\epsilon): \quad -y_1 - 1 = 0, \quad \Rightarrow \quad y_1 = -1, \tag{7.37a}$$

$$O(\epsilon^2): \quad y_1^2 - y_2 = 0, \quad \Rightarrow \quad y_2 = 1, \tag{7.37b}$$

$$O(\epsilon^3): \quad 2y_1y_2 - y_3 = 0, \quad \Rightarrow \quad y_3 = -2, \tag{7.37c}$$

and so on. We have thus constructed the approximate solution for y, namely

$$y \sim -1 - \epsilon + \epsilon^2 - 2\epsilon^3 + \cdots \quad \text{as } \epsilon \to 0, \tag{7.38}$$

and by rescaling $x = y/\epsilon$, we see that we have successfully obtained the second root x_- given by (7.27b)*.*

In Example 7.27, we can find both roots of equation (7.18) as regular asymptotic expansions in integer powers of ϵ, without any rescaling of x. In contrast, in Example 7.28, by seeking a regular expansion, we only manage to obtain one root; to find the other we have to rescale x appropriately. Consequently, one of the roots of equation (7.25) *diverges* like $1/\epsilon$ as $\epsilon \to 0$. This occurs because setting $\epsilon = 0$ reduces the degree of (7.25) from a quadratic to a linear equation, and thus reduces the number of roots from two to one. It is necessary to rescale x to reintroduce the quadratic term ϵx^2 at leading order to recover the second root. A so-called *singular perturbation* is said to occur when setting $\epsilon = 0$ reduces the degree of the problem, and thus the number of solutions that the problem possesses.

Example 7.28 illustrates the following general procedure to find an approximate solution x of an algebraic equation of the form $F(x;\epsilon) = 0$ containing a small parameter ϵ.

1. Scale the variable(s) to get a dominant balance, i.e. so that at least two of the terms (i) balance and (ii) are much bigger than the remaining terms in the equation.

2. Plug in an asymptotic expansion for x. Usually the form of the expansion is clear from the form of the equation (though see below an example where it isn't so clear).

3. By equating the terms multiplying each power of ϵ in the equation, obtain the coefficients in the expansion.

4. Repeat for any other possible dominant balances in the equation to obtain approximations for other roots.

We next try to use the same ideas to solve an equation where there is no exact solution to guide us.

Example 7.29. Find an asymptotic expansion for all the roots of

$$x\mathrm{e}^{-x} = \epsilon \quad \text{as } \epsilon \to 0. \tag{7.39}$$

Figure 7.1 *shows a plot of $x\mathrm{e}^{-x}$ versus x. For small, positive values of ϵ, we expect there to be two roots x of* (7.39)*: one close to $x = 0$ and one with x large.* [**Exercise:** *show that there exist two roots if $\epsilon < \mathrm{e}^{-1}$.*]

We consider the smaller root first. When x is small, we have $\mathrm{e}^{-x} = O(1)$ and, to balance the left- and right-hand sides of (7.39)*, we should therefore scale x with ϵ. We set $x = \epsilon y$, with y assumed to be $O(1)$, and equation* (7.39) *can then be written as*

$$y = \mathrm{e}^{\epsilon y} \sim 1 + \epsilon y + \frac{\epsilon^2 y^2}{2} + \frac{\epsilon^3 y^3}{6} + \cdots \quad \text{as } \epsilon \to 0. \tag{7.40}$$

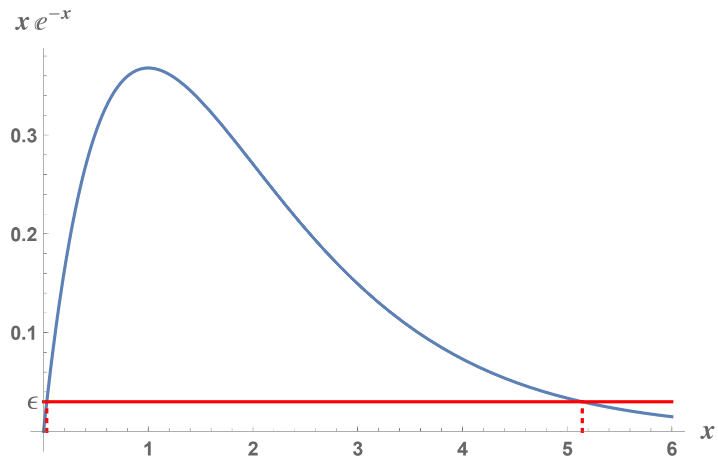


Figure 7.1: The function xe^{-x} plotted versus x, indicating two roots to equation (7.39) with $0 < \epsilon \ll 1$.

The Maclaurin expansion of the right-hand side is valid given our hypothesis that $y = O(1)$.

Now we pose an asymptotic expansion for y*: given that only integer powers of* ϵ *appear in equation* (7.40)*, it is reasonable to assume that* y *may be expanded in the form*

$$y \sim y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots \sim 1 + \epsilon\left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right) + \frac{1}{2}\epsilon^2\left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right)^2 + \cdots. \quad (7.41)$$

We can then easily determine the coefficients:

$$y_0 = 1, \quad (7.42a)$$

$$y_1 = y_0 = 1, \quad (7.42b)$$

$$y_2 = y_1 + \frac{1}{2}y_0^2 = \frac{3}{2}, \quad (7.42c)$$

and so on, and therefore the smaller root of (7.39) *is given by the asymptotic expansion*

$$x \sim \epsilon + \epsilon^2 + \frac{3}{2}\epsilon^3 + \ldots \quad \text{as } \epsilon \to 0. \quad (7.43)$$

An asymptotic expansion for the larger root of (7.39) *is a lot harder to find. As a first step, we take logs of both sides of* (7.39) *to get*

$$x - \log x = -\log\epsilon = |\log\epsilon|. \quad (7.44)$$

Health warning: *examples like this with logs are notoriously awkward: the solution of the apparently innocuous algebraic equation* (7.44) *is just about as bad as one will ever encounter!*

Since ϵ *is assumed to be very small (and positive),* $\log\epsilon$ *is large and negative, with* $|\log\epsilon| \to \infty$ *as* $\epsilon \to 0$*. To satisfy* (7.44)*,* x *will need to be large, in which case* $x \gg \log x$*. To get a balance in* (7.44)*, we therefore scale* $x = |\log\epsilon| y$ *to get*

$$|\log\epsilon| y - \log(|\log\epsilon| y) = |\log\epsilon| \quad \Leftrightarrow \quad y - \frac{\log(|\log\epsilon|)}{|\log\epsilon|} - \frac{\log y}{|\log\epsilon|} = 1. \quad (7.45)$$

The difficulty here is that we can't assume a known form *of the asymptotic expansion for* y *and then just solve for the coefficients: it is not obvious in advance what gauge functions we should use. So let us just pose a general expansion of the form*

$$y \sim 1 + \phi_1(\epsilon) + \phi_2(\epsilon) + \cdots, \quad (7.46)$$

assuming only that $\cdots \ll \phi_2 \ll \phi_1 \ll 1$, *and try to calculate what* ϕ_1, ϕ_2, $\ldots$ *should be. Note that* (7.46) *gives*

$$\log y \sim (\phi_1 + \phi_2 + \cdots) - \frac{1}{2}(\phi_1 + \phi_2 + \cdots)^2 + \cdots \sim \phi_1 \tag{7.47}$$

to lowest order. Rearranging (7.45), *we therefore obtain*

$$\underbrace{y - 1}_{\sim\phi_1} - \underbrace{\frac{\log y}{|\log \epsilon|}}_{\sim\phi_1/|\log\epsilon|} = \frac{\log(|\log \epsilon|)}{|\log \epsilon|}. \tag{7.48}$$

We observe that the first term dominates the second, and obtain a balance in (7.48) *by choosing*

$$\phi_1(\epsilon) = \frac{\log(|\log \epsilon|)}{|\log \epsilon|}. \tag{7.49}$$

Indeed this does give $\phi_1 \ll 1$, *in the sense that* $\phi_1(\epsilon) \to 0$ *as* $\epsilon \to 0$, *so our assumed form of the expansion* (7.46) *is self-consistent (so far at least).*

Again we rearrange (7.48) *to*

$$\underbrace{y - 1 - \phi_1}_{\sim\phi_2} = \underbrace{\frac{\log y}{|\log \epsilon|}}_{\sim\phi_1/|\log\epsilon|}, \tag{7.50}$$

and a leading-order balance is now obtained by choosing

$$\phi_2(\epsilon) = \frac{\phi_1(\epsilon)}{|\log \epsilon|} = \frac{\log(|\log \epsilon|)}{|\log \epsilon|^2}. \tag{7.51}$$

Again we can verify that $\phi_2 \ll \phi_1$, *i.e. that* $\phi_2(\epsilon)/\phi_1(\epsilon) \to 0$ *as* $\epsilon \to 0$, *so that our expansion is self-consistent. We thus get the early terms in an expansion for the larger root of* (7.39), *namely*

$$x \sim |\log \epsilon| + \log(|\log \epsilon|) + \frac{\log(|\log \epsilon|)}{|\log \epsilon|} + \cdots \quad \text{as } \epsilon \to 0. \tag{7.52}$$

Exercise: *Show that the next term in the expansion is of order* $\big(\log(|\log \epsilon|)/|\log \epsilon|\big)^2$.

Chapter 8

Asymptotic analysis - Part 2

These lecture notes are based on material written by Derek Moulton and Peter Howell. *Please send any corrections or comments to* Renaud Lambiotte.

8.1 Regular perturbations in ODEs

We have shown how to use asymptotic methods to systematically approximate the roots of algebraic and transcendental equations. Now we explore how the same ideas may be used to find approximate solutions to ODEs.

Example 8.30. Find the approximate solution $y(x)$ of the following problem when $0 < \epsilon \ll 1$:

$$y''(x) = -\frac{1}{1+\epsilon y(x)^2}, \quad 0 < x < 1, \qquad y(0) = y(1) = 0. \tag{8.1}$$

The solution $y(x;\epsilon)$ depends on both x and ϵ. Since the problem (8.1) *contains only ϵ, and no other powers or functions of ϵ, it is reasonable to assume that the solution may be expressed as an asymptotic expansion in integer powers of ϵ, i.e.*

$$y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots . \tag{8.2}$$

Putting this into the ODE (8.1)*, we get*

$$\begin{aligned} y_0'' + \epsilon y_1'' + \cdots &= -\frac{1}{1+\epsilon(y_0+\epsilon y_1+\cdots)^2} \\ &\sim -1 + \epsilon y_0^2 + \cdots , \end{aligned} \tag{8.3}$$

with boundary conditions

$$0 = y(0,\epsilon) \sim y_0(0) + \epsilon y_1(0) + \cdots , \qquad 0 = y(1,\epsilon) \sim y_0(1) + \epsilon y_1(1) + \cdots . \tag{8.4}$$

By setting in turn the coefficient of each power of ϵ to zero, we get

$$\begin{aligned} O(1): \quad & y_0'' = -1, \quad y_0(0) = y_0(1) = 0 \\ \Rightarrow \quad & y_0(x) = \frac{1}{2}\,x(1-x), \end{aligned} \tag{8.5a}$$

$$\begin{aligned} O(\epsilon): \quad & y_1''(x) = y_0(x)^2 = \frac{1}{4}\,x^2(1-x)^2, \quad y_1(0) = y_1(1) = 0 \\ \Rightarrow \quad & y_1(x) = -\frac{1}{240}\,x(1-x)\left(2x^4 - 4x^3 + x^2 + x + 1\right), \end{aligned} \tag{8.5b}$$

and so on.

Example 8.31. Small oscillations of a pendulum

Let us return to the problem (7.5) *from* Example 7.20, *in the limit where the dimensionless parameter* α, *which measures the strength of the initial impulse, is small. To cast the problem in a more familiar form, set* $\alpha = \epsilon \ll 1$ *and* $u(\tau) = y(x)$ *so the problem reads*

$$y''(x) + \frac{\sin\big(\epsilon y(x)\big)}{\epsilon} = 0, \qquad y(0) = 0, \qquad y'(0) = 1. \tag{8.6}$$

Note that

$$\frac{\sin(\epsilon y)}{\epsilon} \sim y - \frac{1}{6}\,\epsilon^2 y^3 + \frac{1}{120}\,\epsilon^4 y^5 + \cdots \quad \text{as } \epsilon \to 0, \tag{8.7}$$

and the problem (8.6) *therefore contains only* even *powers of* ϵ. *It follows that we can seek the solution for* y *as an asymptotic expansion of the form*

$$y(x;\epsilon) \sim y_0(x) + \epsilon^2 y_2(x) + \epsilon^4 y_4(x) + \cdots \quad \text{as } \epsilon \to 0. \tag{8.8}$$

(If we included intermediate terms like $\epsilon y_1(x)$ *in the expansion* (8.8), *then on substitution into* (8.6) *we would find that they are identically zero.)*

Now we substitute (8.8) *into* (8.6) *and equate the coefficients of each power of* ϵ *as usual. At leading order we have the problem*

$$y_0'' + y_0 = 0, \qquad y_0(0) = 0, \qquad y_0'(0) = 1, \tag{8.9}$$

whose solution is given by

$$y_0(x) = \sin x. \tag{8.10}$$

At order ϵ^2, *we get*

$$y_2'' + y_2 = \frac{y_0^3}{6}, \qquad y_2(0) = 0, \qquad y_2'(0) = 0. \tag{8.11}$$

The right-hand side of (8.11) *can be written in the form*

$$\frac{1}{6}\sin^3(x) = \frac{1}{8}\sin(x) - \frac{1}{24}\sin(3x), \tag{8.12}$$

and we thus find the general solution for y_2 *to be*

$$y_2(x) = \frac{1}{192}\sin(3x) - \frac{1}{16}\,x\cos(x) + c_1\sin(x) + c_2\cos(x). \tag{8.13}$$

The integration constants are determined by applying the initial conditions, and thus we obtain

$$y_2(x) = \frac{3}{64}\sin(x) + \frac{1}{192}\sin(3x) - \frac{1}{16}\,x\cos(x). \tag{8.14}$$

The asymptotic expansion of the solution of the problem (8.6) *is thus given by*

$$y(x;\epsilon) \sim \sin(x) + \epsilon^2\left[\frac{3}{64}\sin(x) + \frac{1}{192}\sin(3x) - \frac{1}{16}\,x\cos(x)\right] + \cdots \tag{8.15}$$

as $\epsilon \to 0$.

Example 8.31 illustrates a potential difficulty that may be encountered when we try to write a function of *two* variables $y(x;\epsilon)$ as an asymptotic expansion in the limit $\epsilon \to 0$. The approximate solution (8.15) is a valid asymptotic expansion provided each term in the series is much smaller than the previous terms. This is certainly true if $x = O(1)$ and $\epsilon \ll 1$, but what happens when x gets very large? Eventually, when $x = O\left(1/\epsilon^2\right)$, the term proportional to $\epsilon^2 x$ becomes the same order as the leading-order term, and the expansion (8.15) ceases to be asymptotic. When x becomes sufficiently large, the expansion (8.15) is said to become *nonuniform*. In this example, the nonuniformity arises from the *secular term* proportional to $x\cos(x)$ in the solution for $y_2(x)$, which itself was a consequence of the forcing term proportional to $\sin(x)$ on the right-hand side of (8.11). In general, in problems like (8.11), we expect to find a secular term in the solution whenever the right-hand side contains a term that is in the complentary function (i.e. in the kernel of the differental operator on the left-hand side).

One can modify the solution (8.15) to a form that is valid for larger values of x by using the *method of multiple scales* — see §8.4.3 for a simple implementation of the method or C5.5 Perturbation Methods for the more general version. For the moment we consider another example where taking an infinite interval for the independent variable leads to trouble.

Example 8.32. Find the approximate solution of the IVP

$$y'(x) = y(x) - \epsilon y(x)^2, \quad x > 0, \qquad y(0) = 1, \tag{8.16}$$

as a regular asymptotic expansion in the limit $\epsilon \to 0$.

Writing the solution as an asymptotic expansion

$$y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x) + \cdots, \tag{8.17}$$

and equating powers of ϵ in the usual way gives us

$$y_0(x) = \mathrm{e}^x, \tag{8.18}$$

and then

$$y_1'(x) = y_1(x) - \mathrm{e}^{2x}, \quad y_1(0) = 0 \qquad \Rightarrow \quad y_1(x) = \mathrm{e}^x - \mathrm{e}^{2x}. \tag{8.19}$$

We thus obtain the following asymptotic expansion for the solution:

$$y(x;\epsilon) \sim \mathrm{e}^x + \epsilon\left(\mathrm{e}^x - \mathrm{e}^{2x}\right) + \cdots \quad \text{as } \epsilon \to 0. \tag{8.20}$$

Now we see that the expansion becomes nonuniform when $\epsilon\mathrm{e}^{2x} \sim \mathrm{e}^x$, i.e. when $x = O\left(|\log\epsilon|\right)$.

In this case, we can solve the simple ODE (8.16) exactly to get

$$y(x;\epsilon) = \frac{\mathrm{e}^x}{1 + \epsilon\left(\mathrm{e}^x - 1\right)}. \tag{8.21}$$

Expansion of the solution (8.21) in powers of ϵ indeed reproduces the approximation (8.20), assuming that $x = O(1)$. However, the exact solution (8.21) satisfies $y(x) \to 1/\epsilon$ as $x \to \infty$, while the approximate solution (8.20) suggests that $y(x)$ grows without bound. Evidently the asymptotic approximation is valid only if x is not too large (specifically if $x \ll |\log\epsilon|$), and a different approach would be needed to approximate the solution for larger value of x. [Try substituting $x = \log(1/\epsilon) + X$ into (8.21) before expanding in powers of ϵ.]

8.2 Boundary layers

8.2.1 A first example

The solution of an ODE like (8.16), containing a parameter ϵ, is a function of *two* variables, namely ϵ and the independent variable x of the ODE. To obtain an approximate solution when ϵ is small, our starting point is generally to seek the solution as a regular asymptotic expansion

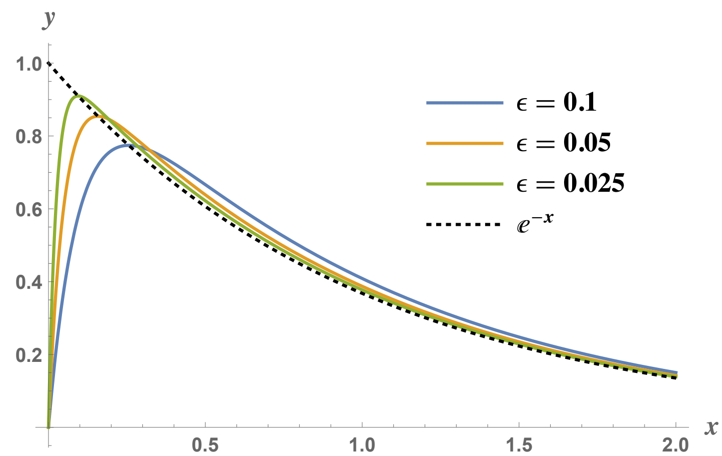


Figure 8.1: The function $y(x;\epsilon)$ given by (8.24) plotted versus x with three different values of ϵ. The leading-order outer solution e^{-x} is plotted as a black dotted curve.

of the form $y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x) + \cdots$. However, the previous examples demonstrate that such an expansion may only be valid for a limited range of values of x. This may reduce the usefulness of the approximation. Even worse, it is not even clear how to determine the solution uniquely if a boundary condition is imposed in a region where the asymptotic expansion is not valid, as illustrated by the following simple example.

Example 8.33. Find the approximate solution of the IVP

$$\epsilon y'(x) + y(x) = e^{-x}, \quad x > 0, \qquad y(0) = 0. \tag{8.22}$$

If we seek the solution as a regular asymptotic expansion of the form $y \sim y_0 + \epsilon y_1 + \cdots$, then we find

$$\begin{aligned} y_0(x) &= e^{-x}, \\ y_1(x) &= -y_0'(x) = e^{-x}, \end{aligned} \tag{8.23}$$

and so on. The problem is that we can never satisfy the boundary condition $y(0) = 0$!

The difficulty that in Example 8.33 occurs because the small parameter ϵ multiplies the highest derivative in the problem. In the limit $\epsilon \to 0$, the ODE (8.22) reduces to an *algebraic* equation, namely $y(x) \sim e^{-x}$, and it becomes impossible to impose any initial condition.

The *exact solution* of (8.22) is given by

$$y(x;\epsilon) = \frac{e^{-x}}{1-\epsilon} - \frac{e^{-x/\epsilon}}{1-\epsilon}, \tag{8.24}$$

which is plotted versus x for small but nonzero values of ϵ in Figure 8.1. We see that $y(x) \sim e^{-x}$ *does* provide a good approximation to the exact solution for nearly all values of x. However, e^{-x} stops being a good approximation to $y(x)$ in a narrow region, called a *boundary layer*, close to $x = 0$, where the solution rapidly adjusts to satisfy the boundary condition $y(0) = 0$. Examining the exact solution (8.24), we can see that the rapid variation near $x = 0$ is caused by the second term containing $e^{-x/\epsilon}$ ceasing to be negligible. Hence we expect the boundary layer to occur when $x = O(\epsilon)$.

To solve problems like (8.22), we use the *method of matched asymptotic expansions*. We construct *two different* asymptotic expansions for the solution $y(x)$: one in the *outer region* where $x = O(1)$, and the other in the very narrow boundary layer near $x = 0$, also known as the *inner region*. Since these two expansions are approximating the *same function* $y(x)$, they must be self-consistent, and this allows them to be joined up by asymptotic *matching*.

8.2.2 Inner and outer expansions

To get the ideas clear, consider the example above where the exact solution (8.24) is known, and we want to find the inner and outer expansions. When $x = O(1)$, the second term in (8.24) is exponentially small, and thus

$$\begin{aligned} y(x;\epsilon) &\sim \frac{\mathrm{e}^{-x}}{1-\epsilon} + \text{exp small} \\ &\sim \mathrm{e}^{-x} + \epsilon \mathrm{e}^{-x} + \cdots \quad \text{as } \epsilon \to 0, \end{aligned} \tag{8.25}$$

which reproduces the first two terms in the asymptotic expansion found in Exercise 8.33. This is the *outer expansion*, which applies when $x = O(1)$.

We can see from the exact solution (8.24) that the second term proportional to $\mathrm{e}^{-x/\epsilon}$ stops being negligible when $x = O(\epsilon)$. We therefore examine the inner region by *rescaling* the independent variable. If we set $x = \epsilon X$ and $y(x;\epsilon) = Y(X;\epsilon)$, and now assume that $X = O(1)$ (corresponding to $x = O(\epsilon)$), then the exact solution (8.24) becomes

$$\begin{aligned} Y(X;\epsilon) &= \frac{\mathrm{e}^{-\epsilon X} - \mathrm{e}^{-X}}{1-\epsilon} \\ &\sim \left(1 - \mathrm{e}^{-X}\right) + \epsilon\left(1 - X - \mathrm{e}^{-X}\right) + \cdots \quad \text{as } \epsilon \to 0. \end{aligned} \tag{8.26}$$

This is the *inner expansion*, which is valid when $X = x/\epsilon = O(1)$.

8.2.3 Matching

In the previous section we showed how to create different asymptotic expansions of a single function which hold in different regions. Now we check that the two different approximations are self-consistent, in that they connect smoothly as x increases from $O(\epsilon)$ to $O(1)$. This method of joining two asymptotic expansions in different regions is called *matching*. For simplicity we restrict attention to only the leading-order terms outer and inner expansions (8.25) and (8.26), namely

$$y_0(x) = \mathrm{e}^{-x}, \qquad\qquad Y_0(X) = 1 - \mathrm{e}^{-X}, \tag{8.27}$$

with $X = x/\epsilon$. The two approximations are plotted in Figure 8.1. We see that the outer and inner solutions do indeed give good approximations to the exact solution (8.24) when $x = O(1)$ and when $x = O(\epsilon)$ respectively. The underlying principle of asymptotic matching is that both approximations should be valid in an intermediate *overlap region*.

To examine such an overlap region, let us rescale $x = \delta\xi$ and $X = (\delta/\epsilon)\xi$, where δ is chosen to be intermediate between the inner and outer scalings for x, i.e. $\epsilon \ll \delta \ll 1$. The (8.27) becomes

$$y_0(\delta\xi) = \mathrm{e}^{-\delta\xi} \sim 1 + O(\delta) \quad \text{as } \delta \to 0, \tag{8.28a}$$

$$Y_0(\delta X/\epsilon) = 1 - \mathrm{e}^{-\delta X/\epsilon} \sim 1 + \text{exp small} \quad \text{as } \frac{\epsilon}{\delta} \to 0, \tag{8.28b}$$

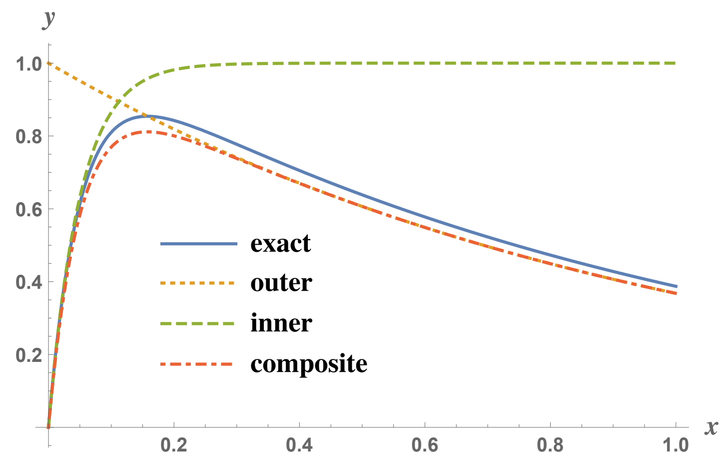


Figure 8.2: The exact expression (8.24) for $y(x;\epsilon)$, the leading-order inner and outer approximations (8.27), and the composite approximation (8.31), plotted with $\epsilon = 0.05$. plotted versus x with three different values of ϵ. The leading-order outer solution e^{-x} is plotted as a black dotted curve.

and we see that the two approximations do agree and are both equal to 1 at lowest order in the overlap region.

A general statement of the leading-order matching principle demonstrated by (8.28) is

$$\lim_{x\to 0} y_0(x) = \lim_{X\to\infty} Y_0(X). \tag{8.29}$$

Loosely interpreted: the behaviour of the outer solution as we go *into* the boundary layer must equal the behavour of the inner solution as we go *out* of the boundary later. More complicated versions of the matching principle (8.29) can be formulated to match inner and outer expansions up to arbitrary orders in ϵ, but we will only consider leading-order matching here.

Figure 8.2 demonstrates that the outer approximation works well when $x = O(1)$ but not when x is close to zero. Similarly, the inner approximation is good when x is small but not when $x = O(1)$. It is sometimes helpful to create a single function that gives a reasonable approximation for all values of x. Such a *composite* expansion can be constructed by forming

$$\text{composite expansion} = \text{inner expansion} + \text{outer expansion} - \text{common limit}, \tag{8.30}$$

where the common limit refers to components shared by the inner and outer approximations, which must subtracted to remove double-counting. At leading order, the common limit is given by $\lim_{x\to 0} y_0(x)$ or by $\lim_{X\to\infty} Y_0(X)$, and these two expressions are equal by the matching principle (8.29).

A composite expansion combining the inner and outer approximations (8.27) is given by

$$\begin{aligned} y_{\text{comp}}(x) &= \underbrace{y_0(x)}_{\text{outer}} + \underbrace{Y_0(X)}_{\text{inner}} - \underbrace{1}_{\text{common limit}} \\ &= e^{-x} - e^{-x/\epsilon}. \end{aligned} \tag{8.31}$$

Figure 8.2 verifies that (8.31) gives a good approximation to the exact solution (8.24) for all values of x.

8.2.4 Getting the expansion from the ODE

So far, we have constructed inner and outer approximations to a *known* solution (8.24). Now let us see whether we could have obtained the same approximations directly from the problem (8.22), if we did not have the exact solution to guide us. We have already seen that substitution of a naïve regular expansion of the form $y \sim y_0 + \epsilon y_1 + \cdots$ into (8.22) produces the outer approximation (8.25).

We note that (8.25) does not satisfy the boundary condition $y(0) = 0$, and we infer that the boundary condition can only be imposed if *the solution has a boundary layer at* $x = 0$. To examine this boundary layer, we have to rescale x: let us set $x = \delta X$ and $y(x) = Y(X)$ where $\delta \ll 1$ is to be determined. Then in terms of these inner variables, the problem (8.22) becomes

$$\frac{\epsilon}{\delta} Y'(X) + Y(X) = \mathrm{e}^{-\delta X}, \quad X > 0, \qquad Y(0) = 0. \tag{8.32}$$

We can balance all three terms in (8.32) by choosing $\delta = \epsilon$. We already know that the boundary layer thickness is of order ϵ from the exact solution (8.24), but here we determine the appropriate choice of δ directly by seeking a *dominant balance* in the ODE (8.32).

Once we have chosen $\delta = \epsilon$, the governing equation (8.32) in the inner region becomes

$$Y'(X) + Y(X) = \mathrm{e}^{-\epsilon X} \sim 1 - \epsilon X + \cdots . \tag{8.33}$$

Now we can seek an inner expansion of the usual form $Y \sim Y_0 + \epsilon Y_1 + \cdots$ and solve for each term successively. At leading order, we get

$$Y_0'(X) + Y_0(X) = 1, \qquad Y_0(0) = 0, \tag{8.34}$$

whose solution is easily found to be $Y_0(X) = 1 - \mathrm{e}^{-X}$, in agreement with (8.26). Thus we have successfully found the leading-order inner and outer approximations directly from the ODE and boundary conditions.

Before proceeding to apply the same ideas to more general BVPs, we note some general ideas that this simple example has illustrated.

(i) The boundary layer in the solution to (8.22) occurs because the small parameter ϵ multiplies the highest derivative in the ODE. When $x = O(1)$, we have

$$\underbrace{\epsilon y'(x)}_{\text{small}} + \underbrace{y(x) - \mathrm{e}^{-x}}_{\text{balance}} = 0 \tag{8.35}$$

and thus, in the limit as $\epsilon \to 0$, the *order* of the ODE is reduced, and we are no longer able to impose the boundary condition.

(ii) However, when there is a boundary layer, the derivative $y'(x)$ becomes very big (see e.g. Figure 8.1), such that the first term in (8.35) is no longer negligible at leading order.

(iii) This magnification of the gradient is represented by the change to the local variable $X = x/\epsilon$; by the chain rule we get $y'(x) = \epsilon^{-1} Y'(X)$.

(iv) The correct boundary layer scaling for x is found by seeking a dominant balance in the ODE; in particular, we want to bring the highest derivative back into the problem so that we are able to impose the boundary condition.

(v) The solutions of the inner and outer problems give us two alternative approximations for $y(x;\epsilon)$ — one that holds when $x = O(1)$ and one that holds when $x = O(\epsilon)$.

(vi) The leading-order inner and outer approximations can be reconciled by using the matching condition (8.29): the limit of the outer solution as we go into the boundary layer must equal the limit of the inner solution as we go out of the boundary layer.

In general, we can expect boundary layers (or something even worse) to occur whenever the small parameter ϵ multiplies the highest derivative in an ODE. The situation is analogous to Example 7.28, where we had to solve a quadratic equation with ϵ multiplying x^2. In both cases, if we set $\epsilon = 0$, the degree of the problem is reduced, and we do not obtain the full family of solutions. In both cases, the difficulty is resolved by rescaling x to get a dominant balance in the equation. In general, problems where setting ϵ to zero reduces the degree of the problem are called *singular perturbation* problems.

8.3 Boundary layers in BVPs

8.3.1 A simple example

In Example 8.33, we were unable to impose the boundary condition $y(0) = 0$ on the outer solution, and we deduced that there must be a boundary layer at $x = 0$. Once we found the inner and outer solutions, the matching condition (8.29) was satisfied identically: we could use it to verify that the inner and outer solutions are self-consistent, but it did not give us any further information about the solution. For higher-order BVPs, the situation is less clear. The location of any boundary layers may not be obvious in advance, and in general we will need to match the inner and outer approximations to determine the solution uniquely. We will illustrate the issues involved by solving a simple example.

Example 8.34. Find the leading-order solution of the BVP

$$\epsilon y''(x) + y'(x) = 1, \quad 0 < x < 1, \qquad y(0) = y(1) = 0 \tag{8.36}$$

in the limit $\epsilon \to 0$.

It is easy to solve (8.36) *exactly, but let us try to proceed using asymptotic expansions without assuming that we have the exact solution to hand.*

Outer solution *We try for a regular expansion with* $y \sim y_0 + \epsilon y_1 + \cdots$ *and obtain at leading order*

$$y_0'(x) = 1 \quad \Rightarrow \quad y_0(x) = x + A, \tag{8.37}$$

where A *is an integration constant. Since the limit* $\epsilon \to 0$ *has reduced* (8.36) *from a second-order to a first-order ODE, we are unable to impose both of the boundary conditions. We deduce that there is a boundary layer somewhere, but where?*

Let us assume for the moment that the boundary layer is at $x = 0$. *This means that we can apply the boundary condition* $y(1) = 0$ *directly to the outer solution* (8.37) *and thus obtain*

$$y_0(x) = x - 1. \tag{8.38}$$

Then the outer solution does not satisfy the boundary condition $y(0) = 0$, *and we hope to resolve this by examining a boundary layer at* $x = 0$.

Boundary layer *We find the size of the boundary layer by scaling $x = \delta X$ and $y(x) = Y(X)$, where $\delta \ll 1$ is to be determined. Putting this change of independent variables into the problem* (8.36)*, we get*

$$\frac{\epsilon}{\delta^2} Y''(X) + \frac{1}{\delta} Y'(X) = 1. \tag{8.39}$$

Now we choose δ to achieve a dominant balance, in particular one that makes the highest derivative term no longer negligible. In this case this we achieve this by balancing the first two terms and thus taking $\delta = \epsilon$, so the ODE (8.39) *becomes*

$$Y''(X) + Y'(X) = \epsilon. \tag{8.40}$$

Now we can assume a simple expansion for the inner solution with $Y(X) \sim Y_0(X) + \epsilon Y_1(X) + \cdots$. At leading order we get

$$Y_0''(X) + Y_0'(X) = 0, \tag{8.41}$$

along with the boundary condition $Y_0(0) = 0$ (coming from the boundary condition for y at $x = 0$). The leading-order solution of the inner problem is thus given by

$$Y_0(X) = B\left(1 - \mathrm{e}^{-X}\right), \tag{8.42}$$

where B is an integration constant. Here we cannot solve for B, and therefore cannot determine the inner solution uniquely, using only the information in the boundary layer. To proceed, we must ensure that the inner and outer solutions match.

Matching *Now we impose the matching principle* (8.29)*. In this case, the inner limit of the outer solution is $\lim_{x\to 0} y_0(x) = -1$, and the outer limit of the inner solution is $\lim_{X\to\infty} Y_0(X) = B$. The matching principle tells us that these must be equal, and hence $B = -1$ and the leading-order inner solution is given by*

$$Y_0(X) = -1 + \mathrm{e}^{-X}. \tag{8.43}$$

We can construct a composite expansion by combining (8.38) *and* (8.43)*, noting that the common limit here is equal to -1, to get*

$$y_{\mathrm{comp}}(x) = y_0(x) + Y_0(X) - (-1) = x - 1 + \mathrm{e}^{-x/\epsilon}, \tag{8.44}$$

which is a very good approximation of the exact solution of (8.36)*, namely*

$$y(x) = x - \frac{1 - \mathrm{e}^{-x/\epsilon}}{1 - \mathrm{e}^{-1/\epsilon}}. \tag{8.45}$$

8.3.2 Locating the boundary layer

In Example 8.34, to get the leading-order solution, we assumed that the boundary layer is at $x = 0$, and therefore applied the boundary condition at $x = 1$ directly to the outer solution. The resulting leading-order approximation is in good agreement with the exact solution, but how could we have known in advance where to look for a boundary layer without having the exact solution to guide us?

Well, suppose that we had instead assumed the boundary layer to be at $x = 1$. We could attempt to analyse such a layer by using a local variable ξ such that $x = 1 - \delta\xi$ and $y(x) = \eta(\xi)$, with $\delta \ll 1$ to be determined. (It is not necessary to include the minus sign in the definition of ξ, but doing so means that we are dealing with $\xi > 0$ rather than $\xi < 0$.) Then equation (8.36) is transformed to

$$\frac{\epsilon}{\delta^2} \eta''(\xi) - \frac{1}{\delta} \eta'(\xi) = 1, \tag{8.46}$$

and a dominant balance between the first two terms is again achieved by choosing $\delta = \epsilon$. The leading-order problem in the inner region is thus

$$\eta_0''(\xi) - \eta_0'(\xi) = 0, \quad \xi > 0, \qquad \eta_0(0) = 0, \tag{8.47}$$

whose general solution is

$$\eta_0(\xi) = A\left(\mathrm{e}^{\xi} - 1\right), \tag{8.48}$$

where A is an integration constant. The problem is that the proposed inner solution (8.48) *grows exponentially* as ξ tends to infinity, and it is therefore impossible to match this solution to the solution in the outer region.

Note: In the above analysis, we assume that $0 < \epsilon \ll 1$. If $\epsilon = -|\epsilon|$ is negative, then the boundary layer *is* at $x = 1$, and the analysis in §8.3.1 needs to be redone.

There is a general principle for locating the boundary layers in simple two-point boundary-value problems like (8.36). Consider the general ODE

$$\epsilon y''(x) + P_1(x)y'(x) + P_0(x)y(x) = R(x), \quad a < x < b, \tag{8.49}$$

with boundary conditions given at $x = a$ and $x = b$. Assume that the coefficients P_0, P_1 and R are smooth and bounded, and that $P_1(x)$ *is non-zero* for $x \in [a, b]$.

The leading-order outer solution is found via a regular asymptotic expansion of the form $y \sim y_0 + \epsilon y_1 + \cdots$, which leads to

$$y_0'(x) + \frac{P_0(x)}{P_1(x)}\, y_0(x) = \frac{R(x)}{P_1(x)}. \tag{8.50}$$

This can be solved without difficulty on $[a, b]$ because of our assumptions about P_0, P_1 and R. However, because (8.50) is just a first-order ODE, we will be unable to impose both boundary conditions: there must be a boundary layer at one end of the domain, but which end?

Suppose we look for a boundary layer at $x = a$, via the re-scaling $x = a + \delta X$ and $y(x) = Y(X)$. It is clear that a dominant balance between the first two terms in (8.49) is achieved when $\delta = \epsilon$, and the leading-order inner equation is then

$$Y_0''(X) + P_1(a)Y_0'(X) = 0, \quad X > 0. \tag{8.51}$$

This has solutions of the form $Y_0(X) = A + B\mathrm{e}^{-P_1(a)X}$, and we can match with the outer only if the inner solution has a *decaying* exponential, i.e. if $P_1(a) > 0$.

Similarly, we can look for a boundary layer at $x = b$ with the scaling $x = b - \epsilon\xi$ and $y(x) = \eta(\xi)$, and get to leading order

$$\eta_0''(\xi) - P_1(b)\eta_0'(\xi) = 0, \quad \xi > 0. \tag{8.52}$$

Now the inner solution $\eta_0(\xi) = A + B\mathrm{e}^{P_1(b)\xi}$ can match with the outer only if $P_1(b) < 0$. Given our assumption that P_1 does not change sign, we conclude that:

- the boundary layer is at the *left-hand* boundary (i.e. $x = a$) if $P_1(x) > 0$, or

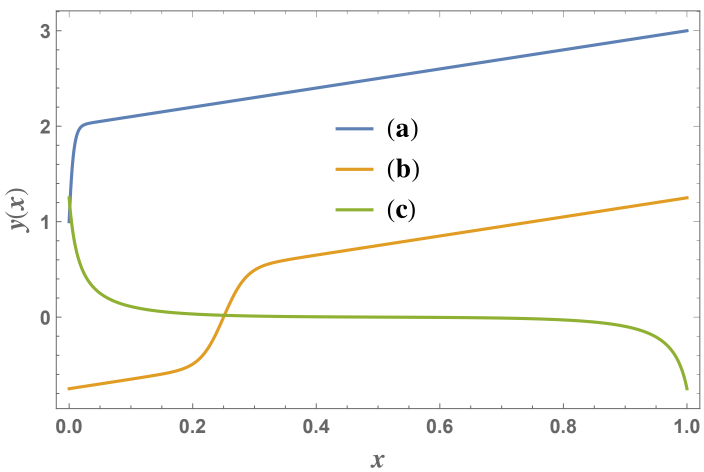


Figure 8.3: Solutions of the ODE (8.53) satisfying each of the boundary conditions (8.54), computed with $\epsilon = 0.01$.

- at the *right-hand* boundary (i.e. $x = b$) if $P_1(x) < 0$.

One can imagine that more complicated behaviour is possible if $P_1(x)$ *does* change sign. The solution may have *two* boundary layers — one at each end of the domain — or an *internal* boundary layer somewhere in $a < x < b$ (and even more complicated structures are possible: see below).

8.4 More general perturbation methods for ODEs

8.4.1 Introduction

We have seen some examples of asymptotic methods applied to simple algebraic equations and ODE problems. More generally, ODEs containing small parameters can exhibit much more complicated behaviour than we have seen so far, and a range of asymptotic techniques have been developed to deal with them, which can be studied in more detail in C5.5 Perturbation methods. Here we give a brief (non-examinable) survey of some of the possible generalisations of the theory that has been developed so far.

8.4.2 Multiple or interior boundary layers

We argued in §8.3.2 that, in a second-order singular BVP, the location of the boundary layer can be predicted from the sign of the coefficient of the first derivative of y. But what happens if that coefficient changes sign somewhere in the domain? Here is a (relatively) simple example that illustrates what kind of behaviour can happen.

Example 8.35. Find the leading-order solution to the ODE

$$\epsilon y''(x) + y(x)y'(x) - y(x) = 0, \quad 0 < x < 1, \tag{8.53}$$

in the limit $\epsilon \to 0$, subject to each of the following sets of boundary conditions:

$$y(0) = 1, \qquad y(1) = 3; \tag{8.54a}$$

$$y(0) = -3/4, \qquad y(1) = 5/4; \tag{8.54b}$$

$$y(0) = 5/4, \qquad y(1) = -3/4. \tag{8.54c}$$

In case (8.54a)*, the coefficient of* y' *in* (8.53) *is* y*, which is positive at both ends of the domain. The argument used in §8.3.2 works: there is a boundary layer only at* $x = 0$*. The leading-order inner and outer solutions may be found and matched in the usual way (with boundary layer thickness* ϵ*).*

In case (8.54b)*, the coefficient of* y' *in* (8.53) changes sign*, and it appears that a boundary layer is not allowed at either end of the domain. In this case, there is an* internal *boundary layer, at* $x = x_*$ *say, somewhere between* $x = 0$ *and* $x = 1$*. To solve the problem, we have to solve* two *outer problems: one in* $0 < x < x_*$ *and one in* $x_* < x < 1$*, and also solve for the boundary layer at* $x = x_*$*. By matching all three regions together, one can determine the location of the interior boundary layer (namely* $x_* = 1/4$*).*

Case (8.54c) *is even worse. In this case the signs of the coefficient of* y' *in* (8.53) *suggest that there might be a boundary layer at both ends of the domain. Indeed this turns out to be true, but the structure in this case is more complicated. The leading-order outer solution is given by* $y_0(x) = 0$ *(i.e. the other root of the leading-order outer equation* $y_0\left(y_0' - 1\right) = 0$*). The boundary layer at* $x = 0$ *has thickness* ϵ *again, but the inner solution in the boundary layer does not match directly with the outer solution. Instead, there is a further intermediate region in which* $x = O\left(\epsilon^{1/2}\right)$ *and* $y = O\left(\epsilon^{1/2}\right)$*. This is a so-called "triple deck" structure with one boundary layer nested inside another one. The boundary layer at* $x = 1$ *has an analogous structure.*

Numerically computed solutions to (8.53) *with* $\epsilon = 0.01$ *satisfying each of the boundary conditions in* (8.54) *are plotted in* Figure 8.3*. The structure of each solution is exactly as predicted: in case* (a) *there is just a boundary layer at* $x = 0$*; in case* (b) *there is an internal boundary layer close to* $x = 1/4$*; and in case* (c) *there is a boundary layer at both ends of the domain.*

Example 8.35 illustrates several issues that can arise in more complicated boundary layer problems. First: it may not be clear in advance where to look for boundary layers. Second: in general, the boundary layer analysis may require us to rescale the dependent variable y as well as the independent variable x. Finally: in the intermediate region encountered in Case (8.54c), we end up having to solve the full ODE, with no simplification (Try rescaling the ODE (8.53) with $x = \epsilon^{1/2}\xi$ and $y(x) = \epsilon^{1/2}\eta(\xi)$).

8.4.3 Slowly varying oscillations

In Example 8.31, we analysed small oscillations of a pendulum and found that we get spurious "secular" terms in the solution if we try a naïve regular asymptotic expansion. The origin of these terms can be understood by considering a very simple example.

Example 8.36. Solve the IVP

$$y''(x) + (1 + \epsilon)y(x) = 0, \quad x > 0, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{8.55}$$

The exact solution is

$$y(x) = \cos\left(x\sqrt{1 + \epsilon}\right), \tag{8.56}$$

but if we try to expand this solution for small ϵ*, we get*

$$y(x) \sim \cos x - \frac{\epsilon}{2} x \sin x + \cdots. \tag{8.57}$$

Thus a secular term has appeared in the expansion, meaning that the expansion stops being valid when $x = O(1/\epsilon)$*. The fact that the exact solution* (8.56) *is a periodic function of* x *has become lost in our particular choice of asymptotic expansion.*

The difficulty encountered in Example 8.36 can be fixed relatively easily using the *Poincaré–Lindstedt method.* Here we know that we are seeking periodic solutions, but with a period that is a function of ϵ. The trick is to make the substitution

$$X = \omega x, \tag{8.58}$$

where the frequency ω is not known in advance, but is chosen to make the solution 2π-periodic as a function of X.

With $y(x) = Y(X)$, the problem (8.55) is transformed to

$$\omega^2 Y''(X) + (1+\epsilon)Y(X) = 0, \quad X > 0, \qquad Y(0) = 1, \qquad Y'(0) = 0. \tag{8.59}$$

Now we expand *both* Y *and* ω in powers of ϵ:

$$Y(X) \sim Y_0(X) + \epsilon Y_1(X) + \cdots, \qquad \omega \sim 1 + \epsilon\omega_1 + \cdots, \tag{8.60}$$

where we have anticipated that the leading-order frequency of oscillations is equal to 1.

At $O(1)$, we get

$$Y_0''(X) + Y_0(X) = 0, \quad X > 0, \qquad Y_0(0) = 1, \qquad Y_0'(0) = 0, \tag{8.61}$$

whose solution is

$$Y_0(X) = \cos X. \tag{8.62}$$

At $O(\epsilon)$, we find that $Y_1(X)$ satisfies the ODE

$$Y_1''(X) + Y_1(X) = -2\omega_1 Y_0''(X) - Y_0(X) = (2\omega_1 - 1)\cos X, \tag{8.63}$$

along with the initial conditions $Y_1(0) = Y_1'(0) = 0$, Now we insist that $Y_1(X)$ should be a 2π-periodic function of X, which means that it cannot contain any secular terms like $X \sin X$. We must therefore eliminate the "resonant" term proportional to $\cos X$ from the right-hand side of (8.63) by choosing $\omega_1 = 1/2$. Thus the oscillation frequency is given by an asymptotic expansion of the form

$$\omega \sim 1 + \frac{\epsilon}{2} + \cdots \quad \text{as } \epsilon \to 0, \tag{8.64}$$

which indeed agrees with the exact frequency $\omega = \sqrt{1+\epsilon}$ from equation (8.56).

The same method works for the problem of small oscillations of a pendulum from Example 8.31. Again the secular terms in the expansion can be suppressed and one can determine an asymptotic expansion for the frequency of the form $\omega \sim 1 - \epsilon^2/16 + O\left(\epsilon^4\right)$. The Poincaré–Lindstedt method is a simplified version of the more general *method of multiple scales*, which can describe oscillations that are not precisely periodic but instead vary slowly with x.

8.4.4 Fast oscillations

When our small parameter ϵ multiplies the highest derivative in an ODE, it does not always lead to the formation of boundary layers: it is also possible for the solution to exhibit rapid oscillations instead, as the following simple example shows

Example 8.37. Solve the BVP

$$\epsilon^2 y''(x) + y(x) = 0, \qquad y(0) = 1, \qquad y(1) = 0. \tag{8.65}$$

Note that, from the Fredholm Alternative*, we expect there to be problems whenever* $\epsilon = 1/\left(n^2\pi^2\right)$ *where* n *is an integer, but let's ignore that for the moment.*

If we try to proceed in the usual way by seeking the solution of (8.65) *as an asymptotic expansion in powers of* ϵ*, we just get* $y(x) \sim 0$*, to all algebraic orders in* ϵ*. Thus it appears to be impossible to impose the boundary conditions, and we might guess that there is a boundary layer at* $x = 0$*. But the inner rescaling* $x = \epsilon X$ *doesn't help, because the inner equation just gives oscillatory solutions which cannot match with the outer.*

One way to tackle problems like (8.65) is to use the *WKBJ method*. We seek the solution in the form

$$y(x) = A(x)\mathrm{e}^{\mathrm{i}u(x)/\epsilon}, \tag{8.66}$$

where both the *phase* $u(x)$ and the *amplitude* $A(x)$ are to be determined. By plugging the ansatz (8.66) into the ODE (8.65), we obtain

$$A(x)\left[1 - u'(x)^2\right] + \mathrm{i}\epsilon\left[2A'(x)u'(x) + A(x)u''(x)\right] + \epsilon^2 A''(x) = 0. \tag{8.67}$$

At leading order we get the *eikonal equation* $u'(x)^2 = 1$, and we deduce that the phase is simply given by $u(x) = \pm x$ (plus an irrelevant constant). We can then write the amplitude as a regular asymptotic expansion $A(x) \sim A_0(x) + \epsilon A_1(x) + \cdots$. In this simple problem, we just get $A'(x) = 0$, at all orders in ϵ, and indeed the ODE is solved exactly by $y(x) = A\mathrm{e}^{\pm\mathrm{i}x/\epsilon}$, with $A =$ constant. The general solution is then a linear combination of the form

$$y(x) = C_1\mathrm{e}^{\mathrm{i}x/\epsilon} + C_2\mathrm{e}^{-\mathrm{i}x/\epsilon}, \tag{8.68}$$

and the arbitrary constants can be determined from the boundary conditions.

Here is a slightly less trivial example, where we determine the asymptotic behaviour of the zeroth order Bessel functions as the argument tends to infinity.

Example 8.38. Find the asymptotic behaviour of solutions to Bessel's equation of order zero:

$$y''(x) + \frac{1}{x}y'(x) + y(x) = 0, \tag{8.69}$$

in the limit as $x \to \infty$.

We can consider the behaviour for large x *by making the rescaling* $x = X/\epsilon$ *and* $y(x) = Y(X)$*, where* $\epsilon \ll 1$ *and* $X = O(1)$*. Then* (8.69) *is transformed to*

$$\epsilon^2 Y''(X) + \frac{\epsilon^2}{X}Y'(X) + Y(X) = 0. \tag{8.70}$$

Now we apply the WKBJ ansatz by writing $Y(X) = A(X)\mathrm{e}^{\mathrm{i}u(X)/\epsilon}$*, and* (8.70) *is transformed to*

$$\left[1 - u'(X)^2\right] + \mathrm{i}\epsilon\left[\left(\frac{2A'(X)}{A(X)} + \frac{1}{X}\right)u'(X) + u''(X)\right] + \epsilon^2\left[\frac{A''(X)}{A(X)} + \frac{A'(X)}{XA(X)}\right] = 0. \tag{8.71}$$

In this example, we get the same eikonal equation for $u(X)$ *as above, with solution* $u(x) = \pm X$*, and we are then left to solve*

$$\pm\left[\frac{2A'(X)}{A(X)} + \frac{1}{X}\right] - \mathrm{i}\epsilon\left[\frac{A''(X)}{A(X)} + \frac{A'(X)}{XA(X)}\right] = 0. \tag{8.72}$$

The leading-order amplitude therefore satisfies

$$\frac{A_0'(X)}{A_0(X)} = -\frac{1}{2X}, \tag{8.73}$$

whose solution is $A_0(X) = const/X^{1/2}$. *Thus solutions to* (8.70) *take the form*

$$Y(X) \sim \frac{C_1 \mathrm{e}^{\mathrm{i}X/\epsilon} + C_2 \mathrm{e}^{-\mathrm{i}X/\epsilon}}{\sqrt{X}} \quad \text{as } \epsilon \to 0. \tag{8.74}$$

In terms of the unscaled variable x, *we can write*

$$y(x) \sim \frac{c_1}{\sqrt{x}} \sin(x) + \frac{c_2}{\sqrt{x}} \cos(x) \quad \text{as } x \to \infty, \tag{8.75}$$

for some constants c_1 *and* c_2.

(The standard Bessel functions of the first and second kind *are normalised such that*

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right), \qquad Y_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right) \tag{8.76}$$

as $x \to \infty$.*)*