B5.4 Waves & Compressible Flow

Question Sheet 0 Solutions

1. See Chapter 1 of lecture notes.

2. (a) Change variables to t and $\eta = x - ct$ using the chain rule

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - c\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial x} \to \frac{\partial}{\partial \eta}. \tag{1}$$

to obtain

$$\frac{\partial u}{\partial t} = 0, \qquad u = f(\eta) \text{ at } t = 0. \tag{2}$$

and so $u = f(\eta) = f(x - ct)$.

(b) Change variables to obtain

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0, \tag{3}$$

and perform the change of variables $\xi = x + ct$, $\eta = x - ct$. By applying the chain rule to the derivatives, and substituting into the equation, we obtain

$$-4c^2 \frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0. \tag{4}$$

Hence

$$\frac{\partial \phi}{\partial \xi} = f(\xi), \tag{5}$$

a constant w.r.t. η, and an arbitrary function of ξ. Therefore,

$$\phi = F(\xi) + G(\eta), \tag{6}$$

where $G(\eta)$ is an arbitrary function of η and

$$F(\xi) := \int^{\xi} f(\zeta)\, \mathrm{d}\zeta. \tag{7}$$

3. See Chapter 1 of lecture notes for first part.

Entropy is defined by

$$S = S_0 + c_v \log\left(p/\rho^\gamma\right). \tag{1}$$

Start from conservation of energy and mass,

$$\rho c_v \frac{\mathrm{D}T}{\mathrm{D}t} = -p\left(\boldsymbol{\nabla} \cdot \mathbf{u}\right) + \boldsymbol{\nabla} \cdot \left(k \boldsymbol{\nabla} T\right), \tag{2}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\left(\boldsymbol{\nabla} \cdot \mathbf{u}\right) = 0, \tag{3}$$

together with the ideal gas law

$$p = \rho R T, \tag{4}$$

and the expression of the gas constant in terms of the specific heat,

$$R = c_v(\gamma - 1). \tag{5}$$

Eliminate p and $\boldsymbol{\nabla}\cdot\mathbf{u}$ from (2) using (3) and (4) to give

$$\rho c_v \frac{\mathrm{D}T}{\mathrm{D}t} = \rho R T \frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \boldsymbol{\nabla}\cdot\left(k\boldsymbol{\nabla}T\right). \tag{6}$$

Eliminate R using (5) to give

$$\boldsymbol{\nabla}\cdot\left(k\boldsymbol{\nabla}T\right) = \rho c_v \frac{\mathrm{D}T}{\mathrm{D}t} - c_v\left(\gamma - 1\right) T \frac{\mathrm{D}\rho}{\mathrm{D}t}. \tag{7}$$

Therefore,

$$\begin{aligned}
\frac{1}{\rho}\boldsymbol{\nabla}\cdot\left(k\boldsymbol{\nabla}T\right) &= c_v T\left(\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{\gamma-1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right), \\
&= c_v T \frac{\mathrm{D}}{\mathrm{D}t}\left(\log T + \log \rho^{1-\gamma}\right), \\
&= T\frac{\mathrm{D}}{\mathrm{D}t}\left(c_v \log \frac{RT\rho}{\rho^{\gamma}}\right) \\
&= T\frac{\mathrm{D}S}{\mathrm{D}t}.
\end{aligned} \tag{8}$$

Using the Reynolds Transport Theorem,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iiint_D \rho S \,\mathrm{d}V &= \iiint_D \rho \frac{\mathrm{D}S}{\mathrm{D}t}\,\mathrm{d}V, \\
&= \iiint_D \frac{1}{T}\boldsymbol{\nabla}\cdot\left(k\boldsymbol{\nabla}T\right)\,\mathrm{d}V, \\
&= \iiint_D \boldsymbol{\nabla}\cdot\left(\frac{k\boldsymbol{\nabla}T}{T}\right) - k\left(\boldsymbol{\nabla}T\right)\cdot\boldsymbol{\nabla}\left(\frac{1}{T}\right)\,\mathrm{d}V, \\
&= \iint_{\partial D}\frac{k}{T}\left(\boldsymbol{\nabla}T\right)\cdot\mathbf{n}\,\mathrm{d}S + \iiint_D \frac{k}{T^2}\left(\boldsymbol{\nabla}T\right)\cdot\left(\boldsymbol{\nabla}T\right)\,\mathrm{d}V.
\end{aligned}$$

For an insulating container there is no flux of heat through the walls so $\mathbf{n}\cdot\boldsymbol{\nabla}T = 0$ on ∂D and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_D \rho S\,\mathrm{d}V = \iiint_D \frac{k}{T^2}\left|\boldsymbol{\nabla}T\right|^2\,\mathrm{d}V \geq 0. \tag{9}$$

Hence the entropy increases whenever the temperature is non-uniform.