Question Sheet 3: Solutions Optional Questions

(i) (a) Define

$$I = \int_{-R}^{R} \mathrm{e}^{\pm \mathrm{i}s^2}\,\mathrm{d}s, \tag{1}$$

and deform the contour into three components (see the figure):

$$\gamma_1 : s = R\mathrm{e}^{i(\theta+\pi)}, \quad \gamma_2 : s = \frac{(1 \pm i)\,t}{\sqrt{2}}, \quad \gamma_3 : s = R\mathrm{e}^{i\theta}. \tag{2}$$

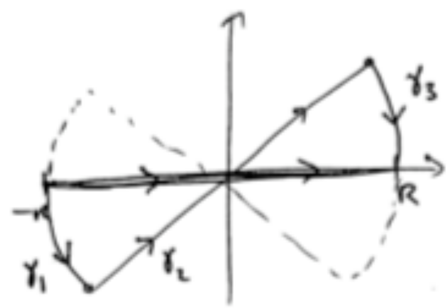

Figure 1: Contours for question (i)

Then,

$$\begin{aligned} I = &\int_0^{\pm\pi/4} \mathrm{e}^{iR^2(\cos 2\theta + i\sin 2\theta)} 2iR\mathrm{e}^{i(\theta+\pi)}\,\mathrm{d}\theta + \int_{-R}^{R} \mathrm{e}^{(\pm i)^2 t^2} \frac{(1 \pm i)}{\sqrt{2}}\,\mathrm{d}t \\ &+ \int_0^{\pm\pi/4} \mathrm{e}^{\pm iR^2(\cos 2\theta + i\sin 2\theta)} 2iR\mathrm{e}^{i\theta}\,\mathrm{d}\theta, \\ =& I_1 + I_2 + I_3. \end{aligned} \tag{3}$$

Now,

$$I_1 = \int_{-\infty}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t \frac{(1 \pm i)}{\sqrt{2}} = \sqrt{\pi}\frac{(1 \pm i)}{\sqrt{2}} = (1 \pm i)\sqrt{\frac{\pi}{2}}. \tag{4}$$

Also,

$$I_2 = -2\int_R^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}r \frac{(1 \pm i)}{\sqrt{2}}, \tag{5}$$

so that (substituting $t = R + s$),

$$|I_2| \leq 2\mathrm{e}^{-R^2} \int_0^{\infty} \mathrm{e}^{-Rs}\mathrm{e}^{-s^2}\,\mathrm{d}s \leq 2\mathrm{e}^{-R^2}\frac{\sqrt{\pi}}{2} = \mathcal{O}\left(\mathrm{e}^{-R^2}\right). \tag{6}$$

Finally,

$$I_3 = 4iR\int_0^{\pm\pi/4} \mathrm{e}^{\pm iR^2(\cos\theta + i\sin 2\theta)}\mathrm{e}^{i\theta}\,\mathrm{d}\theta, \tag{7}$$

so that (using $\sin 2\theta \geq 2\theta/\pi$),

$$\begin{aligned}|I_3| &\leq 4R \int_0^{\pm\pi/4} \left| \mathrm{e}^{\pm R^2 \sin 2\theta} \right| \mathrm{d}\theta \\ &\leq 4R \int_0^{\pi/4} \mathrm{e}^{-R^2 \cdot 2\theta/\pi} \,\mathrm{d}\theta, \\ &\leq 4R \frac{\pi}{2R^2} \left[\mathrm{e}^{-2R^2\theta/\pi} \right]_0^{\pi/4}, \\ &= \mathcal{O}(1/R). \end{aligned} \tag{8}$$

Hence,

$$I = (1 \pm i) \sqrt{\frac{\pi}{2}} + \mathcal{O}(1/R). \tag{9}$$

(b) Define

$$I(t) = \int_a^b f(k) \mathrm{e}^{ikt} \,\mathrm{d}k. \tag{10}$$

We integrate by parts by setting $u = \mathrm{e}^{ikt}/it$ and $v = f(k)$ to give

$$I(t) = \left[\frac{1}{it} \mathrm{e}^{ikt} f(k) \right]_a^b - \int_a^b \frac{1}{it} \mathrm{e}^{ikt} f'(k) \,\mathrm{d}k. \tag{11}$$

Therefore

$$|I(t)| \leq \frac{1}{t} \left(|f(a)| + |f(b)| \right) + \frac{1}{t} (b - a) \sup_k \{ |f'(k)| \}, \tag{12}$$

and providing f and f' are bounded,

$$|I(t)| \leq \frac{M}{t} \quad \text{for some } M. \tag{13}$$

Therefore $I(t) = \mathcal{O}(1/t)$ as $t \to \infty$.

(c) See lecture notes.

(ii) The Fourier transform of $\epsilon/(x^2 + \epsilon^2)$ is given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{\epsilon}{x^2 + \epsilon^2} \mathrm{e}^{-ikx} \,\mathrm{d}x. \tag{1}$$

The integrand has poles at $x = \pm i\epsilon$, and can be written in the form

$$\frac{\epsilon \mathrm{e}^{-ikx}}{(x + i\epsilon)(x - i\epsilon)}. \tag{2}$$

Therefore, the residues are

$$\frac{\epsilon \mathrm{e}^{\pm k\epsilon}}{\pm 2i\epsilon} = \mp \frac{1}{2} i \mathrm{e}^{\pm k\epsilon}. \tag{3}$$

For $k > 0$ we close the contour in the lower half plane, and for $k < 0$ we close the contour in the upper half plane (see figure 2). Jordan's Lemma tells us that the semi-circular contour doesn't contribute as $R \to \infty$, so we just pick up the residue at the pole. Therefore,

$$\hat{f}(k) = \begin{cases} (-2\pi i) \left(\frac{i}{2} \mathrm{e}^{-k\epsilon} \right) & \text{for } k > 0, \\ (2\pi i) \left(-\frac{i}{2} \mathrm{e}^{k\epsilon} \right) & \text{for } k < 0, \end{cases} \tag{4}$$

so that

$$\hat{f}(k) = \pi \mathrm{e}^{-|k|\epsilon}. \tag{5}$$

Also note that the case $k = 0$ follows by continuity, or by direct integration:

$$\hat{f}(0) = \int_{-\infty}^{\infty} \frac{\epsilon}{x^2 + \epsilon^2} \,\mathrm{d}x = \left[\arctan\left(\frac{x}{\epsilon} \right) \right]_{-\infty}^{\infty} = \pi. \tag{6}$$

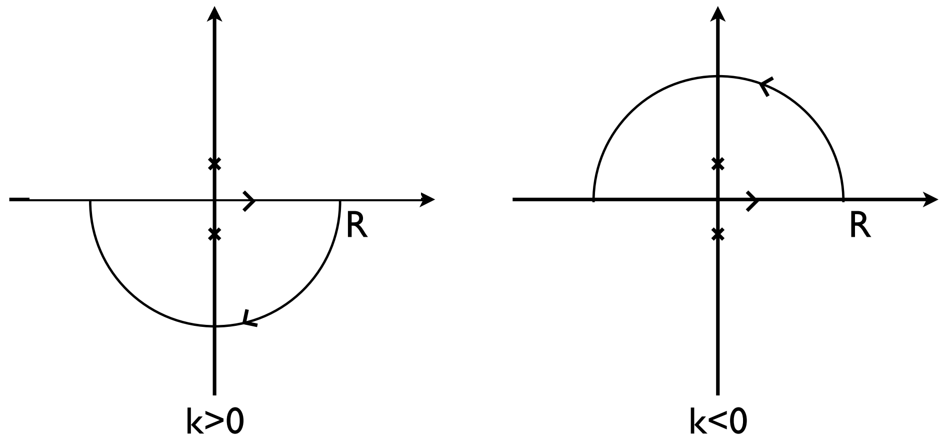

Figure 2: The contours for Question (ii).

Now we assume that fluid occupying the half-space $z < 0$ starts from rest with the initial free surface profile $\eta_0(x) = -a\epsilon/\pi(x^2+\epsilon^2)$. The solution in the lecture notes gives

$$\hat{\eta}(k,t) = \hat{\eta}_0(k)\cos(w(k)t), \tag{7}$$

with $w(k) = \sqrt{g|k|}$. Using the result above we have that $\hat{\eta}_0(k) = -a\mathrm{e}^{-\epsilon|k|}$. We invert the transform to get

$$\begin{aligned}\eta(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\eta}(k,t)\,\mathrm{e}^{ikx}\,\mathrm{d}k,\\ &= -\frac{a}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\epsilon|k|}\cos\left(t\sqrt{g|k|}\right)\mathrm{e}^{ikx}\,\mathrm{d}k,\\ &= I_+ + I_-,\end{aligned} \tag{8}$$

where

$$I_\pm = -\frac{a}{4\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\epsilon|k|}\mathrm{e}^{i\left(kx/t\pm\sqrt{g|k|}\right)}\,\mathrm{d}k. \tag{9}$$

The method of stationary phase tells us that the main contribution to the integral comes from $\psi'(k) = 0$ where

$$\psi(k) = \frac{kx}{t} \pm \sqrt{g|k|}, \tag{10}$$

i.e. where

$$\frac{x}{t} \pm \frac{\sqrt{g|k^\star|}}{2k^\star} = 0 \quad \Rightarrow \quad k^\star = \mp\frac{gt^2}{4x^2}. \tag{11}$$

Further,

$$\psi''(k^\star) = \mp\frac{\sqrt{g|k^\star|}}{4k^{\star 2}} = \mp\frac{\sqrt{g}}{4}\frac{8x^3}{g^{3/2}t^3} = \mp\frac{2x^3}{gt^3}. \tag{12}$$

Therefore the method of stationary phase gives

$$I_\pm \sim -\frac{a}{4\pi}\mathrm{e}^{-\epsilon|k^\star|}\mathrm{e}^{i\left(\psi(k^\star)t+\frac{\pi}{4}\mathrm{sgn}(\psi''(k^\star))\right)}\sqrt{\frac{2\pi}{|\psi''(k^\star)|\,t}}. \tag{13}$$

Now, for ϵ sufficiently small, $\mathrm{e}^{-\epsilon|k^\star|} \sim 1$, so

$$\begin{aligned} I_\pm &\sim -\frac{a}{4\pi}\mathrm{e}^{i\left[\mp\frac{gt^2}{4x}\pm\frac{gt^2}{2x}\mp\frac{\pi}{4}\right]}\sqrt{\frac{2\pi g t^3}{2x^3 t}} \\ &\sim -\frac{at}{4}\mathrm{e}^{\pm i\left[\frac{gt^2}{4x}-\frac{\pi}{4}\right]}\sqrt{\frac{g}{\pi x^3}}. \end{aligned} \tag{14}$$

Therefore,

$$\eta\left(x,t\right) = I_+ + I_- \sim -\frac{at}{2}\sqrt{\frac{g}{\pi x^3}}\cos\left(\frac{gt^2}{4x}-\frac{\pi}{4}\right). \tag{15}$$