Waves and Compressible Flow

Lecture 1

Wares and Compressible Flow

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- Part A : inviscid incompressible flow
  B5.3 : viscous incompressible flow
  B5.4 : inviscid compressible flow

- Resources: online lecture notes by Peter Howell
  Ockendon & Ockendon, 'Waves & Compressible Flow' (Springer)
  Acheson, 'Elementary Fluid Dynamics', Ch.3 (OUP)

1. Equations of motion

- Fluid motion described by state variables:

  density $\rho(\underline{x},t)$

  pressure $p(\underline{x},t)$

  velocity $\underline{u}(\underline{x},t)$

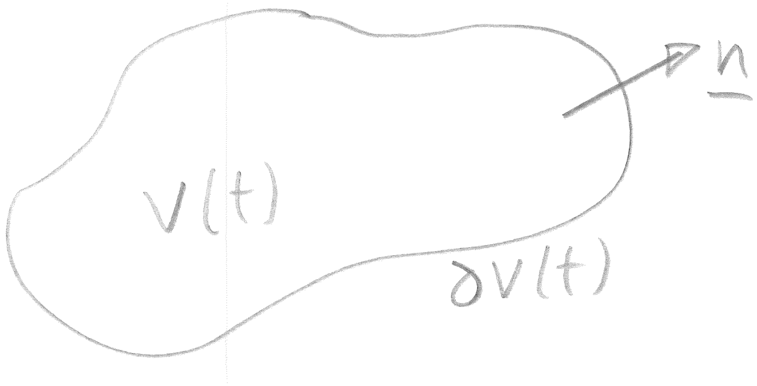
  temperature $T(\underline{x},t)$

- For now, assume these are continuously differentiable (i.e. C^1) functions of position $\underline{x}$ and time t.

- <u>Convective derivative</u>: $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\underline{u} \cdot \nabla) f$ represents rate of change of $f(\underline{x}, t)$ following the flow.

- <u>Reynolds' Transport Theorem</u>: For a material volume $V(t)$ and $f \in C^1$,

$$\frac{d}{dt} \iiint_{V(t)} f(\underline{x},t)\, dV = \iiint_{V(t)} \frac{\partial f}{\partial t} + \nabla \cdot (f \underline{u})\, dV \qquad \text{(RTT)}$$



- Conservation of mass: Mass of any $V(t)$ is constant, i.e.

$$\frac{d}{dt}\iiint_{V(t)} \rho \, dV = 0 \quad \underset{(RTT)}{\Longrightarrow} \quad \iiint_{V(t)} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) \, dV = 0$$

As $V(t)$ arbitrary, integrand must vanish as continuous, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0 \qquad (1)$$

- Corollary of RTT: $f = \rho h \Rightarrow \frac{d}{dt}\iiint_{V(t)} \rho h \, dV = \iiint_{V(t)} \rho \frac{Dh}{Dt} \, dV$

- **Conservation of momentum**: Newton's 2nd Law $\Rightarrow$

$$\underbrace{\frac{d}{dt}\iiint_{V(t)} \rho \underline{u}\, dV}_{\text{momentum}} = \underbrace{\iint_{\partial V(t)} -p\underline{n}\, dS}_{\text{internal pressure force (no viscosity)}} + \underbrace{\iiint_{V(t)} \rho \underline{g}\, dV}_{\text{external body force (gravity here)}}$$

$\Longrightarrow$ (corollary RTT and div. thm)

$$\iiint_{V(t)} \rho \frac{Du}{Dt} + \nabla p - \rho \underline{g}\, dV = \underline{0}$$

$\Longrightarrow$ (V(t) arbitrary, integrand cts)

$$\underbrace{\rho \frac{Du}{Dt} = -\nabla p + \rho \underline{g}}_{\text{Euler's equation}} \tag{2}$$

Constant density recap

- If ρ is constant, ① – ② provide 4 equations for $\underline{u}$, p.

- ① $\Rightarrow$ $\nabla \cdot \underline{u} = 0$, i.e. flow is <u>incompressible</u>.

- If also assume flow is <u>irrotational</u> so $\nabla \wedge \underline{u} = \underline{0}$, then $\underline{u} = \nabla \phi$ for some velocity potential $\phi(\underline{x}, t)$ satisfying

$$\nabla^2 \phi = 0.$$

- To find p, let $\underline{g} = -\nabla \chi$ (usually $\chi = gz$ if z points up) and use $(\underline{u}\cdot\nabla)\underline{u} = \nabla\left(\frac{1}{2}|\underline{u}|^2\right) + \underbrace{(\nabla \wedge \underline{u}) \wedge \underline{u}}_{=\underline{0}}$ in ②, giving

$$\frac{\partial \underline{u}}{\partial t} + \nabla\left(\frac{1}{2}|\underline{u}|^2\right) = -\frac{1}{\rho}\nabla p - \nabla \chi$$

$$\Rightarrow \quad \nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + p/\rho + \chi\right) = \underline{0}$$

$$\Rightarrow \quad \underbrace{\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + p/\rho + \chi = F(t)}_{\text{Bernoulli's equation}}$$

- NB: since $\phi + f(t)$ is also a potential with $F(t) \mapsto F(t) - f'(t)$, $F(t)$ is arbitrary and can therefore be picked for convenience wlog.

- Qn: But is it reasonable to assume irrotational flow?

- Ans: Yes! Kelvin's Circulation Theorem implies that if $\nabla \wedge \underline{u} = \underline{0}$ initially, then it remains so for all time.

- If we relax $\rho = \text{const.}$ (e.g. gas), need two extra equations.

Conservation of energy

ⓐ $\iiint_{V(t)} \frac{1}{2}\rho|\underline{u}|^2 dV$ is kinetic energy in $V(t)$.

ⓑ $\iiint_{V(t)} \rho c_v T dV$ is thermal energy in $V(t)$, where c_v is specific heat at constant volume ($J kg^{-1} K^{-1}$).

ⓒ $\iiint_{V(t)} \rho \underline{g} \cdot \underline{u} \, dV$ is rate of working of body force in $V(t)$.

(d) $\iint_{\partial V(t)} (-p\underline{n})\cdot\underline{u}\,dS$ is rate of working of pressure on $\partial V(t)$.

(e) $\iint_{\partial V(t)} (-k\nabla T)\cdot(-\underline{n})\,dS$ is net heat flux into $V(t)$ through $\partial V(t)$ according to Fourier's Law with thermal conductivity k (s.t. $-k\nabla T$ has units $J m^{-2} s^{-1}$).

(f) $\iiint_{V(t)} \rho q\,dV$ is net rate of heating in $V(t)$ (e.g. due to microwaves) at a prescribed rate $q(x,t)$ per unit mass (so that q has units $J kg^{-1} s^{-1}$).

Conservation of energy:

$$\frac{d}{dt}\left(ⓐ + ⓑ\right) = ⓒ + ⓓ + ⓔ + ⓕ$$

$$\Rightarrow \frac{d}{dt}\iiint_{V(t)} \rho\left(\tfrac{1}{2}|\underline{u}|^2 + c_v T\right) dV = \iiint_{V(t)} \rho \underline{g}\cdot\underline{u}\, dV + \iint_{\partial V(t)} (-p\underline{n})\cdot\underline{u}\, dS + \iint_{\partial V(t)} k\nabla T\cdot\underline{n}\, dS + \iiint_{V(t)} \rho q\, dV$$

$\Rightarrow$ (corollary RTT & div. thm.)

$$\iiint_{V(t)} \rho \frac{D}{Dt}\left(\tfrac{1}{2}|\underline{u}|^2 + c_v T\right) dV = \iiint_{V(t)} \rho\underline{g}\cdot\underline{u} - \nabla\cdot(p\underline{u}) + \nabla\cdot(k\nabla T) + \rho q \, dV$$

$\Rightarrow$ (V(t) arbitrary, integrand cts)

$$\underbrace{\rho\underline{u}\cdot\frac{D\underline{u}}{Dt}} + \rho c_v \frac{DT}{Dt} = \underbrace{\rho\underline{g}\cdot\underline{u}} - \underbrace{\nabla p\cdot\underline{u}} - p\nabla\cdot\underline{u} + \nabla\cdot(k\nabla T) + \rho q$$

$\Rightarrow$ (Euler's eqn)

$$\rho c_v \frac{DT}{Dt} = -p\nabla\cdot\underline{u} + \nabla\cdot(k\nabla T) + \rho q$$

- Conduction is usually negligible for gases, so this simplifies to

$$\rho c_v \frac{DT}{Dt} = -p \nabla \cdot \underline{u} + \rho q \qquad (3)$$

- Recall $\nabla \cdot \underline{u}$ measures rate of expansion of the gas.

The equation of state

- For a perfect or ideal gas, relate p, ρ, T via the equation of state

$$p = \rho R T, \qquad (4)$$

where R is the specific gas constant.

- For a fixed mass M of gas of volume V, we have $M = \rho V$ so that

$$pV = MRT$$

- Some corollaries: at fixed T, $p \propto 1/V$ (Boyle's law)
  at fixed p, $V \propto T$ (Charles' law)

<u>Summary</u>

- The 6 scalar unknowns in $\rho, \underline{u}, p, T$ are governed by the 6 scalar equations in ① – ④