Waves and Compressible Flow

Lecture 2

<u>Equations of motion</u>: ρ, $\underline{u}$, p, T are governed by

$$\rho_t + \nabla \cdot (\rho \underline{u}) = 0 \qquad (1)$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g} \qquad (2)$$

$$\rho c_v \frac{DT}{Dt} = -p \nabla \cdot \underline{u} + \rho q \qquad (3)$$

$$p = \rho R T \qquad (4)$$

Entropy

$$\rho q = c_v \rho \frac{DT}{Dt} + p \nabla \cdot \underline{u} \qquad \text{(by ③)}$$

$$= \frac{c_v \rho}{R} \frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{p}{\rho}\frac{D\rho}{Dt} \qquad \text{(by ① \& ④)}$$

$$= \frac{c_v}{R}\left(\frac{Dp}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt}\right) - \frac{p}{\rho}\frac{D\rho}{Dt}$$

$$= \frac{c_v}{R}\left(\frac{Dp}{Dt} - \left(1 + \frac{R}{c_v}\right)\frac{p}{\rho}\frac{D\rho}{Dt}\right)$$

$$= \frac{c_v}{R}\,\rho^\gamma \frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) \qquad \left(\gamma := 1 + \tfrac{R}{c_v}\right)$$

$$\Rightarrow \quad \rho q = \frac{c_v}{R} p \frac{D}{Dt}\left(\log\left(\frac{p}{\rho^\gamma}\right)\right)$$

$$= \rho T \frac{D}{Dt}\left(c_v \log\left(\frac{p}{\rho^\gamma}\right)\right) \qquad \text{(by (4))}$$

- Define <u>entropy</u> $S = S_0 + c_v \log\left(\frac{p}{\rho^\gamma}\right)$, then

$$T \frac{DS}{Dt} = q \qquad (5)$$

- In general $q \geq 0$ (since radiative cooling thought impossible), so $\frac{DS}{Dt} \geq 0$, an example of 2nd law of thermodynamics.

- Often $q = 0 \Rightarrow \frac{Ds}{Dt} = 0$, i.e. entropy conserved following the flow – called <u>isentropic</u> flow.

- If in addition entropy uniform initially, it remains so for all t – called <u>homentropic</u> flow, and for some constant C,

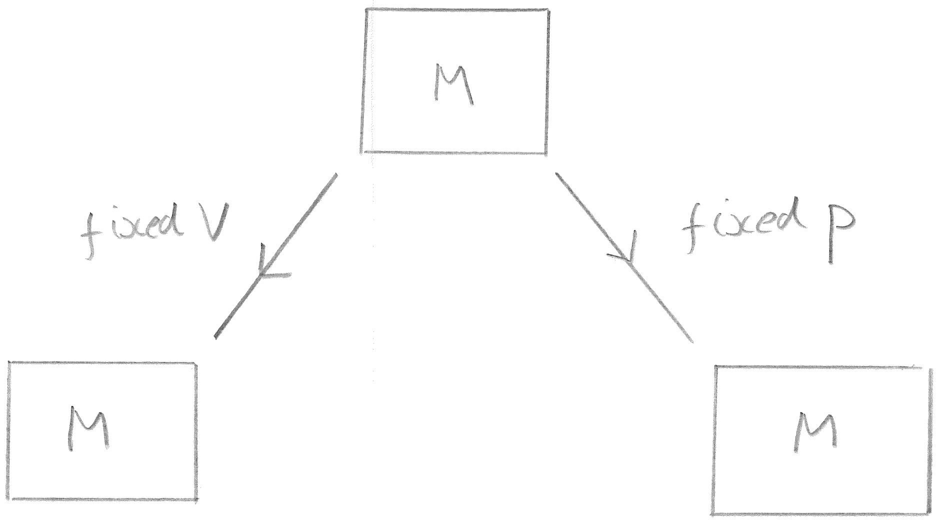
$$p = C\rho^{\gamma} \qquad (6)$$

- (1), (2) and (6) form a closed system for ρ, p and $\underline{u}$.

- (6) is an example of a <u>barotropic</u> flow for which $p = P(\rho)$ for some function P.

Physical interpretation of γ

- Consider energy required to raise temperature of a fixed mass M of gas by ΔT under two conditions.



$\Delta E_v = M c_v \Delta T$ — c_v: specific heat at constant volume

$\Delta E_p = M c_p \Delta T$ — c_p: specific heat at constant pressure

- At fixed p, total energy cost $\Delta E_p = \Delta E_v + \Delta W$, where ΔW is additional work done by gas to expand volume by ΔV.

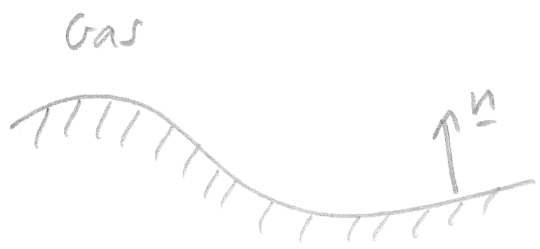
- But $\Delta W = \iint_{\partial V(t)} p \, \underline{u} \cdot \underline{n} \, \Delta t \, dS = p \Delta t \iiint_{V(t)} \nabla \cdot \underline{u} \, dV = p \Delta t \frac{d}{dt} \iiint_{V(t)} dV = p \Delta t \frac{\Delta V}{\Delta t}$ to a first approximation, so $\Delta W = p \Delta V$.

- Recall $p = \rho R T$, $M = \rho V \Rightarrow pV = MRT \Rightarrow \Delta W = MR \Delta T$

- Combo $\Rightarrow$ $M c_p \Delta T = M c_v \Delta T + MR \Delta T \Rightarrow \gamma = 1 + \frac{R}{c_v} = \frac{c_p}{c_v}$,

  i.e. γ is the <u>ratio of specific heats</u>. NB $\gamma \simeq 1.4$ for air.

Boundary conditions



- At a fixed rigid boundary B, have no normal flow $\underline{u} \cdot \underline{n} = 0$ on B.

- If boundary is moving with velocity $\underline{v}_B(\underline{x}, t)$, then $\underline{u} \cdot \underline{n} = \underline{v}_B \cdot \underline{n}$ on B.

- If B given by $f(\underline{x}, t) = 0$, then $\dfrac{Df}{Dt} = 0$ on B — kinematic BC.

- If B is a free boundary (e.g. interface between air and water), also need a dynamic BC — later on.

Rotating fluids

- To describe flows in atmosphere or oceans, easier to work in a frame that rotates with the Earth.

- Let S be an inertial frame and R be a frame rotating with angular velocity $\underline{\Omega}$ relative to S.

- Let $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ be an orthonormal basis fixed in R, then

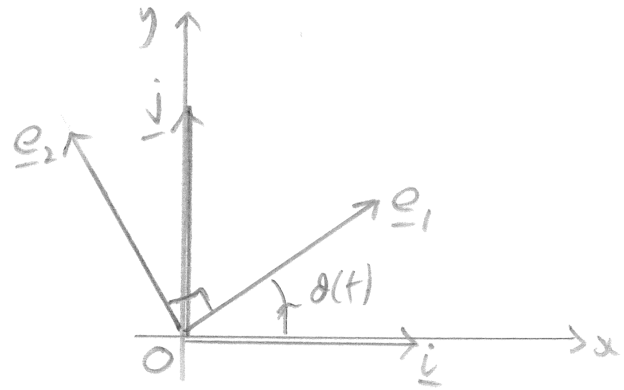
$$\left.\frac{d\underline{e}_i}{dt}\right|_S = \underline{\Omega} \wedge \underline{e}_i$$

Example

- Suppose $S = O\underline{i}\,\underline{j}\,\underline{k}$ and $R = O\underline{e}_1\underline{e}_2\underline{e}_3$, where

$$\underline{e}_1 = \underline{i}\cos\theta(t) + \underline{j}\sin\theta(t), \quad \underline{e}_2 = -\underline{i}\sin\theta(t) + \underline{j}\cos\theta(t), \quad \underline{e}_3 = \underline{k}$$

- Then $\dfrac{d\underline{e}_1}{dt} = \dot{\theta}\,\underline{e}_2 = \underline{\Omega}\wedge\underline{e}_1$, $\dfrac{d\underline{e}_2}{dt} = -\dot{\theta}\underline{e}_1 = \underline{\Omega}\wedge\underline{e}_2$, $\dfrac{d\underline{e}_3}{dt} = \underline{0} = \underline{\Omega}\wedge\underline{e}_3$, where $\underline{\Omega} = \dot{\theta}\underline{k}$, i.e. a rotation with angular speed $|\underline{\Omega}| = |\dot{\theta}|$ about the instantaneous axis of rotation $\dfrac{\underline{\Omega}}{|\underline{\Omega}|} = \underline{k}$.



- This is true in general.

- Since $\left.\frac{d\underline{e}_i}{dt}\right|_S = \underline{\Omega} \wedge \underline{e}_i$, for any vector $\underline{v} = v_i(t)\,\underline{e}_i(t)$,

$$\left.\frac{d\underline{v}}{dt}\right|_S = \frac{dv_i}{dt}\underline{e}_i + v_i \left.\frac{d\underline{e}_i}{dt}\right|_S$$

$$= \left.\frac{d\underline{v}}{dt}\right|_R + v_i \underline{\Omega} \wedge \underline{e}_i$$

$$= \left.\frac{d\underline{v}}{dt}\right|_R + \underline{\Omega} \wedge \underline{v} \qquad \text{(Coriolis formula)}$$

- Apply to a fluid element at position $\underline{x}(t)$

$$\Rightarrow \quad \left.\frac{d\underline{x}}{dt}\right|_S = \left.\frac{d\underline{x}}{dt}\right|_R + \underline{\Omega} \wedge \underline{x}$$

- Apply again

$$\Rightarrow \quad \left.\frac{d^2\underline{x}}{dt^2}\right|_S = \left(\left.\frac{d}{dt}\right|_R + \underline{\Omega}\wedge\right)\left(\left.\frac{d\underline{x}}{dt}\right|_R + \underline{\Omega}\wedge\underline{x}\right)$$

$$= \left.\frac{d^2\underline{x}}{dt^2}\right|_R + \left.\frac{d\underline{\Omega}}{dt}\right|_R \wedge \underline{x} + 2\underline{\Omega}\wedge\left.\frac{d\underline{x}}{dt}\right|_R + \underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{x})$$

- Now $\left.\frac{d\underline{x}}{dt}\right|_S = \underline{u}_S$, $\left.\frac{d\underline{x}}{dt}\right|_R = \underline{u}_R$, $\left.\frac{d^2\underline{x}}{dt^2}\right|_S = \left.\frac{D\underline{u}_S}{Dt}\right|_S$, $\left.\frac{d^2\underline{x}}{dt^2}\right|_R = \left.\frac{D\underline{u}_R}{Dt}\right|_R$.

- Assuming $\underline{\Omega}$ is constant, we deduce that

$$\underline{u}_S = \underline{u}_R + \underline{\Omega}\wedge\underline{x}$$

$$\left.\frac{D\underline{u}_S}{Dt}\right|_S = \left.\frac{D\underline{u}_R}{Dt}\right|_R + 2\underline{\Omega}\wedge\underline{u}_R + \underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{x})$$

- Change from S to R affects only time derivatives of vectors, so mass- and energy-conservation as before.

- Dropping the subscript Rs, Euler's equation becomes

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega}\wedge\underline{u} + \underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{x}) = -\frac{1}{\rho}\nabla p + \underline{g}. \qquad (7)$$

- Consider later ρ = constant, so that flow is incompressible with

$$\nabla\cdot\underline{u} = 0 \qquad (8)$$

- The unknowns $\underline{u}$ and p are then governed by (7) e (8).