Waves and Compressible Flow

Lecture 3

2. Models for linear wave propagation

- Various examples: acoustic waves in a gas

  gravity waves on an interface

  internal gravity waves in a stratified fluid

  inertial waves in a rotating fluid

  elastic waves

- Key assumption is that waves are small amplitude, which allows linearization of the governing equations.

Acoustic waves

- Consider barotropic flow with no body force:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \underline{u} \qquad (1)$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p \qquad (2)$$

$$p = P(\rho) \qquad (3)$$

- Stationary state: $\underline{u} = \underline{0}$, $\rho = \rho_0$, $p = p_0 = P(\rho_0)$

- Introduce small perturbation: $\underline{u} = \underline{0} + \underline{u}'$, $\rho = \rho_0 + \rho'$, $p = p_0 + p'$

- Substitute into ① - ③ and linearize about stationary state by neglecting quadratic and higher order terms.

① $\Rightarrow$ $\dfrac{\partial \rho'}{\partial t} + (\underline{u}' \cdot \nabla)\rho' = -(\rho_0 + \rho')\nabla \cdot \underline{u}'$ $\underset{\text{(linearizing)}}{\Longrightarrow}$ $\dfrac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \underline{u}'$ (1L)

② $\Rightarrow$ $(\rho_0 + \rho')\left(\dfrac{\partial \underline{u}'}{\partial t} + (\underline{u}' \cdot \nabla)\underline{u}'\right) = -\nabla p'$ $\underset{\text{(linearizing)}}{\Longrightarrow}$ $\rho_0 \dfrac{\partial \underline{u}'}{\partial t} = -\nabla p'$ (2L)

③ $\underset{\text{(Taylor expanding)}}{\Rightarrow}$ $P_0 + p' = P(\rho_0 + \rho') = P(\rho_0) + \dfrac{dP}{d\rho}(\rho_0)\rho' + \cdots$ $\underset{\text{(linearizing)}}{\Rightarrow}$ $p' = c_0^2 \rho'$ (3L)

where $c_0^2 = \dfrac{dP}{d\rho}(\rho_0)$, e.g. homentropic flow $p = C\rho^\gamma \Rightarrow c_0^2 = \dfrac{\gamma P_0}{\rho_0}$.

- $\nabla \wedge$ (2L) $\Rightarrow \frac{\partial}{\partial t}(\nabla \wedge \underline{u}') = \underline{0}$, so if we assume $\nabla \wedge \underline{u}' = \underline{0}$ initially, then $\nabla \wedge \underline{u}' = \underline{0}$ for $t \geq 0$ and $\exists\, \phi(\underline{x}, t)$ s.t. $\underline{u}' = \nabla \phi$ (4L)

- Then (2L) $\Rightarrow \nabla\left(\rho_0 \frac{\partial \phi}{\partial t} + p'\right) = \underline{0} \Rightarrow \rho_0 \frac{\partial \phi}{\partial t} + p' = F(t) = 0$ (5L) wlog, which is a linearized Bernoulli equation.

- Hence, $\rho_0 \phi_{tt} \underset{(5L)}{=} -p'_t$, $p'_t \underset{(3L)}{=} c_0^2 \rho'_t$ and $\rho'_t \underset{(1L)}{=} -\rho_0 \nabla \cdot \underline{u}' \underset{(5L)}{=} -\rho_0 \nabla^2 \phi$

$$\Rightarrow \qquad \phi_{tt} = c_0^2 \nabla^2 \phi \qquad (6L)$$

(Wave equation, wave speed c_0)

- (5L) $\Rightarrow$ $p' = -\rho_0 \phi_t$, so $\frac{\partial}{\partial t}$ (6L) $\Rightarrow$ $p'_{tt} = c_0^2 \nabla^2 p'$

- (3L) $\Rightarrow$ $p' = c_0^2 \rho'$, so $\rho'_{tt} = c_0^2 \nabla^2 \rho'$

- (4L) $\Rightarrow$ $\underline{u}' = \nabla \phi$, so ∇ (6L) $\Rightarrow$ $\underline{u}'_{tt} = c_0^2 \nabla^2 \underline{u}'$

- After linearization everything is governed by the same wave equation.

- NB: For air $p = C\rho^\gamma$ with $\gamma \simeq 1.4$ $\Rightarrow$ $c_0 \approx 340 ms^{-1}$, which agrees well with measurements.

Example: 1D waves

- $p' = p'(x,t) \Rightarrow p'_{tt} = c_0^2 p'_{xx} \Rightarrow p' = \underbrace{F(x-c_0t)}_{\text{right travelling wave}} + \underbrace{G(x+c_0t)}_{\text{left travelling wave}}$

- Consider a transducer at $x=0$ imposing a periodic pressure fluctuation s.t.

$$p'(0,t) = a\cos\omega t + b\sin\omega t = \mathrm{Re}(Ae^{-i\omega t})$$

where ω is frequency and $A = a+ib$ is complex amplitude.

- Try separable time-harmonic ansatz $p' = f(x)e^{-i\omega t}$ (real part understood), where the complex-valued function $f(x)$ is TBD.

- Wave equation $\Rightarrow \frac{d^2 f}{dx^2} + \frac{\omega^2}{c_0^2} f = 0 \Rightarrow f = \alpha e^{i\omega x/c_0} + \beta e^{-i\omega x/c_0}$ $(\alpha, \beta \in \mathbb{C})$

- Condition at $x = 0$ satisfied if $A = \alpha + \beta$, so need one more piece of information to determine α and β.

- Impose a radiation condition (RC) that the source at $x = 0$ can only produce outward-travelling waves.

- Since $p' = \alpha e^{i\omega(x - c_0 t)/c_0} + \beta e^{-i\omega(x + c_0 t)/c_0}$, the radiation condition requires $\beta = 0$ for $x > 0$ and $\alpha = 0$ for $x < 0$.

$$\Rightarrow \quad \alpha = A, \ \beta = 0 \quad \text{for } x > 0$$

$$\alpha = 0, \ \beta = A \quad \text{for } x < 0$$

$$\Rightarrow \quad p' = \begin{cases} \mathrm{Re}\left(A e^{i\omega(x - c_0 t)/c_0}\right) & \text{for } x > 0 \\ \mathrm{Re}\left(A e^{-i\omega(x + c_0 t)/c_0}\right) & \text{for } x < 0 \end{cases} \qquad (+)$$

- NB: If we'd used instead $p' = F(x - c_0 t) + G(x + c_0 t)$,

then the radiation condition would require

$$p' = \begin{cases} F(x - c_0 t) & \text{for } x \geq 0, \\ G(x + c_0 t) & \text{for } x \leq 0; \end{cases}$$

then the condition at $x = 0$ gives $F(-c_0 t) = A e^{-i\omega t}$, $G(c_0 t) = A e^{-i\omega t}$

and hence (+).

Example: 3D spherically symmetric waves

- $p' = p'(r,t)$, $r = (x^2+y^2+z^2)^{1/2}$ $\Rightarrow$ $p'_{tt} = c_0^2\left(p'_{rr} + \frac{2}{r}p'_r\right) = \frac{c_0^2}{r}\frac{\partial^2}{\partial r^2}(rp')$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2}(rp') = c_0^2 \frac{\partial^2}{\partial r^2}(rp')$$

$$\Rightarrow \quad p' = \frac{F(r - c_0 t) + G(r + c_0 t)}{r}$$

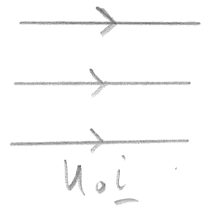
- If waves are generated by a point transducer at origin, then

$$\lim_{r \to 0+} rp'(r,t) = Ae^{-i\omega t} \qquad \text{(real part understood)}$$

- Now radiation condition $\Rightarrow$ only outgoing waves, i.e. $G = 0$

$$\Rightarrow F(-c_0 t) = Ae^{-i\omega t} \quad \Rightarrow \quad p' = \frac{A}{r} e^{i\omega(r - c_0 t)/c_0}$$

Example: acoustic waves in a background flow



- Consider barotropic flow governed by ①-③ but with base state given by $\underline{u} = U_0 \underline{i}$, $\rho = \rho_0$, $p = p_0$, where $U_0 \in \mathbb{R}^+$

- Introduce small perturbation: $\underline{u} = U_0 \underline{i} + \underline{u}'$, $\rho = \rho_0 + \rho'$, $p = p_0 + p'$.

- Before $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}' \cdot \nabla$ replaced by $\frac{\partial}{\partial t}$ upon linearization.

- Now $\frac{D}{Dt} = \frac{\partial}{\partial t} + (U_0 \underline{i} + \underline{u}') \cdot \nabla$ replaced by $\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}$ in (1L) - (3L), so assuming irrotational $\underline{u}' = \nabla \phi$ with $\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)^2 \phi = c_0^2 \nabla^2 \phi$.

Example : $\phi = \phi(x,t)$

- $\phi = \phi(x,t) \Rightarrow \quad \underbrace{1}_{A} \cdot \phi_{tt} + \underbrace{2u_0}_{2B} \phi_{xt} + \underbrace{(u_0^2 - c_0^2)}_{C} \phi_{xx} = 0$

- Characteristics : $\frac{dx}{dt} = \lambda$, $A\lambda^2 - 2B\lambda + C = 0 \Rightarrow \lambda = u_0 \pm c_0$

- Let $\xi = x - (u_0 + c_0)t$, $\eta = x - (u_0 - c_0)t$, then $\phi_{\xi\eta} = 0 \Rightarrow \phi = F(\xi) + G(\eta)$

- Hence, $\phi = \underbrace{F(x - (u_0 + c_0)t)}_{\text{right travelling}} + \underbrace{G(x - (u_0 - c_0)t)}_{\substack{\text{right travelling } u_0 > c_0 \text{ (supersonic)} \\ \text{left travelling } u_0 < c_0 \text{ (subsonic)}}}$

- Transducer at $x = 0$ has "zone of silence" upstream ($x < 0$) in supersonic case!

Example: $\phi = \phi(x,y)$

- $\phi = \phi(x,y) \Rightarrow (1-M^2)\phi_{xx} + \phi_{yy} = 0$, $M = \frac{u_0}{c_0}$ is Mach number

- $M < 1$ (subsonic) $\Rightarrow$ elliptic PDE $\Rightarrow$ localized disturbances felt everywhere.

- $M > 1$ (supersonic) $\Rightarrow$ hyperbolic PDE $\Rightarrow$ localized disturbances propate at finite speed along characteristics s.t. $\frac{dy}{dx} = \pm (M^2-1)^{-1/2}$, the general solution being $\phi = F(y - (M^2-1)^{-1/2}x) + G(y + (M^2-1)^{-1/2}x)$.

- Explore later on these dramatic differences & their implications for the theory of flight.