Waves and Compressible Flow

Lecture 4

Example: Stokes waves on a free surface

- Consider small amplitude waves on surface of constant density fluid, e.g. ripples on a pond.

- ρ = constant $\Rightarrow \nabla \cdot \underline{u} = 0$
  Assume irrotational $\Rightarrow \exists \phi$ s.t. $\underline{u} = \nabla\phi$
  $\Bigg\} \Rightarrow \nabla^2 \phi = 0$ ①

- Euler's equation $\Rightarrow$ Bernoulli's equation $\frac{P}{\rho} + \phi_t + \frac{1}{2}|\underline{u}|^2 + gz = F(t)$ ②

- Note ①-② hold in $-h < z < \eta(x,y,t)$, $-\infty < x, y < \infty$.

- Zero normal velocity on substrate $\Rightarrow$ $\underline{u}\cdot\underline{k} = 0$ or $\phi_z = 0$ on $z = -h$ (3)

- On free surface $z = \eta$ need two BCs because η is unknown.

- Kinematic BC: $\frac{D}{Dt}(z-\eta) = 0$ on $z=\eta \Rightarrow \phi_z = \eta_t + \phi_x\eta_x + \phi_y\eta_y$ on $z = \eta$ (4)

- Dynamic BC: $p = p_{atm}$ on $z = \eta$ $\underset{\uparrow}{\Rightarrow}$ $\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0$ on $z = \eta$ (5)

  ((2) with $F = \frac{p_{atm}}{\rho}$ wlog)

- (1) - (5) nonlinear so hard to solve. To make progress we assume disturbance is small so we can linearize the problem.

- Neglecting quadratic and h.o.t. in ④ suggests $\phi_z = \eta_t$ on $z = \eta$.

- But a Taylor expansion gives $\phi_z|_{z=\eta} = \phi_z|_{z=0} + \eta\,\phi_{zz}|_{z=0} + \text{h.o.t.}$,

  so linearizing ④ in fact gives $\phi_z = \eta_t$ on $z = 0$.

- Similarly, ⑤ gives $\phi_t + g\eta = 0$ on $z = 0$ upon linearization.

- Hence, linearized problem is $\nabla^2 \phi = 0$ in $-h < z < 0$ (1L)

  with $\phi_z = 0$ on $z = -h$ (2L)

  and $\phi_z = \eta_t,\ \phi_t + g\eta = 0$ on $z = 0$ (3L)

- Seek sinusoidal travelling wave solution $\eta = A e^{i(kx-\omega t)}$, where real part is understood, A is constant complex amplitude, ω is frequency and k is wavenumber; note wavelength $\lambda = 2\pi/k$ and phase speed $c_p = \omega/k$.

- Consistent form for ϕ is $\phi = f(z) e^{i(kx-\omega t)} \underset{1L}{\Rightarrow} f'' - k^2 f = 0$ for $-h < z < 0$.

- Since 2L $\Rightarrow f'(-h) = 0$, we deduce that $f = B\cosh k(z+h)$ $(B \in \mathbb{C})$

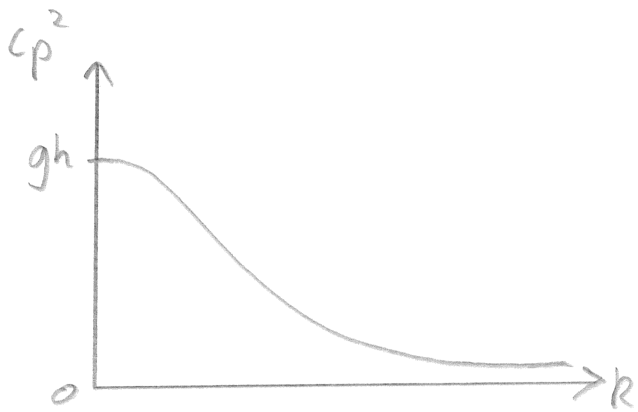
- 3L $\Rightarrow$ $f'(0) = -i\omega A$, $-i\omega f(0) + gA = 0 \Rightarrow$

$$\underbrace{\begin{bmatrix} i\omega & k\sinh(kh) \\ g & -i\omega\cosh(kh) \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- M invertible $\Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \phi = 0, \eta = 0$, i.e. the trivial solution.

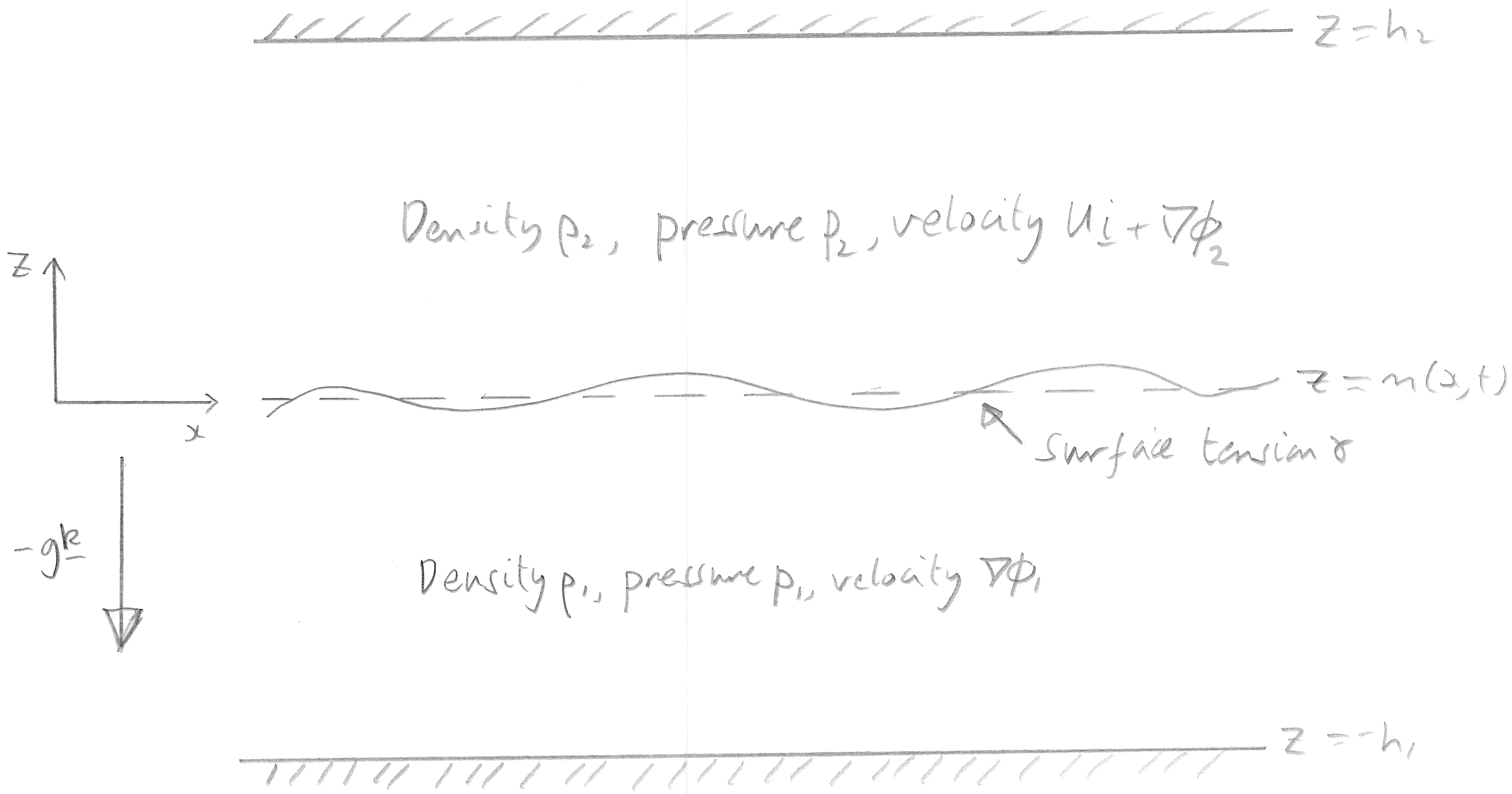
- Hence, a nontrivial solution can only exist if $\det(M) = 0 \Leftrightarrow \omega^2 = gk\tanh(kh)$, which is a dispersion relation (as it relates ω and k).

- Phase speed c_p s.t. $c_p^2 = \frac{\omega^2}{k^2} = \frac{g}{k}\tanh(kh)$, so long waves (small k) travel faster than short waves (large k).



- Waves with different wavelengths travel at different speeds — they are called <u>dispersive</u>, cf. acoustic waves which have constant phase speed $c_p = c_0$.

Example: common generalizations



Governing equations

- $\nabla^2\phi_1 = 0$, $\dfrac{p_1}{\rho_1} + \phi_{1t} + \frac{1}{2}|\nabla\phi_1|^2 + gz = F_1(t)$ for $-h_1 < z < \eta$;

  $\nabla^2\phi_2 = 0$, $\dfrac{p_2}{\rho_2} + \phi_{2t} + \frac{1}{2}|U\underline{i} + \nabla\phi_2|^2 + gz = F_2(t)$ for $\eta < z < h_2$.

- $\phi_{1z} = 0$ on $z = -h_1$; $\phi_{2z} = 0$ on $z = h_2$.

- $\phi_{1z} = \eta_t + \phi_{1x}\eta_x$, $\phi_{2z} = \eta_t + (U + \phi_{2x})\eta_x$, $p_2 - p_1 = \gamma\kappa$ all on $z = \eta$,

  where the curvature $\kappa = \dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}$.

- See part A fluids and online notes for a derivation of the dynamic BC.

Linearized governing equations

- Base state is $\phi_1 = 0$, $\phi_2 = 0$, $\eta = 0$, so convenient to set $F_1 = 0$, $F_2 = \frac{1}{2}U^2$.

- $\nabla^2 \phi_1 = 0$ in $-h_1 < z < 0$; $\nabla^2 \phi_2 = 0$ in $0 < z < h_2$.

- $\phi_{1z} = 0$ on $z = -h_1$; $\phi_{2z} = 0$ on $z = h_2$.

- $\phi_{1z} = \eta_t$, $\phi_{2z} = \eta_t + U\eta_x$, $-\rho_2(\phi_{2t} + U\phi_{2x} + g\eta) + \rho_1(\phi_{1t} + g\eta) = \gamma\eta_{xx}$

  all on $z = 0$; be careful to substitute for p_1 and p_2 before linearizing.

- Seek a sinusoidal travelling wave solution of the form

$$\eta = A e^{i(kx - \omega t)}$$

$$\phi_1 = B e^{i(kx - \omega t)} \cosh k(z + h_1)$$

$$\phi_2 = C e^{i(kx - \omega t)} \cosh k(z - h_2)$$

where real part is understood, the complex amplitudes $A, B, C \in \mathbb{C}$ and ϕ_1, ϕ_2 have been chosen to satisfy Laplace's equation and the boundary conditions on $z = -h_1$ and $z = h_2$.

- BCs on $z = 0 \Rightarrow M\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for some 3×3 matrix M.

- Nontrivial solutions $\Leftrightarrow \det(M) = 0 \Leftrightarrow$ ω and k are related by

$$\omega^2 \rho_1 \coth(kh_1) + (\omega - Uk)^2 \rho_2 \coth(kh_2) = (\rho_1 - \rho_2)gk + \gamma k^3.$$

- Quadratic equation for $\omega(k)$ of the form $a(k)\omega^2 + 2b(k)\omega + c(k) = 0$,

where $a(k) = \rho_1\coth(kh_1) + \rho_2\coth(kh_2)$, $2b(k) = -2U\rho_2 k\coth(kh_2)$ and

$c(k) = U^2k^2\rho_2\coth(kh_2) - (\rho_1 - \rho_2)gk - \gamma k^3.$

- Roots $\omega_\pm(k) = \dfrac{-b \pm \sqrt{b^2 - ac}}{a}$ tough to analyse, so consider two important limiting cases.

Example: $h_1 = \infty$, $h_2 = \infty$, $U = 0$

- In this case, $$\omega^2 = \frac{((\rho_1 - \rho_2)g + \gamma k^2)|k|}{\rho_1 + \rho_2}$$

- $\rho_1 > \rho_2$ $\Rightarrow$ RHS > 0 $\Rightarrow$ two real roots $\Rightarrow$ disturbance stable

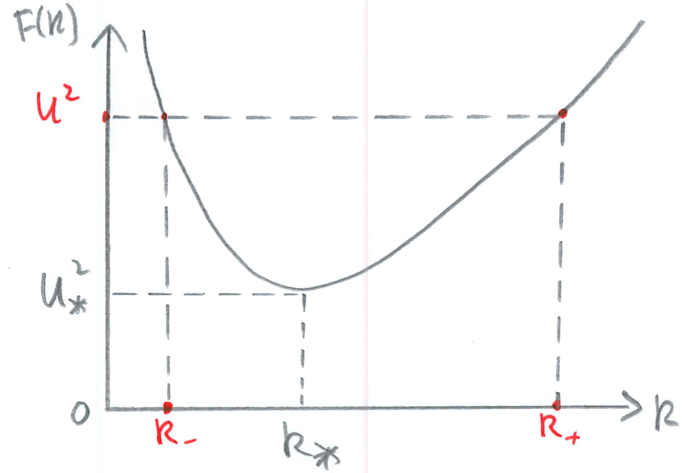
- $\rho_1 < \rho_2$ $\Rightarrow$ RHS < 0 for $|k| < k_c = \left(\frac{(\rho_2 - \rho_1)g}{\gamma}\right)^{1/2}$, and for such wavenumbers, $\omega = \pm i\omega_I(k)$ with $\omega_I \in \mathbb{R}$, giving $\eta = Ae^{ikx \pm \omega_I t}$
$\Rightarrow$ disturbance unstable when denser fluid above lighter fluid.

- This is known as the Rayleigh–Taylor instability.

Example: $h_1 = \infty$, $h_2 = \infty$, $U \neq 0$, $\rho_1 > \rho_2$

- Now discriminant $$b^2 - ac = 4\rho_1\rho_2 k^2 \Big[\underbrace{\frac{\rho_1+\rho_2}{\rho_1\rho_2} \frac{((\rho_1-\rho_2)g + \gamma k^2)}{|k|}}_{= F(k)} - U^2 \Big]$$

- Sketch $F(k)$:



Minimum at

$$k_* = ((\rho_1-\rho_2)g/\gamma)^{1/2}$$

$$U_*^2 = \frac{2(\rho_1+\rho_2)}{\rho_1\rho_2}\left(\gamma(\rho_1-\rho_2)g\right)^{1/2}$$

- $U^2 < U_*^2 \Rightarrow U^2 < F(k)\ \forall k \Rightarrow b^2 - ac > 0\ \forall k \Rightarrow$ stable.

- $U^2 > U_*^2 \Rightarrow U^2 > F(k)$ for a band $k \in (k_-, k_+) \Rightarrow b^2 - ac < 0$ and unstable for such k.

This is the Kelvin-Helmholtz instability — onset of "sea horses"