Waves and Compressible Flow

Lecture 5

Internal gravity waves in a stratified fluid

- Assume incompressible flow with variable density:

$$\nabla \cdot \underline{u} = 0, \quad \rho_t + (\underline{u} \cdot \nabla)\rho = 0, \quad \rho(\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u}) = -\nabla p - \rho g \underline{e}_z$$

- A stratified fluid has density depending on depth, e.g in oceans, where density is a function of salinity.

- Start with stationary state: $\underline{u} = \underline{0}$, $\rho = \rho_0(z)$, $p = p_0(z)$, where $p_{0z} = -\rho_0 g$, so that

$$p_0(z) = p_a - \int_0^z \rho_0(s) g \, ds \qquad (p_a \in \mathbb{R}).$$

- Consider 2D perturbation: $\underline{u} = \underline{0} + (u'(x,z,t), 0, w'(x,z,t))$,

$$\rho = \rho_0(z) + \rho'(x,z,t),$$

$$p = p_0(z) + p'(x,z,t).$$

- Linearized equations:

$$u'_x + w'_z = 0 \qquad (1)$$

$$\rho'_t + w'\rho_{0z} = 0 \qquad (2)$$

$$\rho_0 u'_t = -p'_x \qquad (3)$$

$$\rho_0 w'_t = -p'_z - \rho' g \qquad (4)$$

- Aim to eliminate u', ρ', p' to obtain a single equation for w'.

- $\rho_0 \frac{\partial}{\partial t}$ ① $\Rightarrow \quad \rho_0 u'_{tx} + \rho_0 w'_{zt} = 0 \underset{③}{\Rightarrow} \quad p'_{xx} = \rho_0 w'_{zt}$ ⑤

- $\frac{\partial}{\partial t}$ ④ $\Rightarrow \quad \rho_0 w'_{tt} = -p'_{zt} - \rho'_t g \underset{②}{\Rightarrow} \quad \rho_0 w'_{tt} = -p'_{zt} + g \rho_{0z} w'$ ⑥

- $\frac{\partial^2}{\partial z \partial t}$ ⑤ $\Rightarrow \quad p'_{xxzt} = (\rho_0 w'_{zt})_{zt} = \rho_0 w'_{zztt} + \rho_{0z} w'_{ztt}$ ⑦

- $\frac{\partial^2}{\partial x^2}$ ⑥ $\Rightarrow \quad p'_{xxzt} = -\rho_0 w'_{xxtt} + g \rho_{0z} w'_{xx}$ ⑧

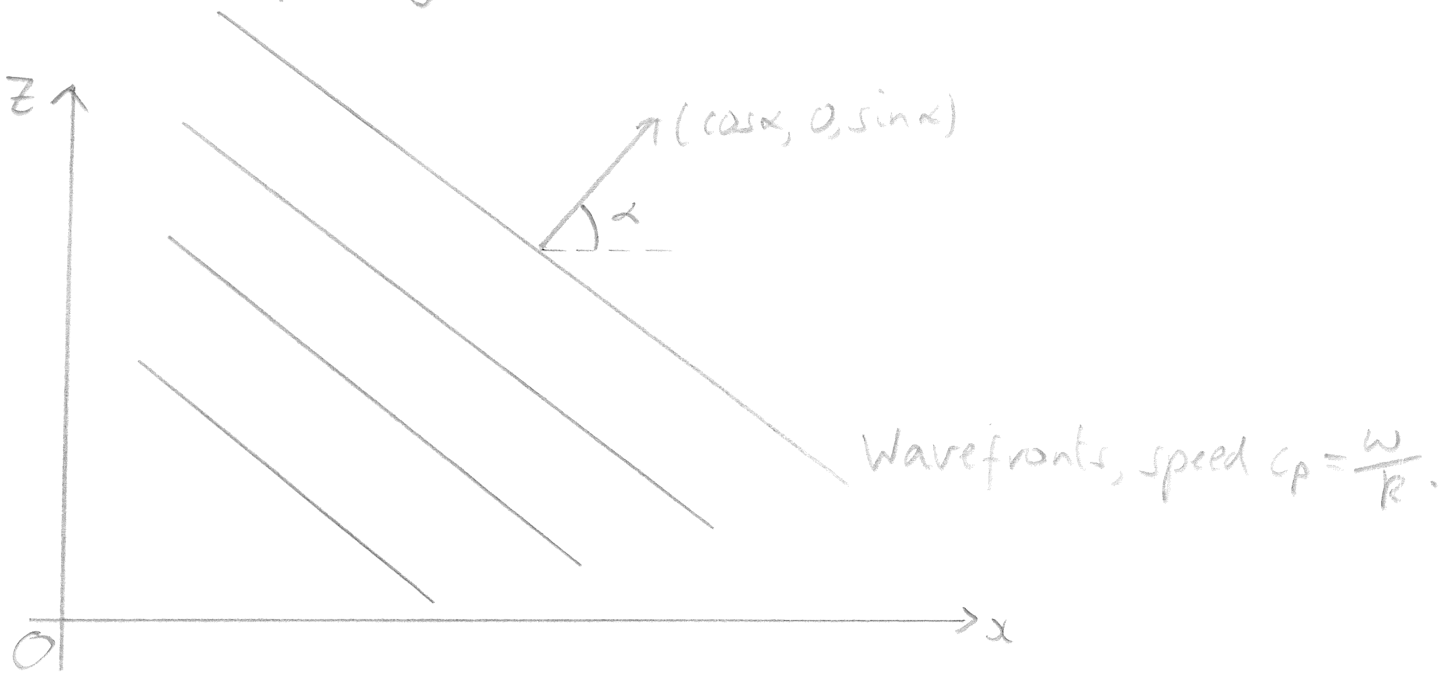
- ⑦ & ⑧ $\Rightarrow \quad (w'_{xx} + w'_{zz})_{tt} = \underbrace{\frac{g \rho_{0z}}{\rho_0}}_{-N^2(z)} \left(w'_{xx} - \frac{1}{g} w'_{ztt} \right)$ ⑨

$-N^2(z)$, where $N(z)$ is the "buoyancy frequency"

- Look for 2D wave travelling at an angle α to x-axis:

$$w' = A\exp\left(i(k\cos\alpha\, x + k\sin\alpha\, y - \omega t)\right) \tag{7}$$

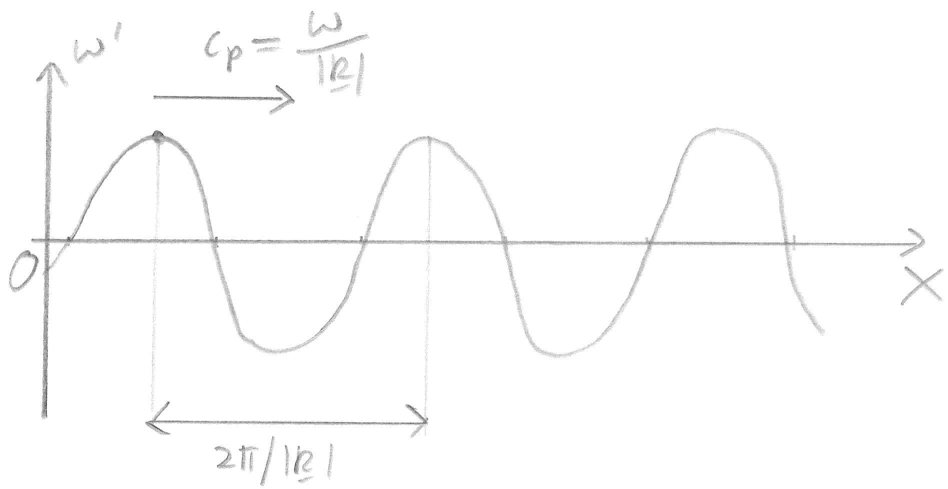
- Wavefronts (lines of constant phase) $k\cos\alpha\, x + k\sin\alpha\, y - \omega t = \text{const.}$ are $\perp$ to direction of propagation.



- For such multidimensional waves usually write $\underline{w' = A e^{i(\underline{k}\cdot\underline{x} - \omega t)}}$, so that $\underline{k} = (k\cos\alpha, 0, k\sin\alpha)$ in (†).

- If we fix $\underline{k} \neq \underline{0}$ and choose new axes OXYZ with $\underline{e}_X = \frac{\underline{k}}{|\underline{k}|}$, then $\underline{k}\cdot\underline{x} = |\underline{k}|\,\underline{e}_X\cdot(X\underline{e}_X + Y\underline{e}_Y + Z\underline{e}_Z) = |\underline{k}|X$, so that $\underline{w' = Ae^{i(kX - \omega t)}}$, where the wavenumber $k = |\underline{k}|$ and phase speed $c_p = \frac{\omega}{k}$.

- Hence, direction of propagation is in direction of <u>wavenumber vector</u> $\underline{k}$ with <u>phase velocity</u> $\underline{c}_p = c_p\underline{e}_X = \frac{\omega\underline{k}}{|\underline{k}|^2}$.



- Back to (9): simplest case is $\rho_0 = \rho_0^* e^{-\beta z}$ $(\rho_0^*, \beta \in \mathbb{R})$ and gravity large, so that $\frac{1}{g}$ term is negligible $\Rightarrow (w'_{xx} + w'_{zz})_{tt} = -\beta g w'_{xx}$

- Substitute (+) $\Rightarrow \left((ik\cos\alpha)^2 + (ik\sin\alpha)^2\right)(-i\omega)^2 w' = -\beta g (ik\cos\alpha)^2 w'$

  $\Rightarrow \underline{\omega^2 = \beta g \cos^2\alpha}$, independent of k, but not α.

- If $\beta > 0$, i.e. $\rho_{0z} < 0$, the fluid is <u>stably</u> stratified

- If $\beta < 0$, i.e. heavier fluid on top, $\omega = \pm i\sqrt{-\beta g \cos^2\alpha} \Rightarrow$ unstable

- NB: With $\frac{1}{g}$ term, $(1 + \frac{i\beta}{k}\sin\alpha)\omega^2 = \beta g \cos^2\alpha \Rightarrow$ unstable for k suff. small ind. of β.

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Inertial waves in a rotating fluid

- Consider constant density fluid rotating with constant angular speed $\underline{\Omega}$.

- Neglecting gravity, $\nabla \cdot \underline{u} = 0$, $\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} + \underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{x}) = -\frac{1}{\rho} \nabla p$.

- But $\underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{x}) = -\frac{1}{2} \nabla \left(|\underline{\Omega} \wedge \underline{x}|^2 \right)$, so $\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho} \nabla P$,
  where $P = p - \frac{1}{2}\rho|\underline{\Omega} \wedge \underline{x}|^2$ is the reduced pressure.

- Small perturbations to stationary state: $\underline{u} = \underline{0} + \underline{u}'$, $P = P_0 + P'$

- Linearized equations: $\underset{(A)}{\nabla \cdot \underline{u}' = 0}$, $\underset{(B)}{\underline{u}'_t + 2\underline{\Omega} \wedge \underline{u}' = -\frac{1}{\rho}\nabla P'}$

- (A) and (B) describe large scale flows in Earth's atmosphere. Note in steady state, $\underline{u}' \cdot \nabla P' = 0$, i.e. flow is along isobars (P' const.) when rotation is important.

- $\nabla \wedge$ (B) $\Rightarrow$ $(\nabla \wedge \underline{u}')_t = -2\nabla \wedge (\underline{\Omega} \wedge \underline{u}') = 2(\underline{\Omega} \cdot \nabla)\underline{u}' - 2\underbrace{(\nabla \cdot \underline{u}')}_{=0}\underline{\Omega}$

$$\underset{\frac{\partial}{\partial t}\nabla\wedge}{\Rightarrow} \quad (\nabla \wedge (\nabla \wedge \underline{u}'))_{tt} = 2(\underline{\Omega} \cdot \nabla)(\nabla \wedge \underline{u}')_t$$

$$\Rightarrow \quad \Big(\nabla \underbrace{(\nabla \cdot \underline{u}')}_{=0} - \nabla^2 \underline{u}'\Big)_{tt} = 2(\underline{\Omega} \cdot \nabla)(2(\underline{\Omega} \cdot \nabla)\underline{u}')$$

$$\Rightarrow \quad \underset{(C)}{\nabla^2 \underline{u}'_{tt} + 4(\underline{\Omega} \cdot \nabla)^2 \underline{u}' = \underline{0}}$$

- Seek a travelling wave solution $\underline{u}' = \underline{A} e^{i(\underline{k}\cdot\underline{x} - \omega t)}$, where $\underline{A}$ is complex amplitude, $\underline{k}$ is wavenumber vector and ω is frequency.

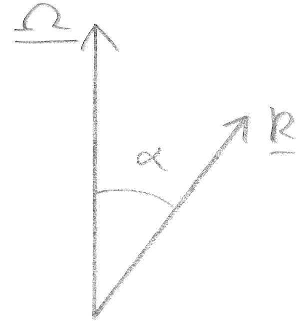
- Compute:

$$\nabla^2 \underline{u}' = (\partial_x^2 + \partial_y^2 + \partial_z^2)\underline{A} e^{i(k_1 x + k_2 y + k_3 z - \omega t)}$$
$$= ((ik_1)^2 + (ik_2)^2 + (ik_3)^2)\underline{u}'$$
$$= -|\underline{k}|^2 \underline{u}',$$

$$(\underline{\Omega}\cdot\nabla)\underline{u}' = (\Omega_1 \partial_x + \Omega_2 \partial_y + \Omega_3 \partial_z)\underline{A} e^{i(k_1 x + k_2 y + k_3 z - \omega t)}$$
$$= (\Omega_1 ik_1 + \Omega_2 ik_2 + \Omega_3 ik_3)\underline{u}'$$
$$= i(\underline{\Omega}\cdot\underline{k})\underline{u}',$$

- Hence, (C) $\Rightarrow$ $-|\underline{k}|^2(-i\omega)^2 + 4(i(\underline{\Omega}\cdot\underline{k}))^2 = 0 \Rightarrow \omega^2 = \dfrac{4(\underline{\Omega}\cdot\underline{k})^2}{|\underline{k}|^2}$

- Let α be the angle between $\underline{k}$ and $\underline{\Omega}$, then



$$\omega^2 = \frac{4|\underline{\Omega}|^2|\underline{k}|^2\cos^2\alpha}{|\underline{k}|^2} = 4|\underline{\Omega}|^2\cos^2\alpha$$

- RHS $\geqslant 0 \Rightarrow \omega^2 \geqslant 0 \Rightarrow \omega$ real $\Rightarrow$ always stable.

- Again frequency ω is independent of wavenumber $k = |\underline{k}|$, but depends on the direction α.

- Note highest frequencies and hence phase speeds (given k) are obtained when $\underline{k}$ is aligned with $\underline{\Omega}$ (i.e. $\cos\alpha = \pm 1$).

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- **Electromagnetic waves**: In free space (no charges or currents), electric field $\underline{E}$ and magnetic field $\underline{B}$ satisfy Maxwell's equations:

$$\nabla \cdot \underline{E} = 0, \quad \nabla \wedge \underline{E} = -\mu_0 \underline{B}_t, \quad \nabla \cdot \underline{B} = 0, \quad \nabla \wedge \underline{B} = \varepsilon_0 \underline{E}_t,$$

where μ_0 / ε_0 are the permeability / permittivity of free space.

- Eliminate $\underline{B}$: $\varepsilon_0 \mu_0 \underline{E}_{tt} = \nabla \wedge \mu_0 \underline{B}_t = -\nabla \wedge (\nabla \wedge \underline{E}) = \nabla^2 \underline{E} - \nabla \underbrace{(\nabla \cdot \underline{E})}_{=0}$

- Similarly, eliminating $\underline{E} \Rightarrow \varepsilon_0 \mu_0 \underline{B}_{tt} = \nabla^2 \underline{B}$.

- Hence, $\underline{E}_{tt} = c^2 \nabla^2 \underline{E}$, $\underline{B}_{tt} = c^2 \nabla^2 \underline{B}$, where $c = (\varepsilon_0 \mu_0)^{-1/2}$ is speed of light.

- <u>Elastic membrane</u>: Small transverse displacement $w(x,y,t)$ satisfies $\sigma w_{tt} = T\nabla^2 w$, where σ is density and T is the tension, with wavespeed $c = (T/\sigma)^{1/2}$.

- <u>Elastic plate</u>: Small transverse displacement $w(x,y,t)$ satisfies $\sigma w_{tt} = -B\nabla^4 w$, where B is the bending stiffness.

$$w = Ae^{i(\underline{k}\cdot\underline{x} - \omega t)} \Rightarrow \omega^2 = \frac{B}{\sigma}|\underline{k}|^4,$$ so waves are dispersive.

- <u>Online notes</u>: waves in elastic solids and quantum mechanics