Waves and Compressible Flow

Lecture 6

3. Methods for linear waves

- For finite domains, superimpose normal modes – find using separation of variables.

- For infinite domains, superimpose time-harmonic travelling wave solutions – find using a Fourier transform.

- Analyse Fourier integrals using method of stationary phase.

- Supersonic thin aerofoil theory – use method of characteristics.

Separation of variables

Example: Acoustic waves in a 1D box

- 1D waves in a box in $0 < x < a$, with fixed ends at $x = 0, a$, are governed by $\phi(x,t)$ s.t. ① $\phi_{tt} = c_0^2 \phi_{xx}$ for $0 < x < a$,

  ② $\phi_x = 0$ at $x = 0, a$.

- Seek normal mode with frequency ω, i.e. $\phi = f(x) e^{-i\omega t}$ (real part understood)

$$\Rightarrow \quad f'' + \left(\frac{\omega}{c_0}\right)^2 f = 0 \text{ for } 0 < x < a, \quad \text{with } f'(0) = f'(a) = 0.$$

- Hence, $f(x) = A\cos\left(\frac{n\pi x}{a}\right)$ $(A \in \mathbb{C})$ and $\omega = \frac{n\pi c_0}{a}$ $(n \in \mathbb{Z})$,

  i.e. a countably infinite set of natural or resonant frequencies.

- ①-② linear so can superimpose (with $n \geq 0$ wlog) to obtain general series solution

$$\phi(x,t) = \frac{B_0}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{a}\right)\left(B_n \cos\left(\frac{n\pi c_0 t}{a}\right) + C_n \sin\left(\frac{n\pi c_0 t}{a}\right)\right)$$

$(B_n, C_n \in \mathbb{R})$.

- If $\phi(x,0) = F(x)$, $\phi_t(x,0) = G(x)$, then theory of Fourier series gives

$$B_n = \frac{2}{a}\int_0^a F(x)\cos\left(\frac{n\pi x}{a}\right)dx, \quad \frac{n\pi c_0}{a} C_n = \frac{2}{a}\int_0^a G(x)\cos\left(\frac{n\pi x}{a}\right)dx.$$

- NB: Can choose F s.t. $B_0 = 0$ wlog (as in online notes).

<u>$Q^{\underline{n}}$</u>: What happens if one end of the box is oscillated?

- Suppose now LH boundary is at $x = \varepsilon e^{-i\omega t}$, where $\varepsilon \ll a$.

- Kinematic BC $\Rightarrow$ $\phi_x = -i\omega\varepsilon e^{-i\omega t}$ at $x=0$ upon linearizing for $\varepsilon \ll a$.

- Try $\phi = e^{-i\omega t} f(x) \Rightarrow f'' + \left(\frac{\omega}{c_0}\right)^2 f = 0$ for $0<x<a$, with $f'(0) = -i\omega\varepsilon$, $f'(a)=0$.

- Solving gives $\quad f(x) = -ic_0\varepsilon \dfrac{\cos \frac{\omega}{c_0}(a-x)}{\sin \frac{\omega a}{c_0}} \quad$ for $\sin\frac{\omega a}{c_0} \neq 0$.

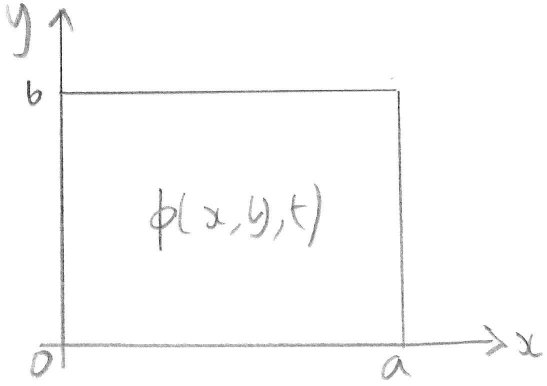
- Amplitude becomes unbounded (resonance) as $\omega \to \frac{n\pi c_0}{a}$ $(n \in \mathbb{Z})$, i.e. as ω approaches one of the natural or resonant frequencies.

- Resonance is fundamental mechanism behind many physical phenomena, e.g. musical instruments.

- If $\omega = \frac{n\pi c_0}{a}$ for some $n \in \mathbb{Z}$, seek instead a secular solution of the form $\phi = (f(x) + t g(x)) e^{-i\omega t}$ – see online notes for f eg.

- Find $g \neq 0$, so linearization breaks down at large times.

Example: Acoustic waves in a 2D box

- 

$$\phi_{tt} = c_0^2 \nabla^2 \phi \text{ for } 0<x<a,\ 0<y<b;$$

$$\phi_x = 0 \text{ at } x = 0, a; \quad \phi_y = 0 \text{ at } y = 0, b.$$

- Try $\phi = f(x)g(y)e^{-i\omega t}$

$$\Rightarrow \quad -\frac{\omega^2}{c_0^2} fg = f''g + fg''$$

$$\Rightarrow \quad \frac{f''}{f} = -\frac{g''}{g} - \frac{\omega^2}{c_0^2} \qquad (fg \neq 0)$$

$$\Rightarrow \quad \frac{f''}{f} = -\lambda^2, \ \frac{g''}{g} = -\mu^2, \ \lambda^2 + \mu^2 = \frac{\omega^2}{c_0^2} \quad (\lambda, \mu \in \mathbb{R})$$

with signs chosen for non-trivial solutions.

- $f'' + \lambda^2 f = 0$ for $0 < x < a$, with $f'(0) = f'(a) = 0 \Rightarrow$ non-trivial solutions

$$f(x) \propto \cos\left(\frac{n\pi x}{a}\right), \quad \lambda = \frac{n\pi}{a} \quad (n \in \mathbb{Z}).$$

- $g'' + \mu^2 g = 0$ for $0 < y < b$, with $g'(0) = g'(b) = 0 \Rightarrow$ non-trivial solutions

$$g(y) \propto \cos\left(\frac{m\pi y}{b}\right), \quad \mu = \frac{m\pi}{b} \quad (m \in \mathbb{Z}).$$

- Combo $\Rightarrow$ $\phi = A\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{-i\omega t}$, $\omega^2 = \pi^2 c_0^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$ $(m, n \in \mathbb{Z})$

- Hence, there is a doubly-infinite set of natural frequencies.

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Example: Acoustic waves in a 3D box

- In $\{0 < x < a,\ 0 < y < b,\ 0 < z < c\}$, normal modes are

$$\phi = A\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{l\pi z}{c}\right)e^{-i\omega t}$$

with

$$\omega^2 = \pi^2 c_0^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{c^2}\right) \qquad (m, n, l \in \mathbb{Z}).$$

- In general, find an infinite family of normal modes for each spatial dimension.

- Can still superimpose to satisfy ICs using Fourier theory.

Example: spherically symmetric acoustic waves

- Suppose gas is contained in the annular region $a < r < b$.

- For spherically symmetric waves $\phi(r,t)$ satisfies

$$\phi_{tt} = \frac{c_0^2}{r}(r\phi)_{rr} \text{ for } a<r<b; \quad \phi_r = 0 \text{ on } r = a, b.$$

- Seek normal modes $\phi = f(r)e^{-i\omega t}$

$$\Rightarrow \quad (rf)'' + \frac{\omega^2}{c_0^2} rf = 0 \text{ for } a<r<b; \quad f'(a) = f'(b) = 0$$

$$\Rightarrow \quad f(r) = \frac{1}{r}\left(A\cos\left(\frac{\omega r}{c_0}\right) + B\sin\left(\frac{\omega r}{c_0}\right)\right) \qquad (A, B \in \mathbb{C})$$

- BCs $\Rightarrow$ $\underbrace{\begin{bmatrix} \cos ka + ka\sin ka & \sin ka - ka\cos ka \\ \cos kb + kb\sin kb & \sin kb - kb\cos kb \end{bmatrix}}_{M(k=\omega/c_0)} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- $\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \det(M) = 0$

$$\Rightarrow (1 + k^2 ab)\tan k(b-a) = k(b-a)$$

$$\Rightarrow \left(1 + \frac{ab}{c_0^2}\omega^2\right)\tan \frac{(b-a)}{c_0}\omega = \frac{(b-a)}{c_0}\omega$$

- Given a, b, c_0 this transendental equation for the natural frequencies ω has a countably infinite set of solutions.

Example: circularly symmetric acoustic waves

- Suppose $\phi(r,t)$: $\quad \phi_{tt} = c_0^2\left(\phi_{rr} + \frac{1}{r}\phi_r\right)$ for $0 < r < a$;

$$\phi = O(1) \text{ at } r = 0; \quad \phi_r = 0 \text{ on } r = a.$$

- Try $\phi = f(r)e^{-i\omega t} \Rightarrow \quad f'' + \frac{1}{r}f' + \frac{\omega^2}{c_0^2}f = 0$ for $0 < r < a$;

$$f(0) = O(1) \text{ and } f'(a) = 0.$$

- Let $r = \frac{c_0}{\omega}\zeta$, $f(r) = F(\zeta)$, then $\zeta^2 F'' + \zeta F' + \zeta^2 F = 0$, which is Bessel's equation of order 0; hence $F(\zeta) = AJ_0(\zeta) + BY_0(\zeta)$

$(A, B \in \mathbb{C})$

- But only J_0 is bounded at the origin, so $f(r) = A J_0\left(\frac{\omega r}{c_0}\right)$.

- Other BC $\Rightarrow$ $J_0'\left(\frac{\omega a}{c_0}\right) = 0$

  $\Rightarrow$ $\frac{\omega a}{c_0} = \xi_{0,m}$ $(m \in \mathbb{Z}^+)$, where $\xi_{0,1} < \xi_{0,2} < \ldots$

  are the countably infinite set of extrema

  of $J_0(\xi)$.

- Straightforward to extend analysis to find normal modes in a pipe $\{r < a, 0 < z < L\}$ — see online notes.