Waves and Compressible Flow

Lecture 7

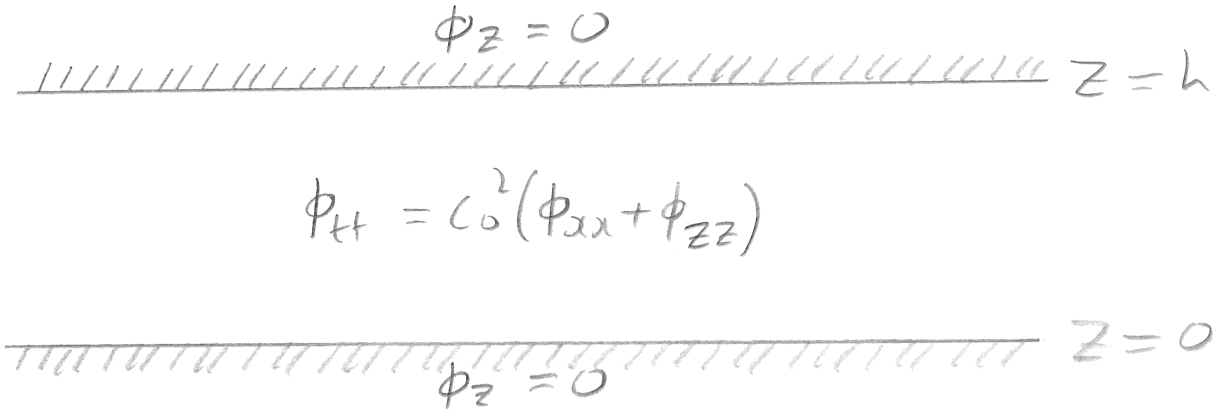
Travelling waves

- Last lecture: normal modes for wave equation in finite domains using separation of variables.

- This lecture: travelling harmonic waves and Fourier transforms for infinite domains.

Example : waveguide

- Acoustic waves in a 2D channel have potential $\phi(x,z,t)$ s.t.



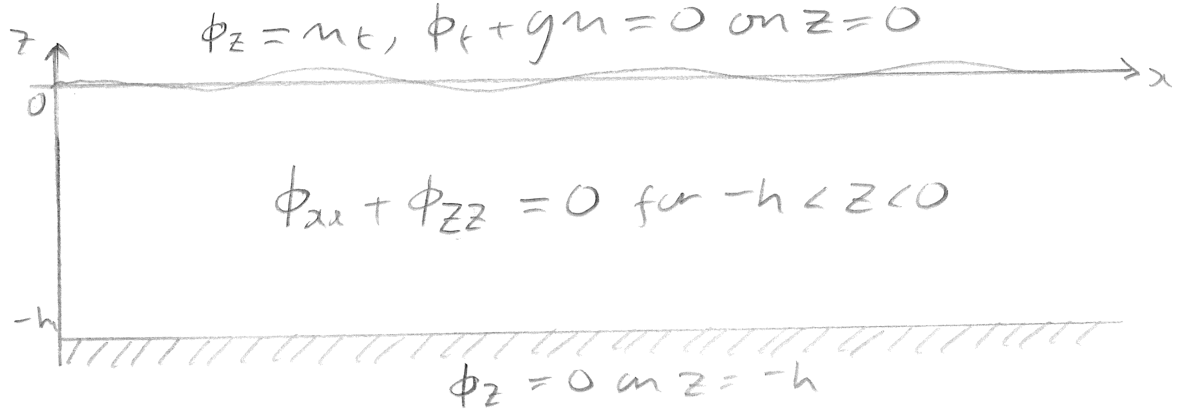
- Try travelling harmonic wave $\phi = f(z)e^{i(kx - \omega t)}$, then

$$f'' + \left(\frac{\omega^2}{c_0^2} - k^2\right) f = 0 \text{ for } 0 < z < h \, ; \; f'(0) = f'(h) = 0.$$

- Hence, $f(z) = A \cos\left(\frac{n\pi z}{h}\right)$, $\omega^2 = c_0^2\left(k^2 + \frac{n^2\pi^2}{h^2}\right)$, where $k \in \mathbb{R}, n \in \mathbb{Z}$.

Example: Stokes waves

- Suppose $\phi(x,z,t)$ and $\eta(x,t)$ satisfy the following linearized problem:



- Try $\eta = A e^{i(kx-\omega t)}$, $\phi = f(z) e^{i(kx-\omega t)}$, then $f'' - k^2 f = 0$ for $-h<z<0$, with $f'(-h) = 0$, $f'(0) = -i\omega A$, $-i\omega f(0) + gA = 0$.

- Hence, $f(z) = \dfrac{gA}{i\omega} \dfrac{\cosh k(h-z)}{\cosh kh}$ and $\omega^2 = gk \tanh(kh)$

- For infinite domain problems, there is a continuous spectrum, i.e. set of allowable wavenumbers $\underline{k}$, with dispersion relation $\omega(\underline{k})$ say.

- Write general solution as an integral over wavenumbers, i.e. a Fourier transform.

Fourier Transforms

- Recall $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$ (for suitably integrable f)

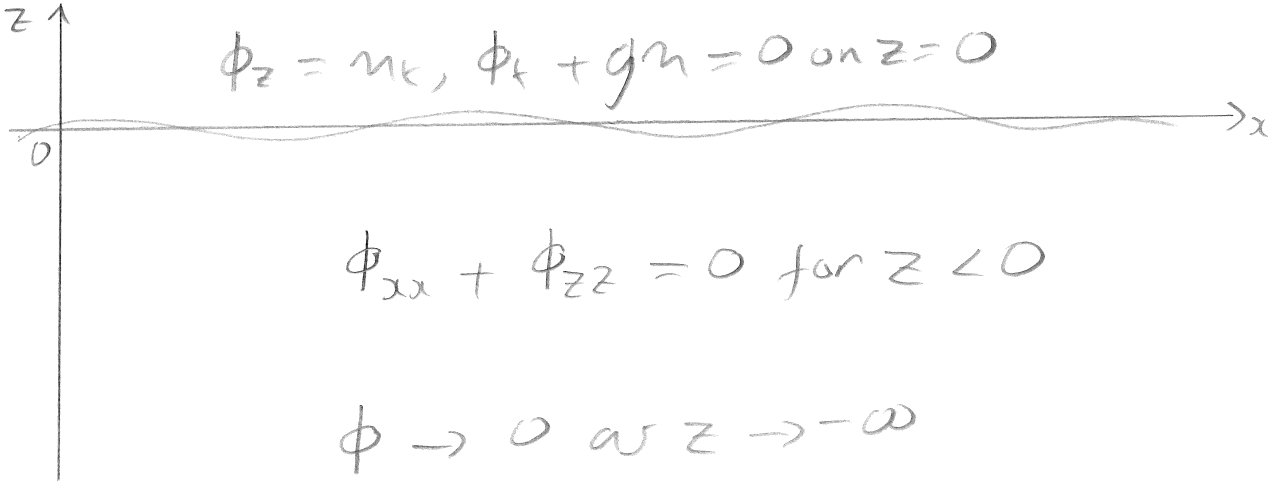
- Inverse: $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk$

- Convolution: $\hat{f}(k) = \hat{g}(k) \hat{h}(k) \Longleftrightarrow f(x) = \int_{-\infty}^{\infty} g(\xi) h(x-\xi) \, d\xi$

- Derivatives: $\widehat{\frac{d^n f}{dx^n}} = (ik)^n \hat{f}$

Stokes waves: IBVP

- Suppose $\phi(x,z,t)$ and $\eta(x,t)$ satisfy the following linearized problem:



$$\phi_z = \eta_t, \quad \phi_t + g\eta = 0 \text{ on } z = 0$$

$$\phi_{xx} + \phi_{zz} = 0 \text{ for } z < 0$$

$$\phi \to 0 \text{ as } z \to -\infty$$

- If fluid starts from rest with free surface displacement $\eta_0(x)$, then problem is closed by the ICs

$$\eta(x,0) = \eta_0(x), \quad \eta_t(x,0) = 0$$

- Take Fourier transform in x:

$$\hat{\phi}(k,z,t) = \int_{-\infty}^{\infty} \phi(x,z,t)e^{-ikx}dx, \quad \hat{\eta}(k,t) = \int_{-\infty}^{\infty} \eta(x,t)e^{-ikx}dx.$$

- $\phi_{xx} + \phi_{zz} = 0$ in $z<0$ $\Rightarrow$ $(ik)^2\hat{\phi} + \hat{\phi}_{zz} = 0$ in $z<0$

- $\phi_z = \eta_t$, $\phi_t + g\eta = 0$ on $z=0$ $\Rightarrow$ $\hat{\phi}_z = \hat{\eta}_t$, $\hat{\phi}_t + g\hat{\eta} = 0$ on $z=0$

- $\phi \to 0$ as $z \to -\infty$ $\Rightarrow$ $\hat{\phi} \to 0$ as $z \to -\infty$

- $\eta = \eta_0$, $\eta_t = 0$ at $t=0$ $\Rightarrow$ $\hat{\eta} = \hat{\eta}_0$, $\hat{\eta}_t = 0$ at $t=0$

- $-k^2\hat{\phi} + \hat{\phi}_{zz} = 0 \implies \hat{\phi} = A(k,t)\,e^{|k|z} + B(k,t)\,e^{-|k|z}$

- Write in this form to simplify algebra: since $\hat{\phi} \to 0$ as $z \to -\infty$, can now set $B \equiv 0$.

- $\hat{\phi}_z = \hat{\eta}_t,\ \hat{\phi}_t + g\hat{\eta} = 0$ on $z=0 \implies |k|A = \hat{\eta}_t,\ A_t + g\hat{\eta} = 0$

- Eliminate A: $\hat{\eta}_{tt} = |k|A_t = -g|k|\hat{\eta}$ for $t>0$

- Solve for $\hat{\eta}$: $\hat{\eta}(k,t) = C(k)\cos\omega t + D(k)\sin\omega t,\ \omega(k) := \sqrt{g|k|}$

- ICs: $\hat{n} = \hat{n}_0$, $\hat{n}_t = 0$ at $t = 0$ $\Rightarrow$ $C(k) = \hat{n}_0(k)$, $D(k) = 0$ $\Rightarrow A(k) = -\frac{\omega \hat{n}_0}{|k|} \sin\omega t$

- Hence, $\hat{n}(k,t) = \hat{n}_0(k) \cos \omega(k) t$, $\hat{\phi}(k,z,t) = -\frac{\omega(k)\hat{n}_0(k)}{|k|} \sin \omega(k) t \, e^{|k|z}$.

- Focus on n. Inverting gives

$$n(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{n}_0(k) \cos \omega(k) t \, e^{ikx} \, dx$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{n}_0(k) \left(e^{i(kx - \omega(k)t)} + e^{i(kx + \omega(k)t)} \right) dk,$$

i.e. superposition of travelling waves with wavenumber k, frequency $\omega(k)$, phase speed $c_p(k) = \frac{\omega(k)}{k}$ travelling left and right.

Multidimensional Fourier transform

- Can generalize Fourier transforms to higher dimensions.

- For e.g. $f(x,y,z)$, can define $\hat{f}(k,l,m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, e^{-ikx-ily-imz}\, dx\,dy\,dz$.

- With wavenumber vector $\underline{k} = (k,l,m)^T$, $\hat{f}(\underline{k}) = \iiint_{\mathbb{R}^3} f(\underline{x})\, e^{-i\underline{k}\cdot\underline{x}}\, d\underline{x}$.

- The inverse then reads $f(\underline{x}) = \frac{1}{(2\pi)^3} \iiint_{\mathbb{R}^3} \hat{f}(\underline{k})\, e^{i\underline{k}\cdot\underline{x}}\, d\underline{k}$.

Example: internal gravity waves.

- Recall that $w(x,y,z,t)$ satisfies (after linearization)

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) = -\beta g\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}-\frac{1}{g}\frac{\partial^3 w}{\partial z \partial t^2}\right)$$

[NB: $\beta \mapsto -\beta$ in online notes.]

- Suppose initially $w = w_0(\underline{x})$, $\frac{\partial w}{\partial t} = 0$ at $t=0$.

- Take a Fourier transform in x, y, z, i.e. $\hat{w}(\underline{k}) = \iiint_{\mathbb{R}^3} w(\underline{x},t)\, e^{-i\underline{k}\cdot\underline{x}}\, d\underline{x}$,
  where $\underline{k} = (k, l, m)^T$.

- PDE $\Rightarrow$ $$\frac{\partial^2}{\partial t^2}\left((ik)^2\hat{\omega} + (il)^2\hat{\omega} + (im)^2\hat{\omega}\right) = -\beta g\left((ik)^2\hat{\omega} + (il)^2\hat{\omega} - \frac{im}{g}\frac{\partial^2\hat{\omega}}{\partial t^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2\hat{\omega}}{\partial t^2} + \Omega(\underline{k})^2\hat{\omega} = 0, \quad \Omega(\underline{k})^2 = \frac{\beta g(k^2+l^2)}{k^2+l^2+m^2+im\beta}$$

$$\Rightarrow \quad \hat{\omega} = A(\underline{k})\cos\Omega(\underline{k})t + B(\underline{k})\sin\Omega(\underline{k})t$$

- ICs $\Rightarrow$ $A(\underline{k}) = \hat{\omega}_0(\underline{k})$, $B(\underline{k}) = 0$, $\hat{\omega}(\underline{k},t) = \hat{\omega}_0(\underline{k})\cos\Omega(\underline{k})t$

- Inverting $\Rightarrow$ $$\omega(\underline{x},t) = \frac{1}{8\pi^3}\iiint_{\mathbb{R}^3} \hat{\omega}_0(\underline{k})\cos(\Omega(\underline{k})t)\, e^{i\underline{k}\cdot\underline{x}}\, d\underline{k}$$

- NB: Since $\mathrm{Im}(\Omega(\underline{k})) \neq 0$ for $m \neq 0$, basestate is stable iff $\hat{\omega}_0$ is ind. m, i.e. ω_0 is ind. z.