Waves and Compressible Flow

Lecture 8

Method of stationary phase

- Recall from last lecture that for Stokes waves on deep water with $\eta = \eta_0$, $\eta_t = 0$ at $t = 0$, a Fourier transform gives

$$\eta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\eta}_0(k)\left(e^{i(kx-\omega(k)t)} + e^{i(kx+\omega(k)t)}\right)dk,$$

where the dispersion relation $\omega(k) = \sqrt{g|k|}$.

- Hence, an observer at $x = Vt$ moving with speed V sees a free surface displacement

$$\eta(Vt,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\eta}_0(k)\, e^{i(kV-\omega(k))t}\,dk + \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\eta}_0(k)\, e^{i(kV+\omega(k))t}\,dk.$$

- This motivates the need to analyse integrals of the form

$$I(t) = \int_a^b f(k)\, e^{i\psi(k)t}\, dk$$

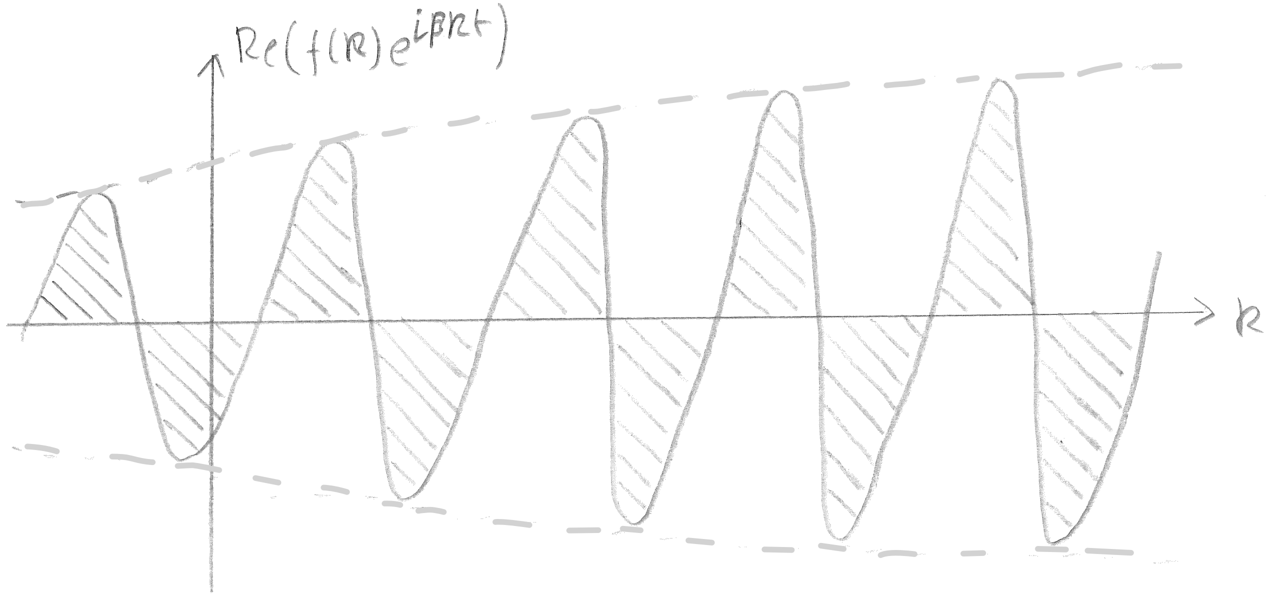
in the limit $t \to \infty$, where $f(k)$ is the <u>amplitude</u> and $\psi(k)$ is the <u>phase</u>, both real and prescribed.

- We begin by building some geometric insight before considering the generic case.

- We will use some ad hoc asymptotic estimates – can be made rigorous.

Example: linear $\psi(k)$

- If $\psi(k) = \beta k$ $(\beta \in \mathbb{R})$, then $I(t) = \int_a^b f(k) e^{i\beta k t}\, dk$



- As $t \to \infty$, the integrand oscillates more and more, so +ve and −ve contributions to the integral almost cancel out.

- Integrate by parts:

$$I(t) = \left[f(k) \frac{e^{i\beta k t}}{i\beta t} \right]_{k=a}^{k=b} - \int_a^b f'(k) \frac{e^{i\beta k t}}{i\beta t} \, dk$$

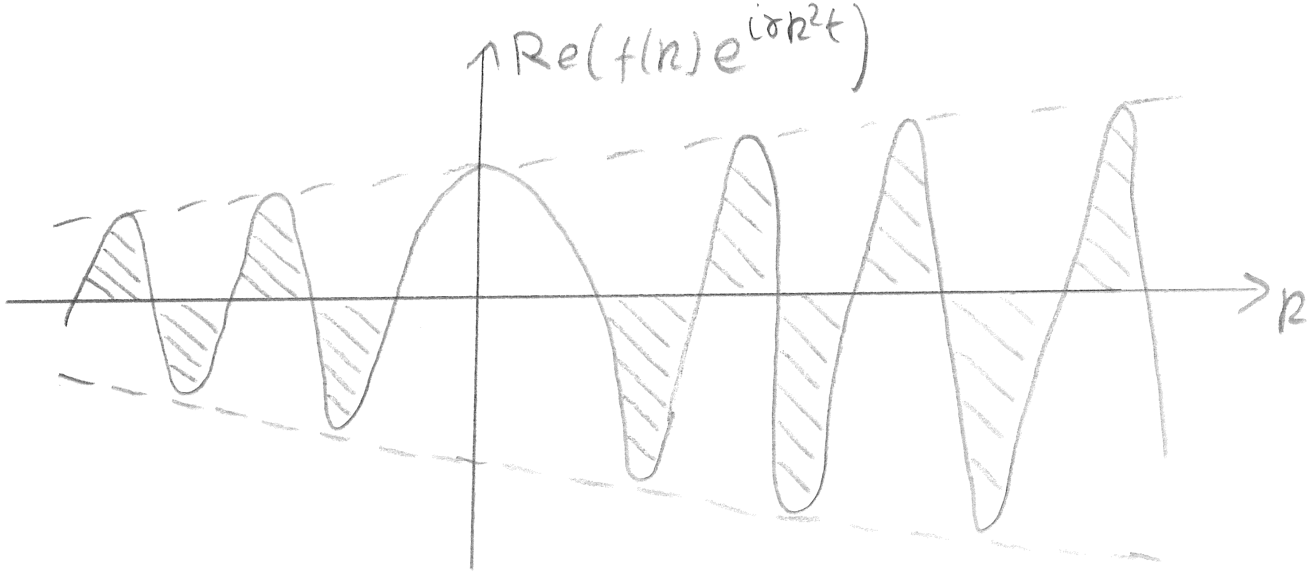
$$= O(1/t) \quad \text{as } t \to \infty$$

- This result is called the <u>Riemann-Lebesgue Lemma</u> (RLL).

- Assuming f' is integrable, it applies for $a = -\infty$, $b = +\infty$.

Example: quadratic $\psi(R)$

- If $\psi(R) = \gamma R^2$ $(\gamma \in \mathbb{R})$, then $I(t) = \int_a^b f(R)\, e^{i\gamma R^2 t}\, dR$.



- Now the cancellation is less effective near $R=0$ (where the phase $\psi(R) = R^2$ changes slowly). As $t \to \infty$, the main contribution to the integral comes from near $R=0$ and we'll show $I(t) = O\left(\frac{1}{\sqrt{t}}\right)$ as $t \to \infty$.

For more general ψ, consider two cases.

<u>(I) $\psi'(x) \neq 0$ for $x \in [a, b]$</u>

- In this case $\psi(x)$ is a monotonic function and has (by the inverse-function theorem) an inverse $x = x(\psi)$.

- Change variables $\Rightarrow$

$$I(t) = \int_{\psi(a)}^{\psi(b)} f(x(\psi))\, x'(\psi)\, e^{i\psi t}\, d\psi = O\left(\frac{1}{t}\right) \text{ as } t \to \infty \text{ by RLL.}$$

(II) $\psi'(k)$ has a single simple zero in (a,b)

- In this case $\psi'(k) = 0$ at one value $k = k_* \in (a,b)$ s.t. $\psi''(k_*) \neq 0$.

- Split up range of integration : $I = I_1 + I_2 + I_3$, where

$$I_1(t) = \int_a^{k_*-\varepsilon} f(k) e^{i\psi(k)t}\,dk, \quad I_2(t) = \int_{k_*-\varepsilon}^{k_*+\varepsilon} f(k) e^{i\psi(k)t}\,dk, \quad I_3(t) = \int_{k_*+\varepsilon}^{b} f(k) e^{i\psi(k)t}\,dk,$$

where ε is a small positive constant.

- Case (I) $\Rightarrow$ $I_1(t), I_2(t) = O(\frac{1}{t})$ as $t \to \infty$.

- Since $\varepsilon \ll 1$, for $I_2(t)$ can use Taylor expansions to approximate $f(k)$ and $\psi(k)$:

$$f(k) \sim f(k_*),$$

$$\psi(k) \sim \psi(k_*) + \psi'(k_*)(k-k_*) + \frac{\psi''(k_*)}{2}(k-k_*)^2,$$

both as $k \to k_*$.

- Substituting and neglecting higher order terms $\Rightarrow$

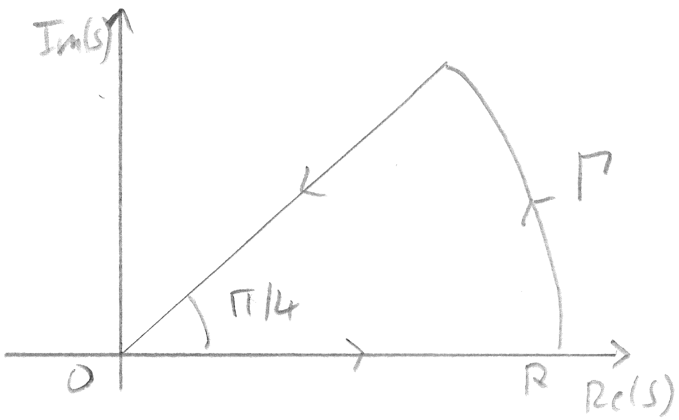
$$I_2(t) \sim f(k_*)\, e^{i\psi(k_*)t} \int_{k_*-\varepsilon}^{k_*+\varepsilon} e^{\frac{i}{2}\psi''(k_*)(k-k_*)^2 t}\, dk$$

• If we let $R = R_* + \frac{s}{\sqrt{\gamma t}}$, where $\gamma = \frac{1}{2}|\psi''(R_*)|$, then

$$\int_{R_*-\varepsilon}^{R_*+\varepsilon} e^{\frac{i}{2}\psi''(R_*)(R-R_*)^2 t}\, dR = \frac{1}{\sqrt{\gamma t}} \int_{-\varepsilon\sqrt{\gamma t}}^{\varepsilon\sqrt{\gamma t}} e^{\pm i s^2}\, ds \qquad (\pm = \mathrm{sgn}(\psi''(R_*)))$$

$$\sim \frac{1}{\sqrt{\gamma t}}\sqrt{\pi}\, e^{\pm i\pi/4} \quad \text{as } t \to \infty.$$

• This follows from contour integration: $\int_\Gamma e^{is^2}\, ds = 0 \Rightarrow$



$$\lim_{R\to\infty} \int_{-R}^{R} e^{is^2}\, ds = \lim_{R\to\infty} 2\int_0^R e^{is^2}\, ds$$

$$= \lim_{R\to\infty} 2\int_0^R e^{-z^2} e^{i\pi/4}\, dz$$

$$= 2e^{i\pi/4}\sqrt{\pi}/2$$

- Hence, $I_2(t) = O(1/\sqrt{t})$ as $t \to \infty$.

- Since $I_1(t), I_3(t) = O(1/t) \ll I_2(t)$ as $t \to \infty$, the contribution from $I_2(t)$ dominates those from $I_1(t)$ and $I_3(t)$.

- Hence,

$$I(t) \sim f(k_*) e^{i(\psi''(k_*)t \pm \pi/4)} \left(\frac{2\pi}{|\psi''(k_*)| t} \right)^{1/2}$$

as $t \to \infty$, where $\pm$ takes the sign of $\psi''(k_*)$.

Remarks

- Argument sufficient for this course and can be made rigorous.
- Dominant contribution comes from a neighbourhood of $k = k_*$ where the phase $\psi(k)$ is stationary – hence name of method.
- If ψ' has multiple simple zeros in (a, b), apply method to each and sum.
- If $k_* = a$ or b and/or k_* is a higher order zero, need a modified version.