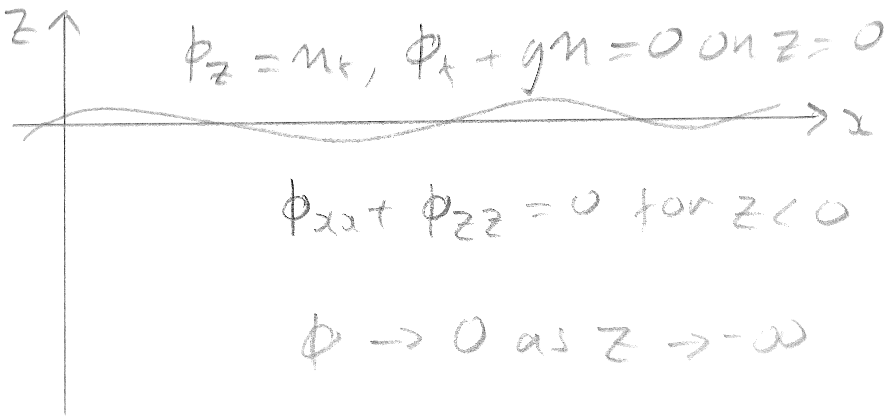
Waves and Compressible Flow

Lecture 9

Group velocity

- Recall that for Stokes waves on a deep pond, $\phi(x,z,t)$ and $\eta(x,t)$ are governed by the linearized problem



- We used a Fourier transform to show that, if $\eta(x,0) = \eta_0(x)$ and $\eta_t(x,0) = 0$, then

$$\eta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\eta}_0(k)\, e^{i(kx - \omega(k)t)}\, dk + \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\eta}_0(k)\, e^{i(kx + \omega(k)t)}\, dk$$

where $\omega(k) = \sqrt{g|k|}$.

- This is a superposition of right- and left-travelling waves with wavenumber k, frequency $\omega(k)$ and phase speed $c_p(k) = \frac{\omega(k)}{k}$.

- Imagine an observer moving with constant speed V by writing $x = Vt$, and then let $x, t \to \infty$, keeping V constant.

- The observer sees a displacement given by

$$\eta(Vt, t) = \underbrace{\frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\eta}_0(k)\, e^{i(kV - \omega(k))t}\, dk}_{I_+(t)} + \underbrace{\frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\eta}_0(k)\, e^{i(kV + \omega(k))t}\, dk}_{I_-(t)}$$

- Hence, $$I_\pm(t) = \int_a^b f(k)\, e^{i\psi(k)t}\, dk,$$

where $a = -\infty$, $b = \infty$, $f(k) = \dfrac{\hat{n}_0(k)}{4\pi}$, $\psi(k) = kV \mp \omega(k)$.

- As $t \to \infty$, method of stationary phase $\Rightarrow$ main contribution to $I_\pm(t)$ comes from near wavenumbers $k = k_*$ where the phase is stationary, i.e. $\psi'(k_*) = 0$, i.e. $V = \pm\omega'(k_*)$ (*)

- Thus, an observer travelling at speed V will see waves of wavenumber k_* satisfying (*).

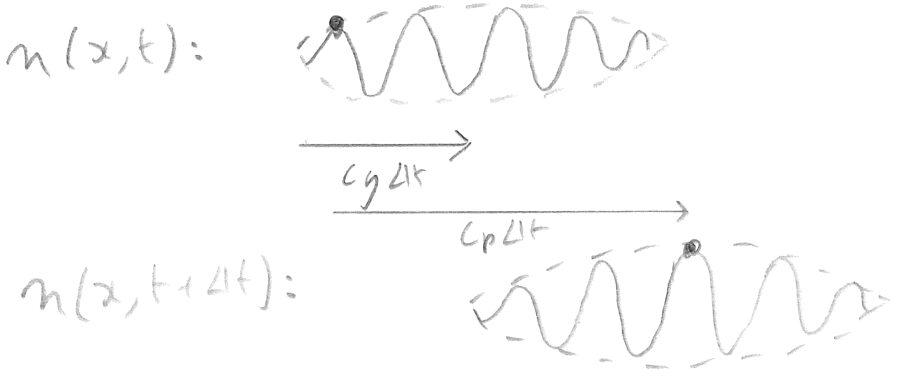
- Equivalently, to see waves of a particular wavenumber k, observer must travel at speed $V = \pm c_g(k)$, where $\underline{c_g(k) = \omega'(k)}$ is the <u>group velocity</u>

- For dispersive waves (i.e. where $c_p(k) = \frac{\omega(k)}{k}$ depends on k), $c_p \neq c_g$.

- This might seem paradoxical: to see waves of wavenumber k we must travel at the group velocity $c_g(k)$ rather than at the phase velocity $c_p(k)$!

- The explanation is that, after a long time, dispersive waves separate into wave packets corresponding to different wavenumbers. Within each packet, individual waves move at speed c_p, but the packet as a whole moves with speed $c_g(k)$.

- For example, if $c_p > c_g$, wave crests move through the wavepacket.



Wave crest • moves distance $c_p \Delta t$

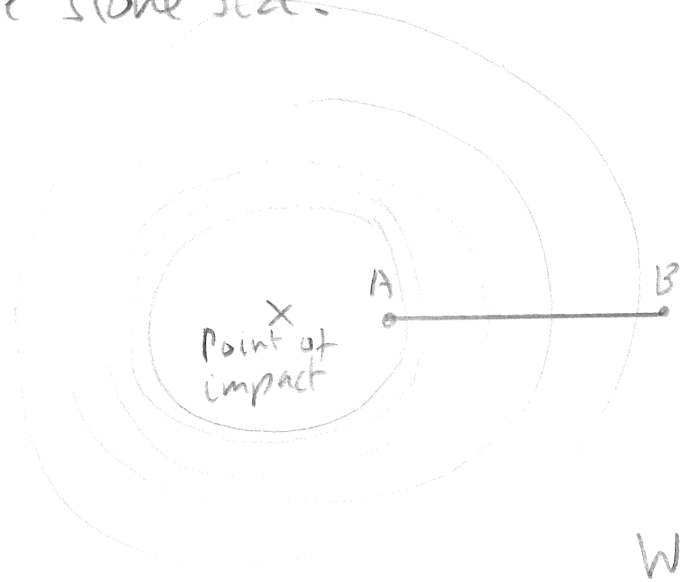
Wavepacket moves distance $c_g \Delta t$

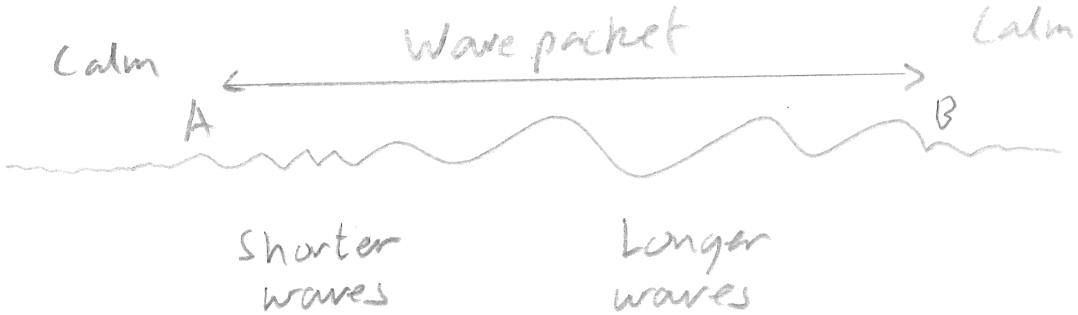
Example: Stokes waves on deep water

- $\omega(k) = \sqrt{g|k|} \Rightarrow c_p = \dfrac{\sqrt{g|k|}}{k}, \quad c_g = \dfrac{\sqrt{g|k|}}{2k} = \frac{1}{2} c_p$

  $\Rightarrow |c_g| < |c_p|$ and crests move through packets.

- A stone dropped in deep water generates wavelengths comparable with the stone size.





Waves appear at A and disappear at B.

Example: Stokes waves on deep water with surface tension

- Now $\omega(k) = \left(g|k| \left(1 + \frac{\gamma k^2}{\rho g}\right)\right)^{1/2} \Rightarrow \frac{c_g}{c_p} = \frac{3}{2} - \left(1 + \frac{\gamma k^2}{\rho g}\right)^{-1}$

- Hence, $|c_g| < |c_p|$ when $|k| < k_c$

  $|c_g| > |c_p|$ when $|k| > k_c$

  where $k_c = (\rho g/\gamma)^{1/2}$

- Corresponding critical wavelength $\lambda_c = \frac{2\pi}{k_c} \approx 1.7\,\text{cm}$ for water.

- Waves longer than λ_c travel faster than their wave packets.

Example: localized disturbance

- We still haven't computed the expansion of $\eta(Vt,t)$ as $t \to \infty$!

- Recall case (II) of the method of stationary phase: if $\psi'(k)$ has a single simple zero at $k = k_* \in (a,b)$, then

$$\int_a^b f(k)\, e^{i\psi(k)t}\, dk \sim f(k_*)\, e^{i(\psi(k_*)t \pm \pi/4)} \left(\frac{2\pi}{|\psi''(k_*)|t} \right)^{1/2}$$

as $t \to \infty$, where $\pm$ takes the sign of $\psi''(k_*)$.

- Apply to $I_\pm(t)$ in which $f(k) = \frac{\hat{\eta}_0(k)}{4\pi}$, $\psi(k) = kV \mp \omega(k)$, $\omega(k) = \sqrt{g|k|}$

- $\omega(k) = \sqrt{g|k|} = \begin{cases} \sqrt{gk} & \text{for } k>0 \\ \sqrt{gk} & \text{for } k<0 \end{cases} \Rightarrow \omega'(k) = \frac{\sqrt{g|k|}}{2k}, \quad \omega''(k) = -\frac{\sqrt{g|k|}}{4k^2}.$

- Hence, $\psi'(k) = V \mp \frac{\sqrt{g|k|}}{2k}, \quad \psi''(k) = \pm\frac{\sqrt{g|k|}}{4k^2}.$

- $\psi'(k_*) = 0 \Rightarrow V = \pm\frac{\sqrt{g|k_*|}}{2k_*} \Rightarrow k_* = \pm\frac{g}{4V^2}$ if $V>0$

- Hence, at the critical wavenumber at which ψ is stationary, we have

$$k_* = \pm\frac{g}{4V^2}, \quad \psi(k_*) = \mp\frac{g}{4V}, \quad \psi''(k_*) = \pm\frac{2V^3}{g},$$

i.e. $\psi'(k)$ has a single simple zero in each case.

- Substituting into the formula gives, as $t \to \infty$,

$$I_+(t) \sim \frac{\hat{n}_0\left(+\frac{g}{4V^2}\right)}{4\pi} \exp\left(i\left(-\frac{gt}{4V} + \frac{\pi}{4}\right)\right)\left(\frac{2\pi}{|2V^3/g|\, t}\right)^{1/2}$$

$$I_-(t) \sim \frac{\hat{n}_0\left(-g/4V^2\right)}{4\pi} \exp\left(i\left(+\frac{gt}{4V} - \frac{\pi}{4}\right)\right)\left(\frac{2\pi}{|2V^3/g|\, t}\right)^{1/2}$$

- Hence,

$$n(Vt,t) \sim \frac{1}{4}\left(\frac{g}{\pi V^3 t}\right)^{1/2}\left\{\hat{n}_0\left(\frac{g}{4V^2}\right)e^{i(\pi/4 - gt/4V)} + \hat{n}_0\left(-\frac{g}{4V^2}\right)e^{-i(\pi/4 - gt/4V)}\right\}$$

as $t \to \infty$, with V held constant.

- For example, if $n_0(x) = \dfrac{a\varepsilon}{\pi(x^2+\varepsilon^2)}$, then

$$\int_{-\infty}^{\infty} n_0(x)\,dx = a \quad \text{and} \quad n_0(x) \to \begin{cases} 0 & \text{for } x \neq 0 \\ +\infty & \text{for } x = 0 \end{cases} \quad \text{as } \varepsilon \to 0,$$

i.e. $n_0(x) \to a\,\delta(x)$ as $\varepsilon \to 0$, where $\delta(x)$ is Dirac's delta function.

- Contour integration gives $\hat{n}_0(k) = a\,e^{-\varepsilon|k|}$.

- Hence, $$n(Vt, t) \sim \frac{a}{2}\left(\frac{g}{\pi V^3 t}\right)^{1/2} e^{-\frac{\varepsilon g}{4V^2}} \cos\left(\frac{gt}{4V} - \frac{\pi}{4}\right)$$

as $t \to \infty$, with $V = O(1)$.

- This agrees well with the exact solution available in this case for $gt/V \gg 1$.

- If $\varepsilon \ll V^2/g$, then $e^{-\varepsilon g/4V^2} \sim 1$, and putting $V = x/t$ then gives

$$\eta(x,t) \sim \frac{at}{2}\left(\frac{g}{\pi x^3}\right)^{1/2} \cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right)$$

as $t \to \infty$ with $\frac{x}{t} = O(1)$.

- It can be shown that this is the universal behaviour far from an initial localized disturbance.