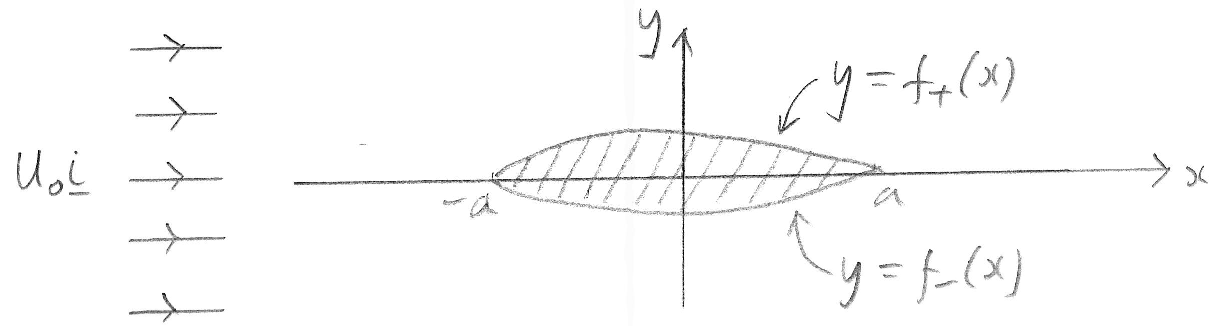
Waves and Compressible Flow

Lecture 10

Steady 2D barotropic flow past a thin wing

- Setup:



- Governing equations: $\nabla \cdot (\rho \underline{u}) = 0$, $\rho(\underline{u} \cdot \nabla)\underline{u} = -\nabla p$, $p = P(\rho)$.

- Thin wing ($|f_\pm| \ll a$) $\Rightarrow$ flow only slightly perturbed from uniform flow in which $\underline{u} = U_0 \underline{i}$, $\rho = \rho_0$, $p = p_0 = P(\rho_0)$.

- Perturb assuming irrotational flow: $\underline{u} = U_0 \underline{i} + \nabla\phi$, $\rho = \rho_0 + \rho'$, $p = p_0 + p'$.

- Substitute and linearize for $\phi \ll LU_0$, $\rho' \ll \rho_0$, $p' \ll p_0$:

$$\nabla \cdot \left((\rho_0 + \rho')(U_0 \underline{i} + \nabla \phi) \right) = 0 \quad \Rightarrow \quad U_0 \rho'_x + \rho_0 \nabla^2 \phi = 0 \quad (1)$$

$$(\rho_0 + \rho')\left((U_0 + \phi_x)\frac{\partial}{\partial x} + \phi_y \frac{\partial}{\partial y} \right) \phi_x = -p'_x \quad \Rightarrow \quad \rho_0 U_0 \phi_{xx} = -p'_x \quad (2)$$

$$(\rho_0 + \rho')\left((U_0 + \phi_x)\frac{\partial}{\partial x} + \phi_y \frac{\partial}{\partial y} \right) \phi_y = -p'_y \quad \Rightarrow \quad \rho_0 U_0 \phi_{xy} = -p'_y \quad (3)$$

$$p_0 + p' = P(\rho_0 + \rho') \quad \Rightarrow \quad p' = c_0^2 \rho' \quad (4)$$

where $c_0^2 = \frac{dP}{d\rho}(\rho_0) = \frac{\gamma p_0}{\rho_0}$ for homentropic flow with $p \propto \rho^\gamma$.

- Recall that c_0 is the speed of sound.

Potential problem

- Eliminate p' & ρ': $\phi_{xx} + \phi_{yy} \underset{\text{①}}{=} -\frac{U_0}{\rho_0}\rho'_x \underset{\text{④}}{=} -\frac{U_0}{\rho_0 c_0^2} p'_x \underset{\text{②}}{=} \frac{U_0^2}{c_0^2}\phi_{xx}$.

- Defining the Mach number $M = \frac{U_0}{c_0}$, we obtain

$$(1 - M^2)\phi_{xx} + \phi_{yy} = 0 \qquad (*)$$

- Character of $(*)$ depends on sign of ϕ_{xx} term, i.e. on whether $M < 1$ (subsonic flow) or $M > 1$ (supersonic flow).

- ② - ③ $\Rightarrow$ $p' = -\rho_0 U_0 \phi_x + \cancel{\text{constant}}^{\,0}$ (wlog), hence having found ϕ we can compute p' and hence the lift and drag on the wing.

Boundary conditions

- The normal velocity on the wing must be zero, i.e.

$$(U_0 + \phi_x, \phi_y) \cdot (-f_\pm', 1) = 0 \quad \text{on } y = f_\pm(x), \ |x| < a.$$

- Upon linearization for $|f_\pm'| \ll 1$ and $|f_\pm| \ll a$, these become

$$\phi_y = U_0 f_\pm' \quad \text{on } y = 0\pm, \ |x| < a.$$

- Elsewhere on $y = 0$, the velocity must be continuous, so that

$$[\phi_x]_-^+ = [\phi_y]_-^+ = 0 \text{ on } y = 0, \ |x| > a$$

- Form of far-field conditions depends on whether $M < 1$ (subsonic flow) or $M > 1$ (supersonic flow) — consider below.

Lift

- $[\phi_x]_-^+ = 0$ on $y = 0$, $|x| > a$ $\Rightarrow$ $[\phi]_-^+$ constant on $x \leq -a$ and $x \geq a$.

- Wlog can choose constant to be zero ahead of wing, i.e. $[\phi]_-^+ = 0$ on $y = 0$, $x \leq -a$.

- But in general constant is not zero and TBD for $x \geq a$: write $[\phi]_-^+ = -\Gamma$ on $y = 0$, $x \geq a$, where the circulation Γ is TBD.

- Since $p' = -\rho_0 U_0 \phi_x$, the lift on the wing is

$$L = \int_{-a}^{a} p'(x,0-) - p'(x,0+)\,dx = -\rho_0 U_0 \left[\phi(x,0-) - \phi(x,0+)\right]_{-a}^{a} = -\rho_0 U_0 \Gamma,$$

which is the Kutta–Joukowski lift theorem for barotropic flows.

Subsonic flow

- $M < 1 \Rightarrow$ (*) elliptic, so impose $\nabla\phi \to \underline{0}$ as $x^2+y^2 \to \infty$.

- Scale $y = Y/\beta$, $\phi = \Phi(x,Y)/\beta$, where $\beta = \sqrt{1-M^2}$, then $\Phi_{xx} + \Phi_{YY} = 0$, with

  $\Phi_Y = U_0 f_\pm'$ on $Y = 0\pm$, $|x| < a$; $[\Phi_x]_-^+ = [\Phi_Y]_-^+ = 0$ on $Y=0$, $|x|>a$; $\nabla\Phi \to \underline{0}$ as $x^2+Y^2 \to \infty$.

- This is identical to the problem of incompressible flow past ~~a~~ a similar wing!

- For symmetric wing ($f_+ = -f_-$), can solve using a Fourier transform (see online notes & problem sheet). Find $\Gamma = 0$, so no lift!

Supersonic flow

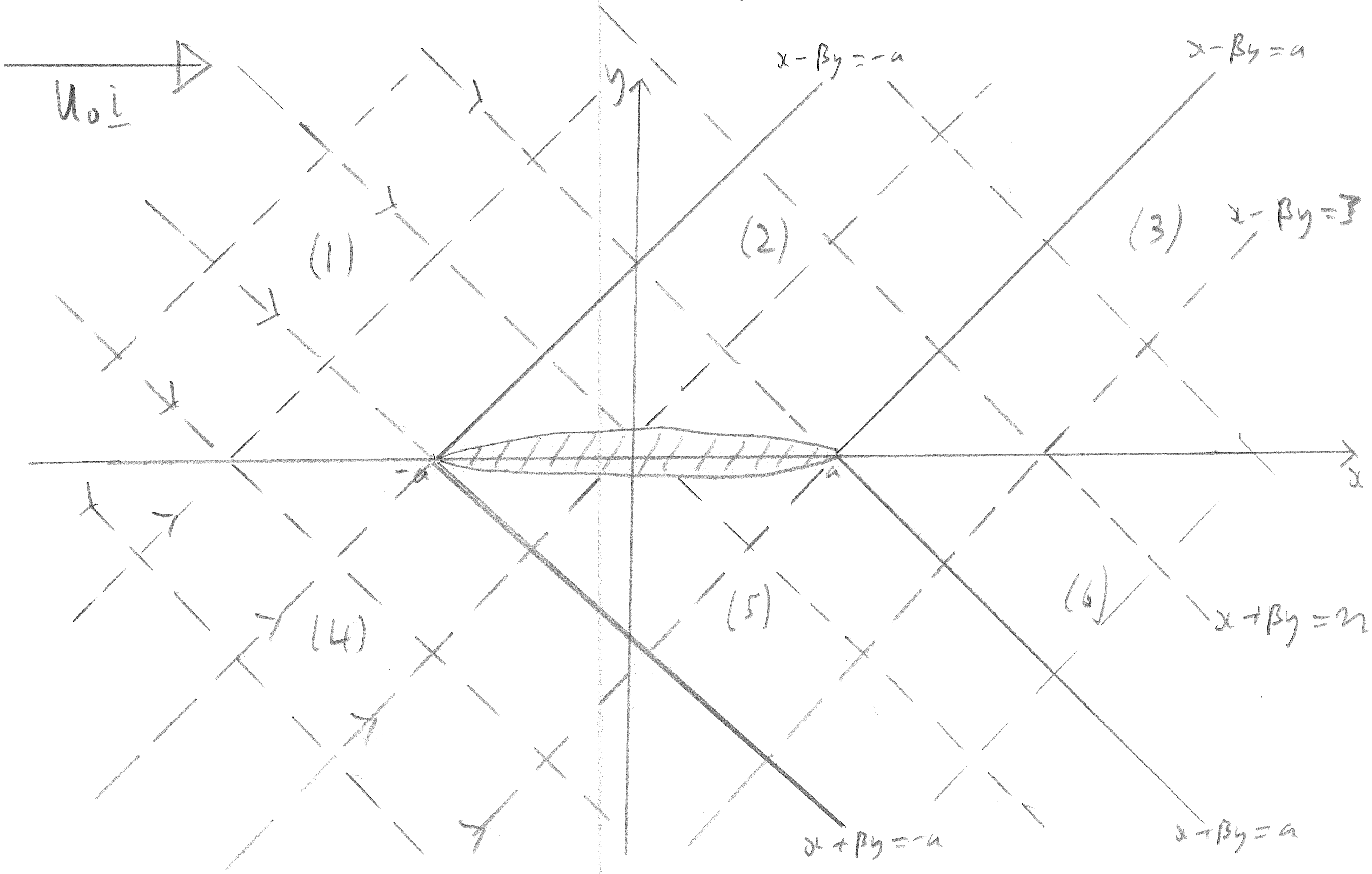
- $M > 1 \Rightarrow$ (*) hyperbolic, since $\phi_{yy} = \beta^2 \phi_{xx}$, where now $\beta = \sqrt{M^2 - 1}$.

- Recall writing $\xi = x - \beta y$, $\eta = x + \beta y$ reduces the equation to the canonical form $\phi_{\xi\eta} = 0$, from which we find the general solution $\phi = F(\xi) + G(\eta)$, i.e.

$$\phi(x, y) = F(x - \beta y) + G(x + \beta y) \qquad (+)$$

- $F(x - \beta y)$ and $G(x + \beta y)$ are constant on the characteristics $x - \beta y = \text{constant}$ and $x + \beta y = \text{constant}$. Causality – that causes happen before effects – means that information travels in the direction of the flow past the wing, i.e. from left to right along the characteristics.

Schematic of the characteristics for supersonic flow



Dividing characteristics $x - \beta y = \pm a \, (y > 0)$ and $x + \beta y = \pm a \, (y < 0)$ called <u>Mach lines</u>

- We suppose that the flow upstream of the wing is undisturbed, i.e.

$$\phi \to 0 \quad \text{as} \quad x \to -\infty. \qquad (\#)$$

- In $y > 0$ all characteristics $x + \beta y = $ constant originate from $x = -\infty$, so fixing $\eta = x + \beta y$ and letting $\xi = x - \beta y \to -\infty$ in $(+)$, it follows from $(\#)$ that $F(-\infty) + G(\eta) = 0 \Rightarrow G = $ constant $\Rightarrow \phi = F(x - \beta y)$ wlog.

- In $y < 0$ all characteristics $x - \beta y = $ constant originate from $x = -\infty$, so fixing $\xi = x - \beta y$ and letting $\eta = x + \beta y \to -\infty$ in $(+)$, it follows from $(\#)$ that $F(\xi) + G(-\infty) = 0 \Rightarrow F = $ constant $\Rightarrow \phi = G(x + \beta y)$ wlog.

- Determine F and G by applying conditions on $y = 0$ and cty of ϕ for $y \neq 0$.

- $[\phi]_-^+ = [\phi_y]_-^+ = 0$ on $y=0$, $x<-a \Rightarrow F(x) = G(x)$, $-\beta F'(x) = \beta G'(x)$ for $x<-a$

  $\Rightarrow F(x) = G(x) = \text{constant} = 0$ by (#) for $x<-a$

  $\Rightarrow \phi = 0$ in regions (1) and (4).

- $\phi_y = U_0 f_+'(x)$ on $y=0+$, $|x|<a \Rightarrow -\beta F'(x) = U_0 f'(x)$ for $|x|<a$

  $\Rightarrow F(x) = -\dfrac{U_0}{\beta}\left(f_+(x) - f_+(-a)\right)$ for $|x|<a$, so ϕ cts on $x-\beta y = -a$ for $y>0$

  $\Rightarrow \phi = -\dfrac{U_0}{\beta}\left(f_+(x-\beta y) - f_+(-a)\right)$ in region (2).

- By a similar argument, we find $G(x) = \dfrac{U_0}{\beta}\left(f_-(x) - f_-(-a)\right)$ for $|x|<a$,

  so that $\phi = \dfrac{U_0}{\beta}\left(f_-(x+\beta y) - f_-(-a)\right)$ in region (5).

- $[\phi_x]_-^+ = [\phi_y]_-^+ = 0$ on $y=0$ for $x>a \Rightarrow F'(x) = G'(x), \; -\beta F'(x) = \beta G'(x)$ for $x>a$

$\Rightarrow \; F(x) = c_1 \in \mathbb{R}, \; G(x) = c_2 \in \mathbb{R}$ for $x > a$

$\Rightarrow \; \phi = c_1$ in region (3) and $\phi = c_2$ in region (6).

- ϕ cts on $x - \beta y = a$ for $y > 0 \Rightarrow \phi = c_1 = -\frac{U_0}{\beta}\left(f_+(a) - f_+(-a)\right)$ in region (3)

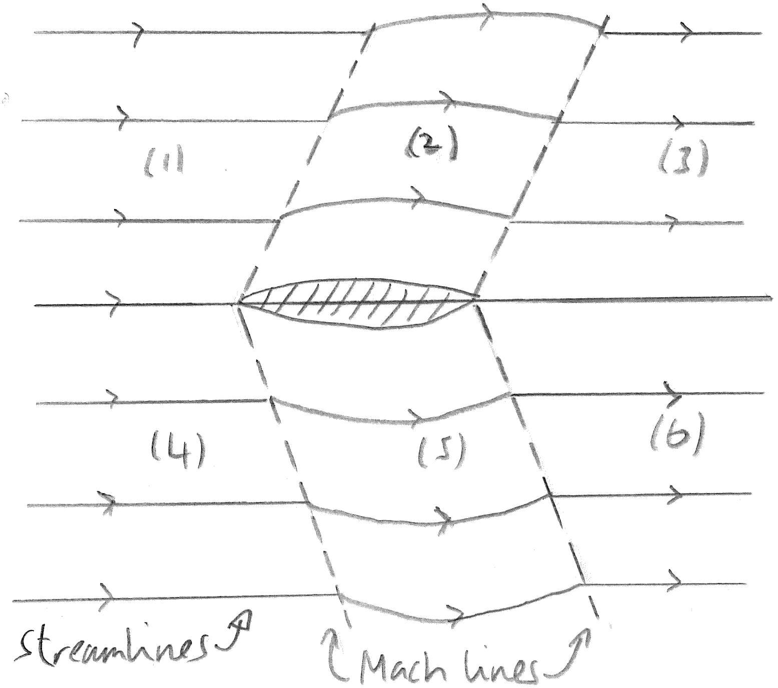
- ϕ cts on $x + \beta y = a$ for $y < 0 \Rightarrow \phi = c_2 = \frac{U_0}{\beta}\left(f_-(a) - f_-(-a)\right)$ in region (6).

- If the wing is smooth at each end, we can set $f_+(-a) = f_-(-a) = 0$ (wlog) and $f_+(a) = f_-(a) = \lambda \in \mathbb{R}$ (say).

- Then $\phi = -\frac{\lambda U_0}{\beta}$ in region (3) and $\phi = \frac{\lambda U_0}{\beta}$ in region (6), so the circulation

$$\Gamma = -\left[\phi(a+, y)\right]_{0-}^{0+} = \frac{2\lambda U_0}{\beta} \Rightarrow \text{lift } L = -\rho_0 U_0 \Gamma = -\frac{2\rho_0 \lambda U_0^2}{\sqrt{M^2-1}}.$$

- As expected $L > 0$ for $\lambda < 0$ and linear theory breaks down with $L \to \infty$ as $M \to 1+$.



- "Zones of silence" with $\nabla\phi = \underline{0}$ in regions (1), (3), (4) and (6).

- Each streamline has same shape as wing between the Mach lines in regions (2) & (5).