Waves and Compressible Flow

Lecture 11

4. Nonlinear waves

- One-dimensional gas dynamics

- Shallow water theory

- Use method of characteristics to construct solutions.

One-dimensional gas dynamics

- If body forces are negligible, the one-dimensional Euler equations are
$$\rho_t + u\rho_x + \rho u_x = 0, \quad u_t + uu_x = -\frac{1}{\rho}p_x.$$

- Assuming homentropic flow, $p = k\rho^\gamma$ for constants $k>0$ and $\gamma>1$.

- Define speed of sound c via $c^2 = \frac{dp}{d\rho} = \gamma k \rho^{\gamma-1} = \frac{\gamma p}{\rho}$, then
$$\rho = \left(\frac{c^2}{\gamma k}\right)^{\frac{1}{\gamma-1}} = A c^{\frac{2}{\gamma-1}} \text{ and } p = \frac{\rho c^2}{\gamma} = \frac{A}{\gamma} c^{\frac{2\gamma}{\gamma-1}},$$
where $A = \left(\frac{1}{\gamma k}\right)^{\frac{1}{\gamma-1}}$, so can eliminate ρ and p in favour of c.

- If $\rho = A c^{\frac{2}{\gamma-1}}$, $p = \frac{A}{\gamma} c^{\frac{2\gamma}{\gamma-1}}$, then

$$\rho_t = \frac{2A}{\gamma-1} c^{\frac{2}{\gamma-1}-1} c_t, \quad \rho_x = \frac{2A}{\gamma-1} c^{\frac{2}{\gamma-1}-1} c_x, \quad p_x = \frac{2A}{\gamma-1} c^{\frac{2\gamma}{\gamma-1}-1} c_x = \frac{2\rho}{\gamma-1} c c_x$$

- $\rho_t + u\rho_x + \rho u_x = 0 \Rightarrow \frac{2}{\gamma-1}(c_t + u c_x) + c u_x = 0$ ①

$u_t + u u_x + \frac{1}{\rho} p_x = 0 \Rightarrow u_t + u u_x + \frac{2}{\gamma-1} c c_x = 0$ ②

- ① ± ② $\Rightarrow \frac{\partial}{\partial t}\left(u \pm \frac{2c}{\gamma-1}\right) + (u \pm c)\frac{\partial}{\partial x}\left(u \pm \frac{2c}{\gamma-1}\right) = 0$

$\Rightarrow \left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)\left(u \pm \frac{2c}{\gamma-1}\right) = 0$

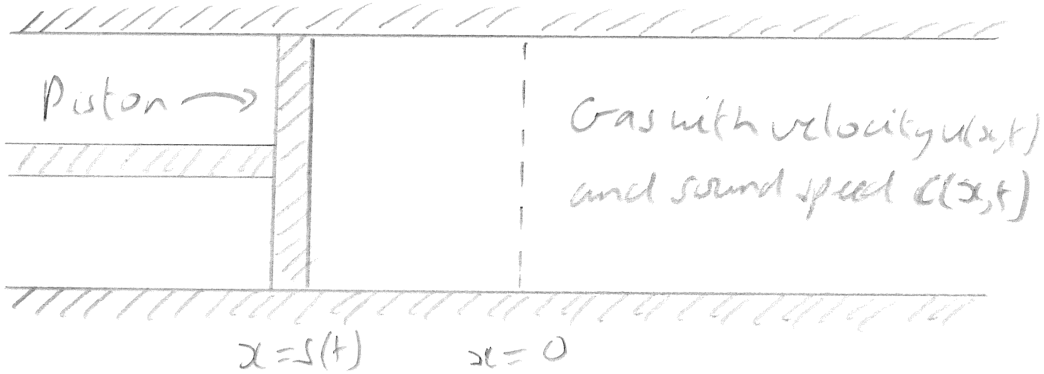
$\Rightarrow \frac{d}{dt}\left(u \pm \frac{2c}{\gamma-1}\right) = 0$ on curves satisfying $\frac{dx}{dt} = u \pm c$

$\Rightarrow \underbrace{R_\pm = u \pm \frac{2c}{\gamma-1}}_{\text{Riemann invariants}}$ are constant on curves on which $\underbrace{\frac{dx}{dt} = u \pm c}_{\pm \text{ characteristics}}$

Example: flow due to a piston

- Suppose the gas is initially at rest in $x > 0$ with $\rho = \rho_0$, $p = p_0$, $c = c_0 = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}$, confined in a tube by a piston initially at $x = 0$.

- The piston is then withdrawn so that its position at time t is $x = s(t)$, where $s(0) = 0$ and $\dot{s}(t) < 0$ for $t > 0$.



- ICs : $u = 0$, $c = c_0$ for $x > 0$ at $t = 0$.

- Assuming the piston does not leave the gas behind (i.e. no vacuum forms), the KBC on the piston says $u = \dot{s}(t)$ on $x = s(t)$ for $t > 0$.

- <u>Characteristic diagram</u>:

– – – – –	+ char$^{\underline{cs}}$
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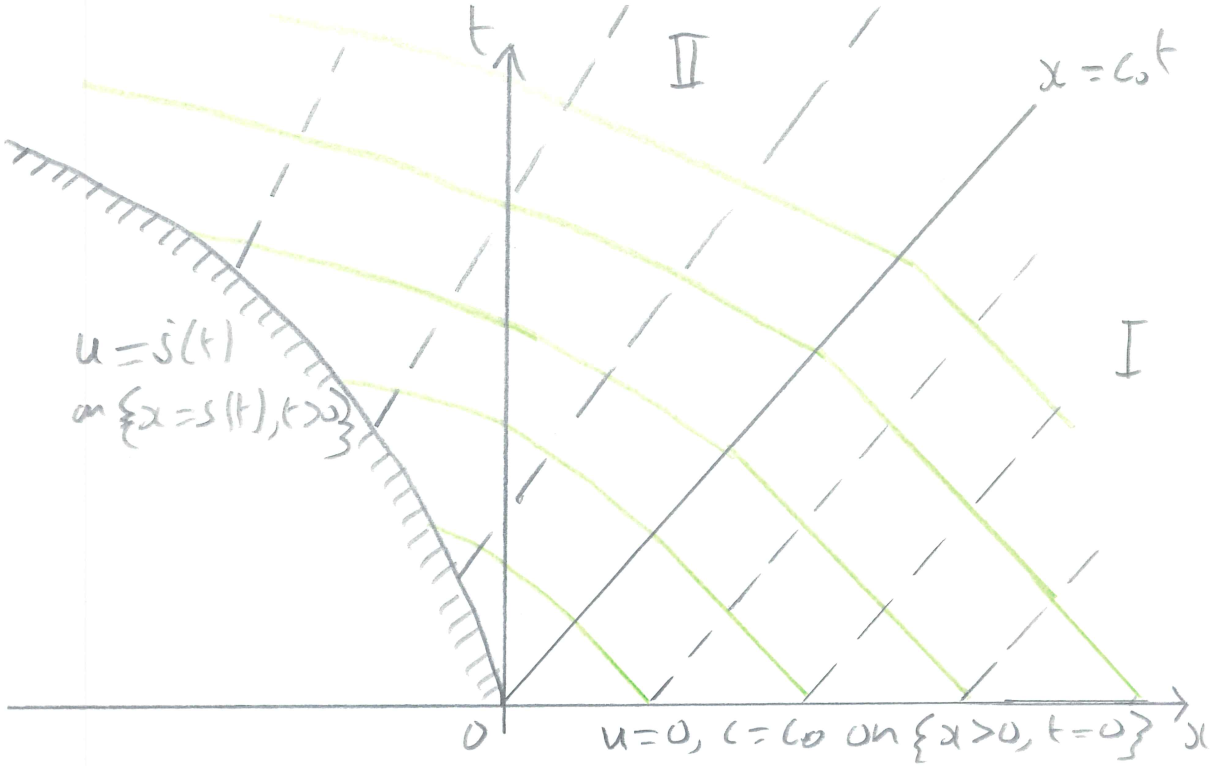


Region I

- Consider $\pm$ characteristics originating from $\{x>0, t=0\}$.

- Where these characteristics intersect,

$$u \pm \frac{2c}{\gamma-1} = 0 \pm \frac{2c_0}{\gamma-1} \Rightarrow u=0, \quad c=c_0$$

- Hence, such $\pm$ characteristics have $\frac{dx}{dt} = 0 \pm c_0$ and are therefore straight lines.

- They therefore map out $x > c_0 t, t > 0$, i.e. $u=0, c=c_0$ in region I.

Region II

- On a + characteristic originating from $(x,t) = (s(\tau), \tau)$ on the piston,

$$u + \frac{2c}{\gamma - 1} = R_+(\tau) = \text{constant}. \qquad (+)$$

- Where this + characteristic intersects the family of − characteristics from $\{x > 0, t = 0\}$, we also have

$$u - \frac{2c}{\gamma - 1} = 0 - \frac{2c_0}{\gamma - 1}. \qquad (\#)$$

- By (+) and (#), u and c are constant on this + characteristic, so by the KBC on the piston, $u = \dot{s}(\tau)$, and so by (#), $c = c_0 + \frac{\gamma - 1}{2}\dot{s}(\tau)$

- Hence, the + characteristic originating from $(s(\tau), \tau)$ on the piston is straight with $\frac{dx}{dt} = u + c = c_0 + \frac{\gamma+1}{2}\dot{s}(\tau)$

- Since it passes through $x = s(\tau)$ at time $t = \tau$, it has equation

$$x - s(\tau) = \left(c_0 + \frac{\gamma+1}{2}\dot{s}(\tau)\right)(t - \tau).$$

- This gives the parametric solution $u = \dot{s}(\tau)$, $c = c_0 + \frac{\gamma-1}{2}\dot{s}(\tau)$ and $x - s(\tau) = \left(c_0 + \frac{\gamma+1}{2}\dot{s}(\tau)\right)(t-\tau)$ in $s(t) < x < c_0 t$, $t > 0$, i.e. in region II, provided $c \geq 0$ (so that $p, \rho \geq 0$) and the + characteristics fill this region, as illustrated in characteristic diagram above.

- But $c \geq 0 \iff c_0 + \frac{\gamma-1}{2}\dot{s}(\tau) \geq 0 \iff -\dot{s}(\tau) \leq \frac{2c_0}{\gamma-1}$

- If the withdrawal speed is ever faster than $\frac{2c_0}{\gamma-1}$, then the piston leaves the gas behind.

- A vacuum then forms near the piston in which $c = \rho = 0$, and it is as if the gas expands freely into a vaccum.

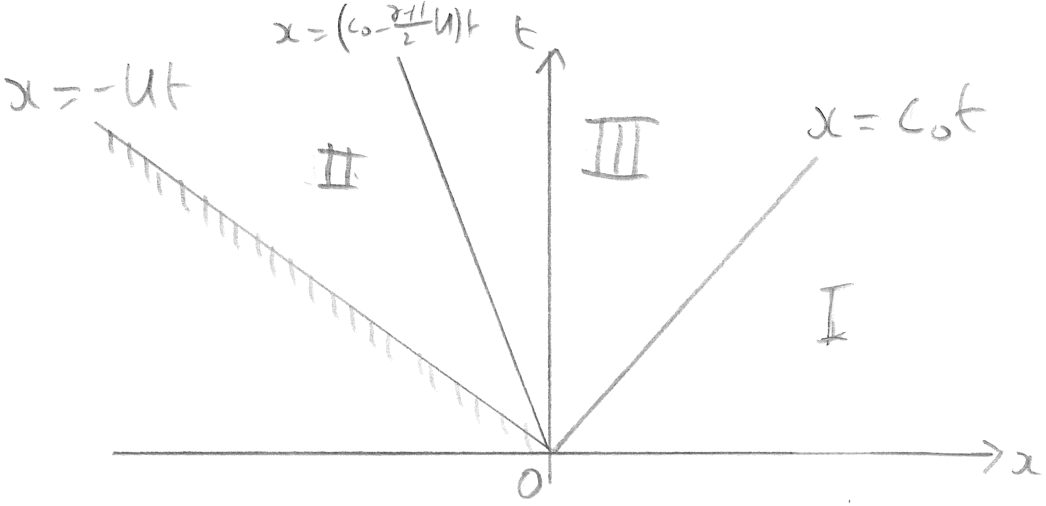
- For example, if $s(t) = -\alpha t^2/2$, where α is the constant acceleration, then the piston leaves the gas behind at time $t = \frac{2c_0}{(\gamma-1)\alpha}$.

Example with expansion fan

- If $s(t)$ particularly simple, can eliminate τ in region II, e.g. if $s(\tau) = -U\tau$ ($U \in \mathbb{R}^+$), then $u = -U$, $c = c_0 - \frac{\gamma-1}{2}U$ where

  $$x = -U\tau + \left(c_0 - \frac{\gamma+1}{2}U\right)(t-\tau) \text{ for } \tau \geq 0.$$

- First such $+$ characteristic has $\tau = 0 \Rightarrow x = \left(c_0 - \frac{\gamma+1}{2}U\right)t < c_0 t$!



What happens in the gap, region III?

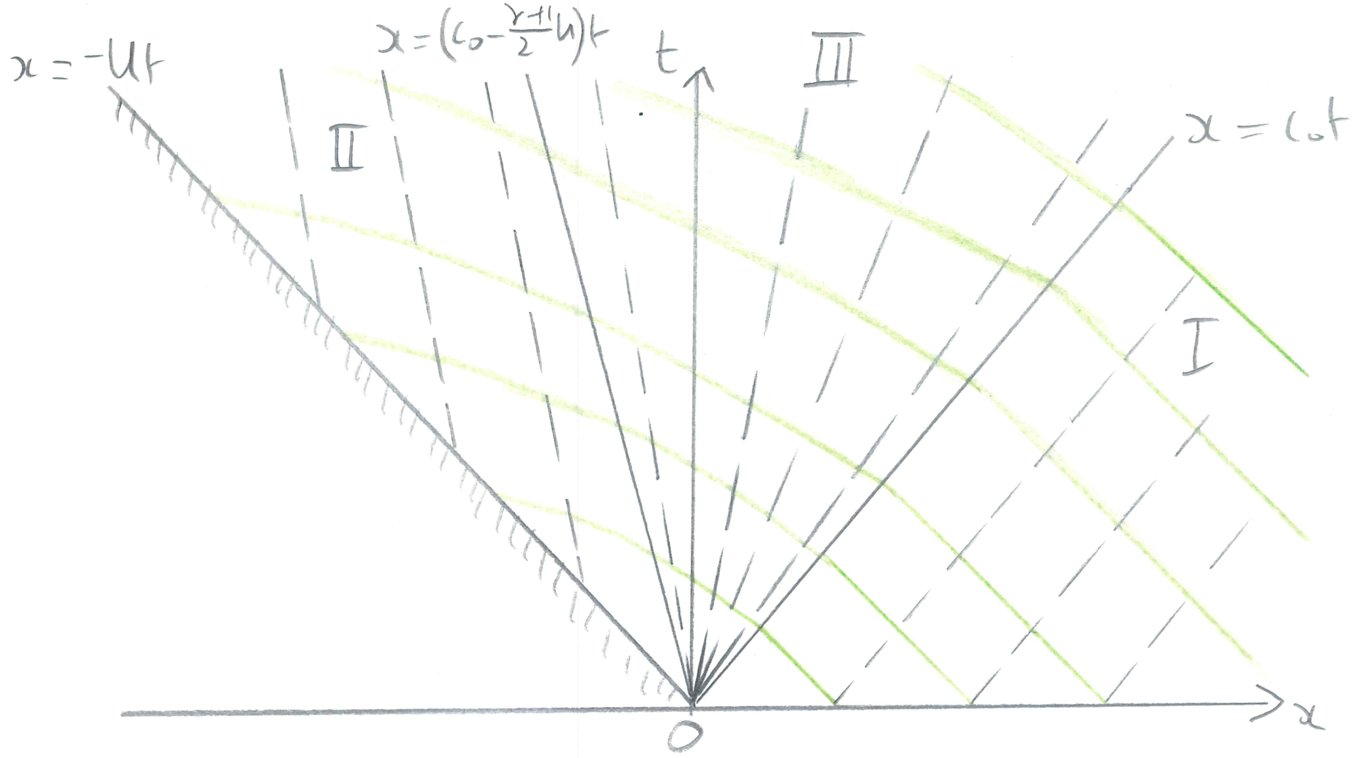
- In <u>region III</u>, still have − characteristics from $\{x>0, t=0\}$, so $u - \frac{2c}{\gamma-1} = \frac{-2c_0}{\gamma-1}$ here.

- On a + characteristic $u + \frac{2c}{\gamma-1} =$ constant, so u, c constant, so it is straight.

- To avoid it crossing + characteristics in region I and II (which would lead to a contradiction), it must pass through the origin and therefore have speed $\frac{dx}{dt} = u + c = x/t$.

- Hence, $u = \frac{2}{\gamma+1}\left(\frac{x}{t} - c_0\right)$, $c = \frac{2c_0 + (\gamma-1)x/t}{\gamma+1}$ in region III, which occupies $\left(c_0 - \frac{\gamma+1}{2}u\right)t < x < -c_0 t$.

- This structure, with the + characteristics radiating out from the origin is known as an <u>expansion fan</u>, as illustrated assuming $U < \frac{2c_0}{\gamma-1}$.



- If $U > \frac{2c_0}{\gamma-1}$, then there is no region II and the expansion fan stops where $c = 0$, i.e. at $x = \frac{-2c_0 t}{\gamma-1}$ — see online notes.