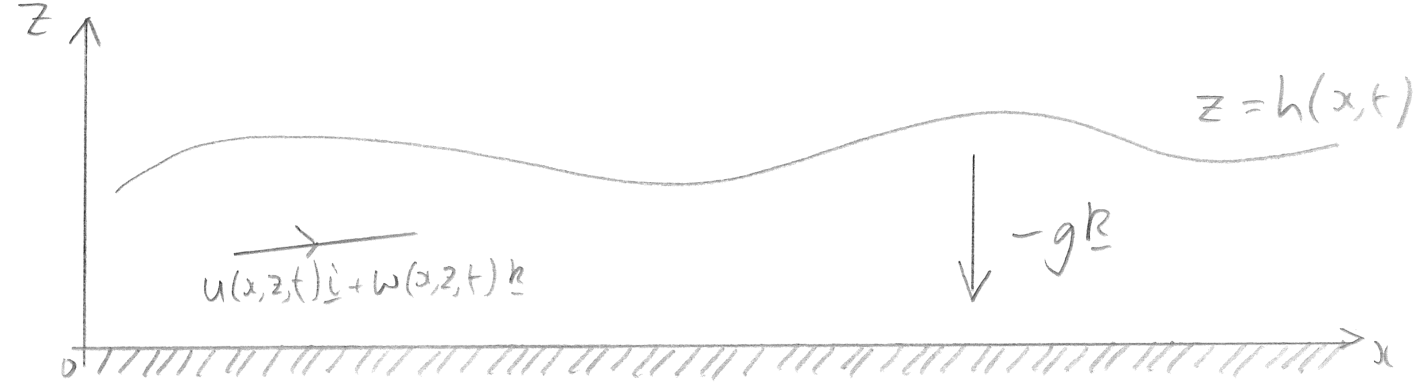
Waves and Compressible Flow

Lecture 12

Shallow water theory

- This is an approximate model for the gravity-driven irrotational flow of a thin layer of inviscid fluid of constant density ρ.

- Setup:



- Governing equations:

$$u_x + w_z = 0 \quad (1)$$

$$u_z - w_x = 0 \quad (2)$$

$$u_t + uu_x + wu_z = -\frac{1}{\rho} p_x \quad (3)$$

$$w_t + uw_x + ww_z = -\frac{1}{\rho} p_z - g \quad (4)$$

- Boundary conditions:

$$w = 0 \quad \text{on } z = 0 \qquad (5)$$

$$w = h_t + u h_x \quad \text{on } z = h \qquad (6)$$

$$p = p_{atm} \quad \text{on } z = h \qquad (7)$$

- Integrating (1) from $z=0$ to $z=h$ gives

$$0 = \int_0^h u_x + w_z \, dz = \frac{\partial}{\partial x}\int_0^h u \, dz - u h_x\big|_{z=h} + [w]_{z=0}^{z=h}$$

by Leibniz's integral rule.

- Hence, defining the average horizontal velocity $\bar{u}(x,t) = \frac{1}{h}\int_0^h u \, dz$, we deduce from (5) and (6) that

$$h_t + (h\bar{u})_x = 0 \qquad (8)$$

- In shallow water theory, we assume that the flow is almost unidirectional, so that $\frac{|z|}{|x|} \sim \frac{|w|}{|u|} \ll 1$.

- This means that we may approximate (2) by $u_z = 0$ and (4) by $0 = -\frac{1}{\rho}p_z - g$.

- $u_z = 0 \Rightarrow u = u(x,t) \Rightarrow h_t + (hu)_x = 0.$

- $p_z = -\rho g$ and (7) $\Rightarrow p = p_{atm} + \rho g(h-z)$, i.e. the pressure is hydrostatic.

- Hence, (3) becomes $u_t + uu_x + w\cancel{u_z}^{\,0} = -\frac{1}{\rho}p_x = -g h_x.$

- We have derived the shallow-water equations

$$h_t + uh_x + hu_x = 0, \quad u_t + uu_x + gh_x = 0,$$

two coupled equations for $h(x,t)$ and $u(x,t)$.

- We have not assumed that the waves have small amplitude (cf. Stokes waves) – can therefore describe e.g. tidal bores.

- This is why the system is nonlinear!

- We define the wave speed $c = \sqrt{gh}$, which is the phase speed of Stokes waves on a shallow fluid layer of constant thickness h.

- $h = \frac{c^2}{g} \Rightarrow \quad 2(c_t + uc_x) + cu_x = 0, \; u_t + uu_x + 2cc_x = 0.$

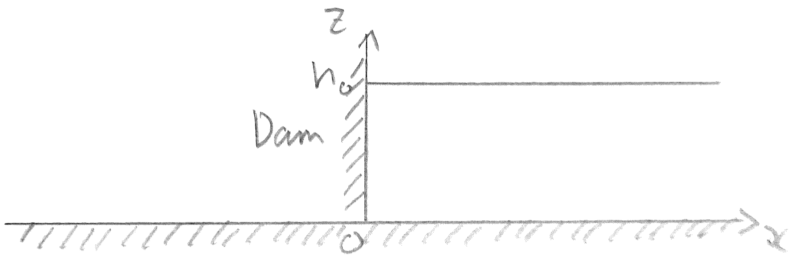
- These are identical to the equations of gas dynamics in last lecture with $\gamma = 2$, so the methods of gas dynamics can be applied directly here.

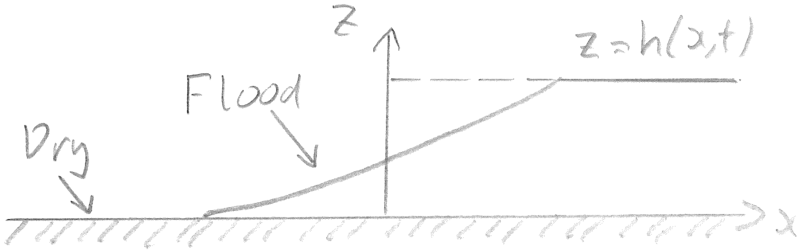
- In particular, adding/subtracting $\Rightarrow \left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm 2c) = 0$,

  i.e. the Riemann invariants $u \pm 2c$ are constant along $\pm$ characteristics on which $\frac{dx}{dt} = u \pm c$.

Example: dam break

- Stationary water with depth h_0 in $x > 0$ is held by a dam at $x = 0$.
  At $t = 0$, the dam is removed – what happens to the water?

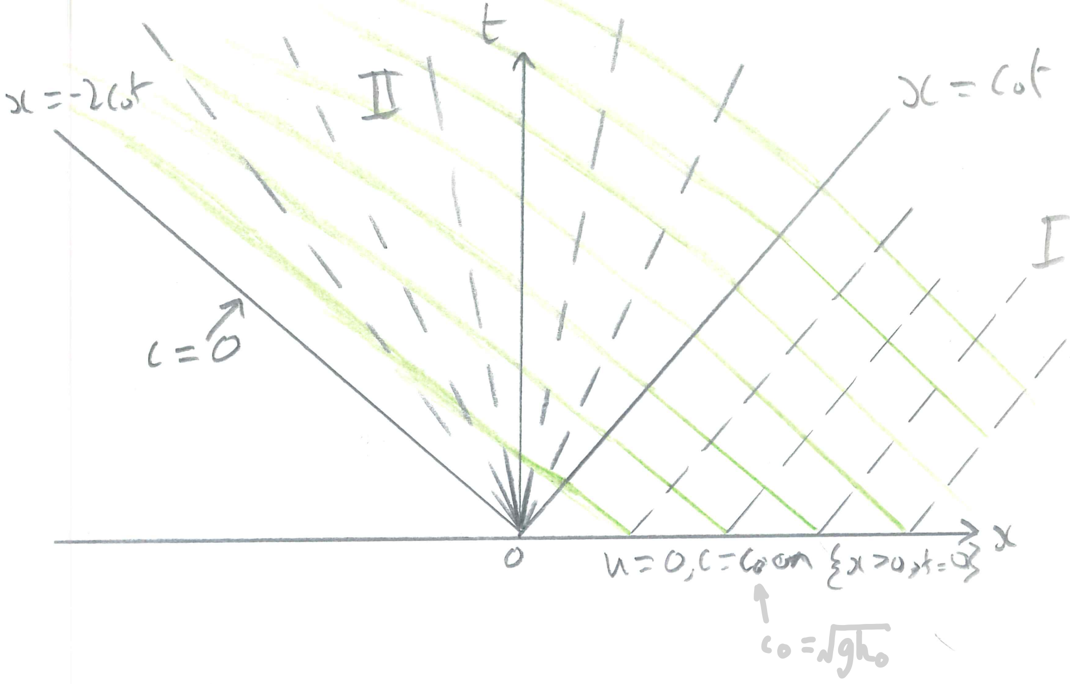




- Equivalent to piston pulling problem in gas dynamics with instantaneous piston withdrawal – hence can set $\gamma = 2$ in solution from last lecture!

- We take opportunity to practice the derivation.

- <u>Characteristic diagram:</u>



<u>Region I:</u>

- Where ± characteristics from $\{x > 0, t = 0\}$ intersect, $u \pm 2c = 0 \pm 2c_0$

$\Rightarrow$ $u = 0$, $c = c_0$, so ± characteristics are straight lines with $\frac{dx}{dt} = \pm c_0$,

and therefore map out $x > c_0 t$, $t > 0$.

Region II

- Still have − characteristics from $\{x>0, t=0\}$, so $u-2c=-2c_0$ here

- On a + characteristic $u+2c=$ constant, so u, c constant and it is straight

- Since fluid occupies $x>0$ at $t=0$, the + characteristics must all start at $x=0, t=0$ to avoid them crossing those that originate from $\{x>0, t=0\}$

- Hence, we have an expansion fan, with the + characteristics having $\frac{dx}{dt}=u+c=\frac{x}{t}$.

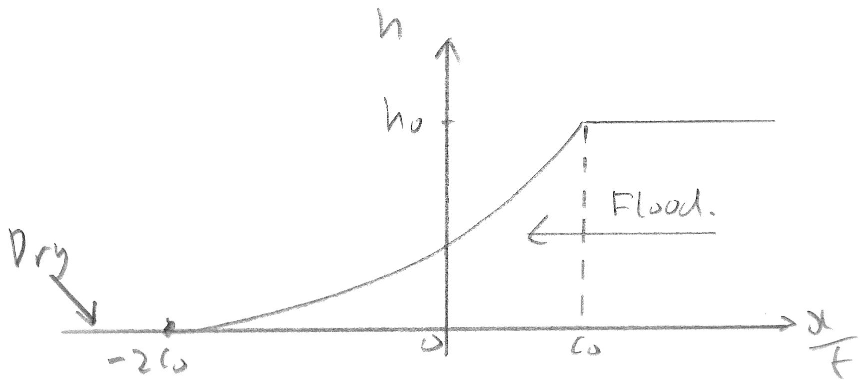
- NB: Can also argue from scale invariance under transformation $x \mapsto \lambda x, t \mapsto \lambda t \ (\lambda>0)$

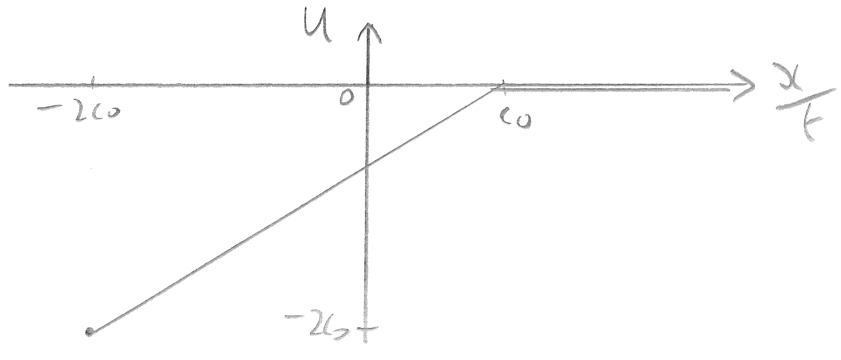
- Solving $u - 2c = -2c_0$ and $u + c = \frac{x}{t}$ gives

$$u = \frac{2}{3}\left(\frac{x}{t} - c_0\right), \quad c = \frac{1}{3}\left(\frac{x}{t} + 2c_0\right) \Rightarrow h = \frac{c^2}{g} = \frac{1}{9g}\left(\frac{x}{t} + 2c_0\right)^2$$

- Expansion fan terminates when $c = \sqrt{gh} = 0$, i.e. at $x = -2c_0 t$, so region II is $-2c_0 t < x < c_0 t$

- Sketches similarity solution:





Multi-valued solutions

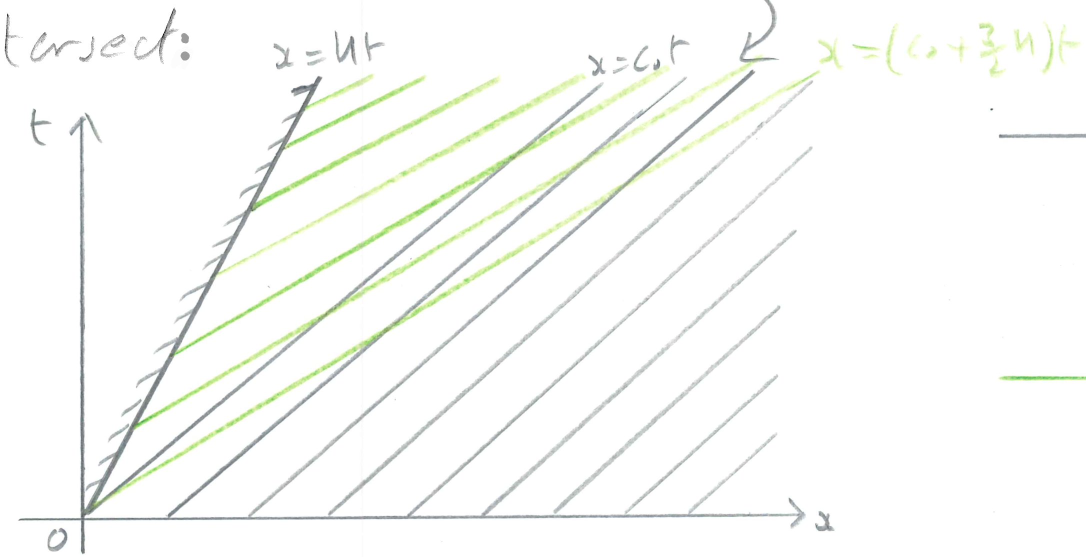
- Suppose instead the dam is pushed into the fluid with constant speed $U < c_0$.

- Expect to still have $u = 0$, $c = c_0$ in $x > c_0 t$.

- In $x < c_0 t$ still have $u - 2c = -2c_0$ from $-$ characteristics.

- On a $+$ characteristic originating from the dam, $u + 2c = $ constant, so u & c are constant and therefore given by $u = U$, $c = c_0 + \frac{U}{2}$; hence all such $+$ characteristics have slope $\frac{dx}{dt} = u + c = c_0 + \frac{3U}{2}$, and therefore map out the region $Ut < x < (c_0 + \frac{3U}{2})t$.

- Hence, we have a multi-valued solution with $u=0$, $c=c_0$ and $u=U$, $c=c_0+\frac{3}{2}U$ in the region $c_0 t < x < (c_0+\frac{3}{2}U)t$ where the + characteristics intersect:



- This is unphysical – instead we must introduce a shock.

- Here solution broke down immediately at $t=0+$. See online notes for an example of breakdown after a finite time.

- More generally, in a <u>simple flow</u> with $u - 2c = -2c_0$ everywhere,

$$c_t + u c_x + \frac{1}{2} c u_x = 0 \quad \Rightarrow \quad c_t + (3c - 2c_0) c_x = 0.$$

- Hence, if $c(x,0) = f(x)$, then $\frac{dc}{dt} = 0$ on characteristics with $\frac{dx}{dt} = 3c - 2c_0$ with $x = \xi$, $c = f(\xi)$ at $t = 0$.

- Thus, $c = f(\xi)$ on $x = \xi + (3f(\xi) - 2c_0)t$.

- $\frac{\partial}{\partial x} \Rightarrow c_x = f'(\xi)\xi_x$ and $1 = \xi_x + 3t f'(\xi)\xi_x \Rightarrow c_x = \dfrac{f'(\xi)}{1 + 3t f'(\xi)}$.

- If $f'(\xi) < 0$ for some ξ initially, then $|c_x| \to \infty$ as $t \to t_c^-$, where $t_c = \min\limits_{\xi : f'(\xi) < 0} \left(-\frac{1}{3f'(\xi)} \right)$.

- The solution breaks down at $t = t_c$ because regions with larger thickness travel more quickly than those with smaller thickness (recall $h = c^2/g$):

- To avoid a multi-valued solution for $t > t_c$, we must introduce a shock.