Waves and Compressible Flow

Lecture 13

5. Shock waves

- Shocks in one-dimensional gas dynamics

- Shocks in one-dimensional shallow water theory

- Weak solutions – a unifying framework

- Two-dimensional steady shocks in gas dynamics

Shocks in one-dimensional gas dynamics

- In conservative form the governing equations for mass, momentum and energy conservation are

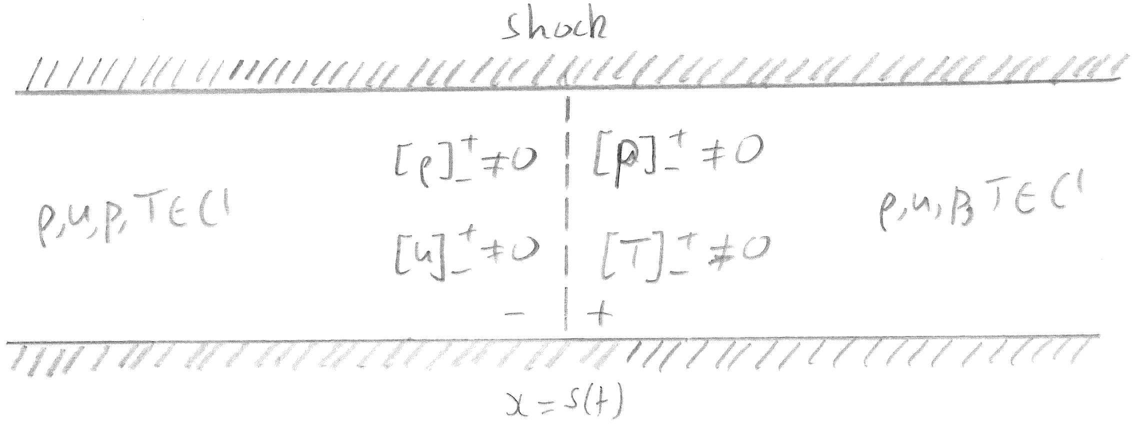
$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2 + p)_x = 0, \quad (\rho e)_t + (\rho e u + p u)_x = 0,$$

where $p = \rho R T$ and $e = \frac{1}{2}u^2 + c_v T$ is the energy per unit mass.

- Of form $f_t + (fu + q)_x = 0$ with $f = \rho, \rho u, \rho e$ and $q = 0, p, pu$ respectively, with corresponding integral conservation law

$$\frac{d}{dt}\int_a^b f \, dx + [fu + q]_a^b = 0 \qquad (a, b \in \mathbb{R}).$$

- Suppose there is a shock at $x = s(t)$ across which ρ, u, p, T are discontinuous.



- If $a < s(t) < b$, then by Leibniz's integral rule

$$\int_a^s f_t \, dx + f|_{s-} \dot{s} + \int_s^b f_t \, dx - f|_{s+} \dot{s} + [fu + q]_a^b = 0.$$

- Assuming f_t is piecewise continuous, in the limit $a \to s-$, $b \to s+$ we obtain the jump condition

$$[f(u - \dot{s}) + q]_-^+ = 0.$$

- Applying to the equations of 1D gas dynamics gives

$$[\rho(u-\dot{s})]_-^+ = 0 \quad (1), \quad [\rho u(u-\dot{s}) + p]_-^+ = 0 \quad (2), \quad [\rho e(u-\dot{s}) + pu]_-^+ = 0 \quad (3)$$

- Let $\hat{u} = u - \dot{s}$ and $\hat{e} = \frac{1}{2}\hat{u}^2 + c_v T$ be velocity and energy density relative to shock.

- (1) $\Leftrightarrow$ $[\rho\hat{u}]_-^+ = 0 \Leftrightarrow \rho_-\hat{u}_- = \rho_+\hat{u}_+$, i.e. rate of mass flow into shock equals rate of mass flow out — mass conservation.

- (2) $- \dot{s}$(1) $\Leftrightarrow [\rho\hat{u}^2 + p]_-^+ = 0 \Leftrightarrow \rho_+\hat{u}_+\hat{u}_+ - \rho_-\hat{u}_-\hat{u}_- = p_- - p_+$, i.e. rate of change of momentum across shock equals net force acting across shock — momentum conservation.

- ③ $- \dot{s}$ ② $+ \dot{s}^2$ ① $\Leftrightarrow [\rho\hat{e}\hat{u} + p\hat{u}]_-^+ = 0 \Leftrightarrow \rho_+\hat{e}_+\hat{u}_+ - \rho_-\hat{e}_-\hat{u}_- = p_-u_- - p_+u_+$,

  i.e. rate of change of energy across shock equals rate of working of pressure on either side – energy conservation.

- Hence, can also derive jump conditions from first principles, as in online notes.

- Since $p = \rho R T = (\gamma - 1)\rho c_v T$, where $R = c_p - c_v$ and $\gamma = \frac{c_p}{c_v}$, then

$$[\rho\hat{e}\hat{u} + p\hat{u}]_-^+ = \left[\rho\hat{u}\left(\frac{1}{2}\hat{u}^2 + \frac{p}{(\gamma-1)\rho} + \frac{p}{\rho}\right)\right]_-^+ = \rho_-\hat{u}_-\left[\frac{1}{2}\hat{u}^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+$$

- If $u_- = \dot{s}$ or $\hat{u}_- = 0$, there is no flow through shock; otherwise $\left[\frac{1}{2}\hat{u}^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0$.

- We have derived the <u>Rankine-Hugoniot conditions</u>

$$\left[\rho(u-\dot{s})\right]_-^+ = 0 \qquad (1')$$

$$\left[\rho(u-\dot{s})^2 + p\right]_-^+ = 0 \qquad (2')$$

$$\left[\frac{1}{2}(u-\dot{s})^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0 \qquad (3')$$

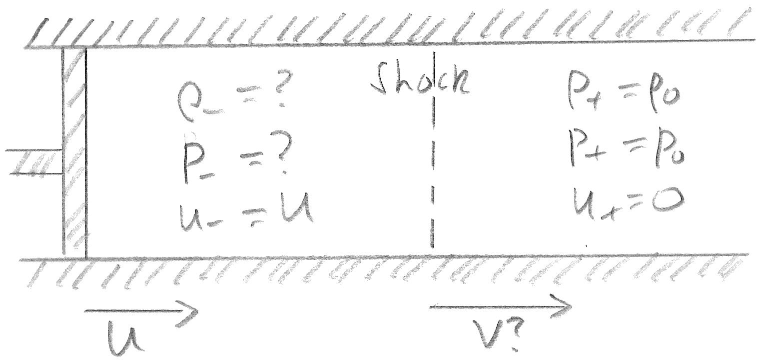
- These are 3 equations for 7 unknowns ($\rho_\pm, u_\pm, p_\pm, \dot{s}$), so in general we must specify 4 of them somewhere else.

- Can work in laboratory or shock frame because PDEs and (1')–(3') are invariant under the transformation $x = \alpha(t) + \hat{x}$, $s(t) = \alpha(t) + \hat{s}(t)$, $u = \dot{\alpha}(t) + \hat{u}$.

Example: piston entry

- Piston pushed into tube at constant speed U in which gas is initially at rest with $\rho = \rho_0$, $p = p_0$, causing a shock to move with speed V.

Gas next to piston moves with speed U



3 remaining unknowns: ρ_-, p_- and V

- (1′)–(3′) $\Rightarrow$ $\rho_0(0-V) = \rho_-(U-V)$, $p_0 + \rho_0(0-V)^2 = p_- + \rho_-(U-V)^2$, $\frac{1}{2}(0-V)^2 + \frac{\gamma p_0}{(\gamma-1)\rho_0} = \frac{1}{2}(U-V)^2 + \frac{\gamma p_-}{(\gamma-1)\rho_-}$.

- Eliminate ρ_-, p_- $\Rightarrow$ $V^2 - \frac{1}{2}(\gamma+1)UV - c_0^2 = 0$, where $c_0^2 = \frac{\gamma p_0}{\rho_0}$.

- $V > U > 0 \Rightarrow V = c_0\left(a + \sqrt{a^2+1}\right)$, where $a = \frac{\gamma+1}{4}\frac{U}{c_0}$, and hence ρ_- and p_-.

Shock relations and entropy

- We use the second law of thermodynamics – that entropy never decreases – to make some general statements about how ρ, p, u change across a shock.

- Consider a stationary shock: after replace u by $u-\dot{s}$ to apply to a moving shock.

- Rewrite (1′) – (3′) using sound speed $c = \left(\frac{\gamma p}{\rho}\right)^{1/2}$ and Mach number $M = \frac{u}{c}$.

- (1′) $\Rightarrow [\rho u]_-^+ = 0 \Rightarrow [\rho c M]_-^+ = 0 \Rightarrow \rho_+ \left(\frac{\gamma p_+}{\rho_+}\right)^{1/2} M_+ = \rho_- \left(\frac{\gamma p_-}{\rho_-}\right)^{\frac{1}{2}} M_- \Rightarrow \frac{p_+ \rho_+}{p_- \rho_-} = \frac{M_-^2}{M_+^2}$ (1″)

- (2′) $\Rightarrow [\rho u^2 + p]_-^+ = 0 \Rightarrow [\rho c^2 M^2 + p]_-^+ = 0 \Rightarrow [\gamma p M^2 + p]_-^+ = 0 \Rightarrow \frac{p_+}{p_-} = \frac{1+\gamma M_-^2}{1+\gamma M_+^2}$ (2″)

• (3') $\Rightarrow \left[\frac{1}{2}u^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0 \Rightarrow \left[\frac{1}{2}c^2M^2 + \frac{c^2}{\gamma-1}\right]_-^+ = 0 \Rightarrow \left[\frac{p}{\rho}\left(2+(\gamma-1)M^2\right)\right]_-^+ = 0$

$$\Rightarrow \frac{p_+}{p_-}\frac{\rho_-}{\rho_+} = \frac{2+(\gamma-1)M_-^2}{2+(\gamma-1)M_+^2} \quad (3'')$$

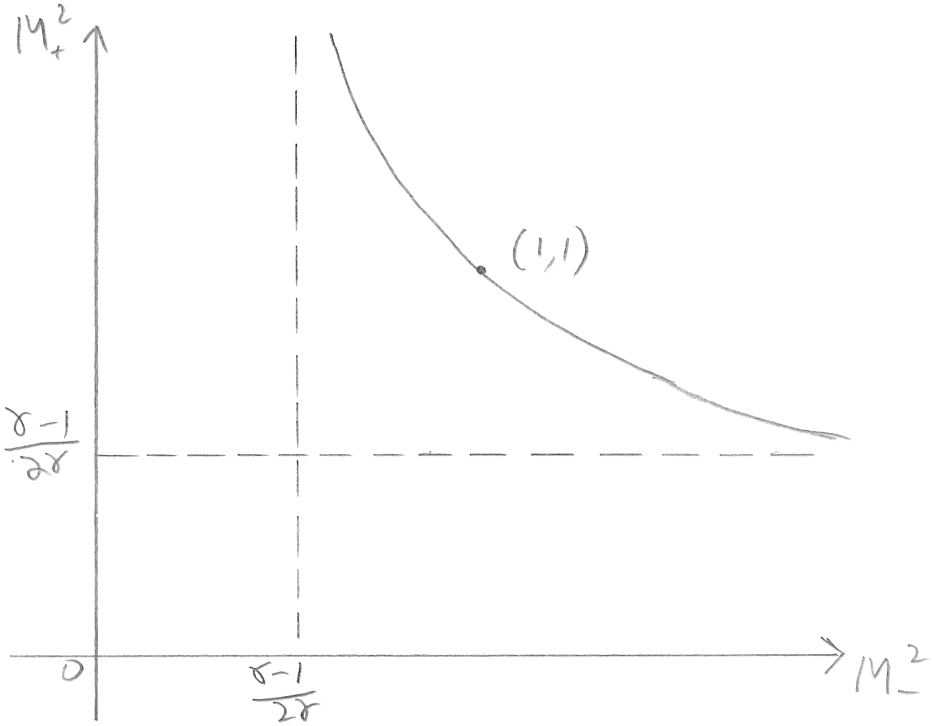
• Now eliminate $\frac{p_+}{p_-}$ and $\frac{\rho_+}{\rho_-}$ via $(1'') \times (3'') = \left(\frac{p_+}{p_-}\right)^2 = (2'')^2$

$$\Rightarrow \frac{M_-^2}{M_+^2}\frac{2+(\gamma-1)M_-^2}{2+(\gamma-1)M_+^2} = \left(\frac{1+\gamma M_-^2}{1+\gamma M_+^2}\right)^2 \Rightarrow (M_+^2 - M_-^2)\left\{2\gamma M_+^2M_-^2 - (\gamma-1)(M_+^2+M_-^2) - 2\right\} = 0$$

• If $M_+ = M_-$, then $p_+ = p_-$, $\rho_+ = \rho_-$ and no shock.

• If $M_+ \neq M_-$, then $M_+^2 = \dfrac{2+(\gamma-1)M_-^2}{2\gamma M_-^2 - (\gamma-1)}$

- Sketch:



- Thus, $M_+ \neq M_- \Rightarrow$ either (i) $M_+^2 < 1 < M_-^2$

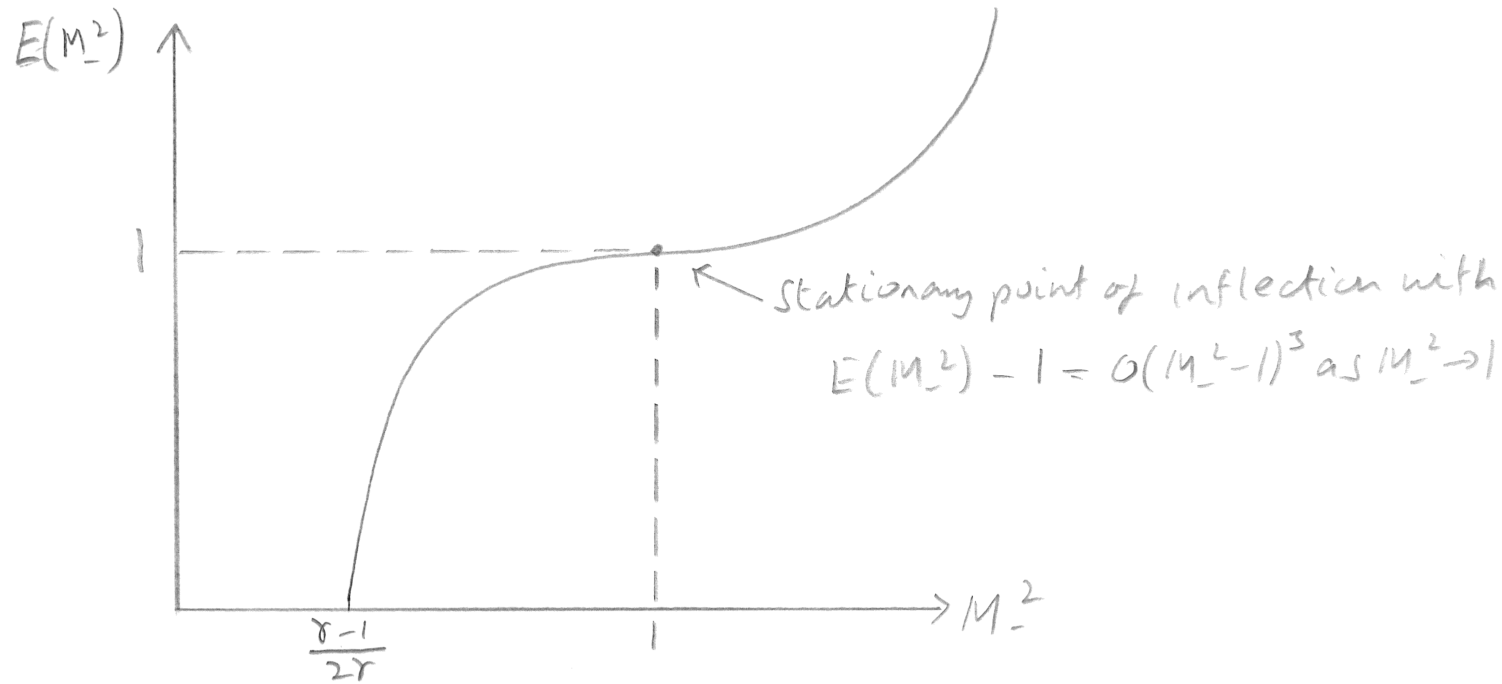
  or (ii) $M_-^2 < 1 < M_+^2$

- I.e. flow transitions between supersonic and subsonic, but which?

- Recall entropy $S = S_0 + c_v \log(P/\rho^\gamma)$. From above,

$$\exp\left(\frac{S_+ - S_-}{c_v}\right) = \frac{P_+/\rho_+^\gamma}{P_-/\rho_-^\gamma} = E(M_-^2) = \left(\frac{2\gamma M_-^2 - (\gamma-1)}{\gamma+1}\right)\left(\frac{2+(\gamma-1)M_-^2}{(\gamma+1)M_-^2}\right)^\gamma$$

- Sketch:



- $M_+ \neq M_- \Rightarrow$ either (i) or (ii) $\Rightarrow M_-^2 \neq 1 \Rightarrow E(M_-^2) \neq 1 \Rightarrow S_+ \neq S_-$,

i.e. entropy must be discontinuous across a shock.

- Suppose flow is left to right, so that $u_I > 0$. Then 2nd law of thermodynamics says $S_+ > S_- \Rightarrow E(M_-^2) > 1 \Rightarrow M_-^2 > 1$ (from graph) $\Rightarrow$ case (i) applies with $M_+^2 < 1 < M_-^2$, i.e. flow changes from supersonic ($M_- > 1$) to subsonic ($M_+ < 1$) as the gas crosses the shock.

- Then (1'') - (3'') $\Rightarrow \frac{p_+}{p_-} > 1, \frac{\rho_+}{\rho_-} > 1, \frac{T_+}{T_-} > 1$, so that p, ρ, T all increase as the gas crosses the shock.

- For a moving shock, the above applies if we define the Mach number relative to the moving shock, i.e. $M \mapsto M = \frac{u - \dot{s}}{c}$.