Waves and Compressible Flow

Lecture 14

Shocks in shallow water theory

- In conservative form, the shallow-water equations are given by

$$h_t + (hu)_x = 0 \quad (1) \qquad (hu)_t + \left(hu^2 + \tfrac{1}{2}gh^2\right)_x = 0 \quad (2)$$

- Of form $f_t + (fu+q)_x = 0$ with $f = h, hu$ and $q = 0, \tfrac{1}{2}gh^2$ respectively.

- Since $[f(u-\dot{s})+q]_-^+ = 0$ at a shock at $x = s(t)$, we deduce from (1) and (2) the jump conditions

$$[h(u-\dot{s})]_-^+ = 0 \quad (1') \qquad \left[hu(u-\dot{s}) + \tfrac{1}{2}gh^2\right]_-^+ = 0 \quad (2')$$

- Introduce the velocity $\hat{u} = u - \dot{s}$ relative to the shock.

- (1') $\Leftrightarrow$ $[h\hat{u}]_-^+ = 0$ $\Leftrightarrow$ $\rho h_- \hat{u}_- = \rho h_+ \hat{u}_+$, i.e rate of flow of mass into shock equals rate of flow out — mass conservation.

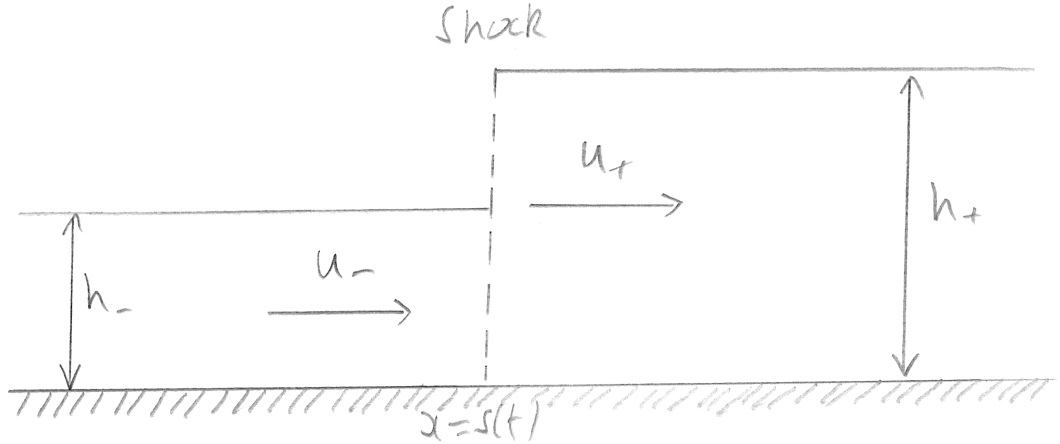
- (2') $- \dot{s}$(1') $\Leftrightarrow$ $[h\hat{u}^2 + \frac{1}{2}gh^2]_-^+ = 0$ $\Leftrightarrow$ $\rho h_+ \hat{u}_+ \hat{u}_+ - \rho h_- \hat{u}_- \hat{u}_- = \int_0^{h_-} p_- - p_a \, dz - \int_0^{h_+} p_+ - p_a \, dz$

since $p = p_a + \rho g(h - z)$, i.e. rate of change of momentum across shock is equal to net force acting across shock — momentum conservation.

- Hence can derive jump conditions from first principles, as in online notes.

- Summary: The Rankine-Hugoniot conditions for a moving shock are

$$[h(u-\dot{s})]_-^+ = 0 \quad (1''') \qquad [h(u-\dot{s})^2 + \tfrac{1}{2}gh^2]_-^+ = 0 \quad (2'')$$

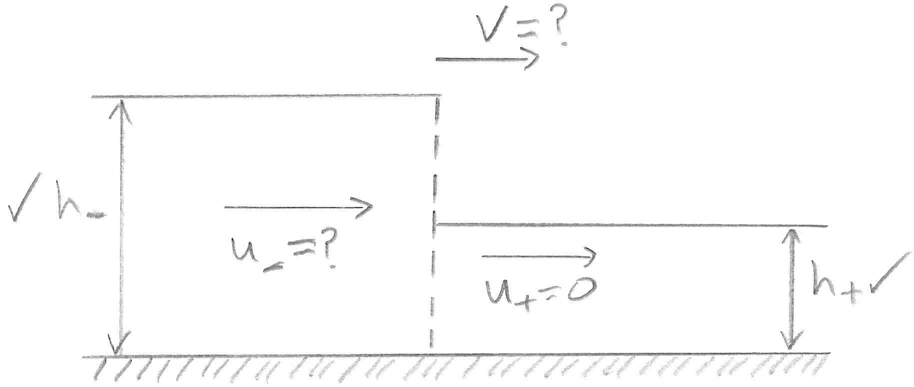


- Note (2'') different from momentum conservation with $\gamma=2$ in gas dynamics, so shocks behave differently.

- Moreover, energy is not conserved.

Example: bore moving into stationary water

- A bore with height h_- moves into stationary water with height $h_- > h_+$, with speed V



- (1'') $\Rightarrow$ $$h_+(0-V) = h_-(u_- - V)$$
  (2'') $\Rightarrow$ $$h_+(0-V)^2 + \tfrac{1}{2}gh_+^2 = h_-(u_- - V)^2 + \tfrac{1}{2}gh_-^2$$

- These give two equations for the two unknowns u_- and V.

- Eliminate $u_- - V$ from second using the first

$$\Rightarrow \quad h_+ V^2 + \tfrac{1}{2} g h_+^2 = h_- \left(\frac{-h_+ V}{h_-} \right)^2 + \tfrac{1}{2} g h_-^2$$

$$\Rightarrow \quad h_+ V^2 \left(1 - \frac{h_+}{h_-} \right) = \tfrac{1}{2} g (h_- - h_+)(h_- + h_+)$$

$$\Rightarrow \quad V^2 = \frac{g h_-}{2 h_+} (h_+ + h_-) \qquad (\text{since } h_+ \neq h_-)$$

$$\Rightarrow \quad V = \pm \left(\frac{g h_-}{2 h_+} (h_+ + h_-) \right)^{1/2}$$

- The sign of V is not determined by the jump conditions.

- We need the analogue of the entropy condition in gas dynamics.

Energy

- The rate at which energy flows out of the shock is

$$Q = \Big[\underbrace{\int_0^h \left(\tfrac{1}{2}\rho \hat{u}^2 + \rho g z\right)\hat{u}\, dz}_{\text{Rate at which KE + PE increase}} + \underbrace{\int_0^h (p - p_a)\,\hat{u}\, dz}_{\text{Rate at which work is done by the pressure}} \Big]_-^+$$

- Since $p = p_a + \rho g (h - z)$, integrating gives

$$Q = \left[\tfrac{1}{2}\rho h \hat{u}^3 + \tfrac{1}{2}\rho g h^2 \hat{u} + \tfrac{1}{2}\rho g h^2 \hat{u}\right]_-^+ = \left[\tfrac{1}{2}\rho h \hat{u}^3 + \rho g h^2 \hat{u}\right]_-^+$$

- But $[h\hat{u}]_-^+ = 0$, so $Q = \rho h_\pm u_\pm \left[\frac{1}{2}\hat{u}^2 + gh \right]_-^+$

- Now simplify using $h_+\hat{u}_+ = h_-\hat{u}_-$ and $h_+\hat{u}_+^2 + \frac{1}{2}gh_+^2 = h_-\hat{u}_-^2 + \frac{1}{2}gh_-^2$

$$\Rightarrow \quad h_+\hat{u}_+^2 + \frac{1}{2}gh_+^2 = h_-\left(\frac{h_+\hat{u}_+}{h_-}\right)^2 + \frac{1}{2}gh_-^2$$

$$\Rightarrow \quad h_+\hat{u}_+^2\left(1 - \frac{h_+}{h_-}\right) = \frac{1}{2}g(h_- - h_+)(h_- + h_+)$$

$$\Rightarrow \quad \hat{u}_+^2 = \frac{gh_-}{2h_+}(h_- + h_+), \quad \hat{u}_-^2 = \frac{gh_+}{2h_-}(h_+ + h_-) \qquad (\text{for } h_- \neq h_+)$$

- Hence, $\left[\frac{1}{2}\hat{u}^2 + gh\right]_-^+ = \frac{g}{4}\left(\frac{h_-}{h_+} - \frac{h_+}{h_-}\right)(h_- + h_+) + g(h_+ - h_-)$

$$= \frac{g(h_- - h_+)^3}{4h_-h_+}$$

- Hence, $Q = (\rho h_- u_-) \dfrac{g(h_- - h_+)^3}{4h_- h_+} \neq 0$ for $h_- \neq h_+$.

- Energy cannot be created by the jump, whereas it can be lost (due to turbulence), so for a physically sensible shock we must have $Q < 0 \iff \mathrm{sgn}(\hat{u}_\pm)(h_- - h_+) < 0$.

- Hence, if the flow relative to the shock is left-to-right, i.e. $\hat{u}_\pm > 0$, then $Q < 0 \iff h_- - h_+ < 0 \iff h_- < h_+$.

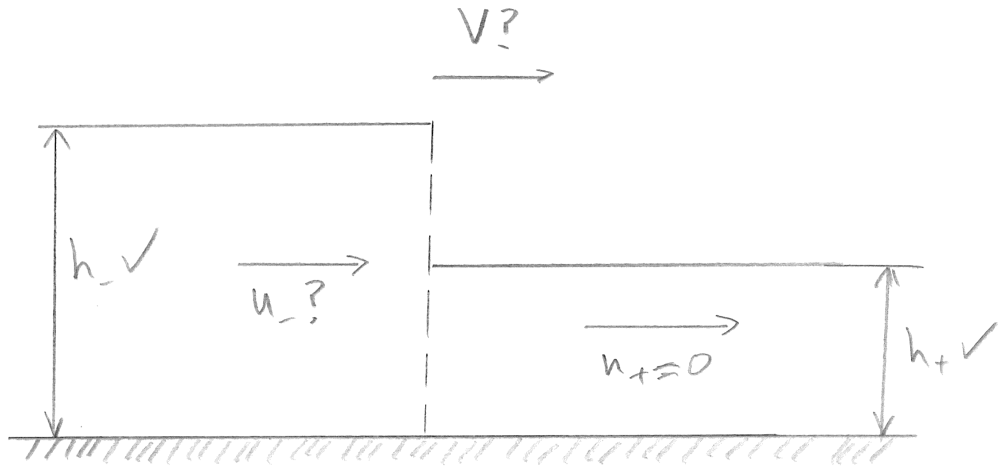
- This means that the depth increases as the fluid passes through the shock.

- If $h_- < h_+$, then $$\frac{\hat{u}_+^2}{gh_+} = \frac{1}{2}\frac{h_-}{h_+}\left(1+\frac{h_-}{h_+}\right) < 1$$ $$\frac{\hat{u}_-^2}{gh_-} = \frac{1}{2}\frac{h_+}{h_-}\left(1+\frac{h_+}{h_-}\right) > 1$$

- Hence if $\hat{u}_\pm > 0$, then $$\frac{\hat{u}_+^2}{gh_+} < 1 < \frac{\hat{u}_-^2}{gh_-},$$

so the flow is supercritical (i.e. faster than $c = \sqrt{gh}$) going into the shock and subcritical (i.e. slower than c) coming out of it, where $c = \sqrt{gh}$ is the wave speed in shallow-water theory

Example: moving bore revisited



- Recall h_-, h_+, u_+ given with $h_- > h_+$ and $u_+ = 0 \Rightarrow V = \pm\left(\frac{g h_-}{2h_+}(h_+ + h_-)\right)^{1/2}$

- Since depth must increase as fluid passes through the shock, the shock must move into shallower water, i.e. $V > 0$, $\hat{u}_- = u_- - V = -\frac{h_+ V}{h_-} < 0$.