Waves and Compressible Flow

Lecture 15

Weak solutions

- When physical variables are:
    - continuously differentiable, have PDEs;
    - discontinuous, have Rankine-Hugoniot conditions.

- A weak formulation provides a way to encompass both.

- Consider a general conservation law

$$\frac{\partial \underline{P}}{\partial t} + \frac{\partial \underline{Q}}{\partial x} = \underline{0},$$

where $\underline{P}$ and $\underline{Q}$ are continuously differentiable functions of $\underline{x}$, t and $\underline{u}(\underline{x}, t)$, the vector of state variables for which we must solve.

Example: one-dimensional gas dynamics

- $\rho_t + (\rho u)_x = 0$, $(\rho u)_t + (\rho u^2 + p)_x = 0$, $(\rho e)_t + (\rho e u + p u)_x = 0$, where $p = \rho R T$, $e = \frac{1}{2}u^2 + C_v T$, may be written in the form $\underline{P}_t + \underline{Q}_x = 0$ with

$$\underline{P} = \begin{pmatrix} \rho \\ \rho u \\ \rho u^2/2 + p/(\gamma-1) \end{pmatrix}, \quad \underline{Q} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u^3/2 + \gamma p u/(\gamma-1) \end{pmatrix}, \quad \underline{u} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}.$$

Example: shallow water equations

- $h_t + (hu)_x = 0$, $(hu)_t + (hu^2 + gh^2/2)_x = 0$ may be written in the form $\underline{P}_t + \underline{Q}_x = \underline{0}$, with

$$\underline{P} = \begin{pmatrix} h \\ hu \end{pmatrix}, \quad \underline{Q} = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \end{pmatrix}, \quad \underline{u} = \begin{pmatrix} h \\ u \end{pmatrix}.$$

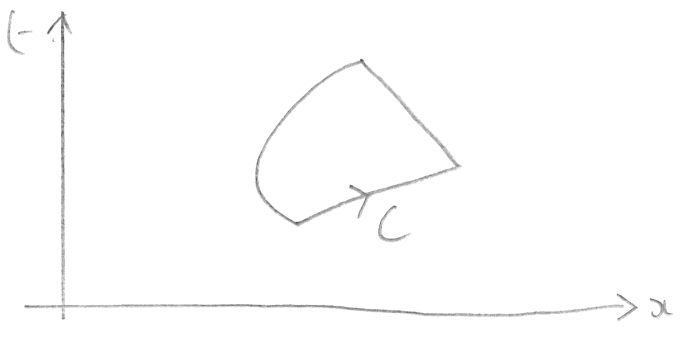
- Not all PDEs can be written in the form

$$\underline{P}_t + \underline{Q}_x = \underline{0}, \qquad (1)$$

but physical models based on conservation principles can almost always.

- We define a weak solution of (1) to be a function $\underline{u}(x,t)$ such that

$$\oint_C \underline{Q}\,dt - \underline{P}\,dx = \underline{0} \text{ for all piecewise-smooth simple closed curves } C. \qquad (2)$$



Classical solutions

- Suppose a weak solution $\underline{u}$ is continuously differentiable. Then Green's theorem gives $\iint_S \underline{P}_t + \underline{Q}_x \, dx\,dt = \underline{0}$ for all regions S whose boundaries C are as in (2)

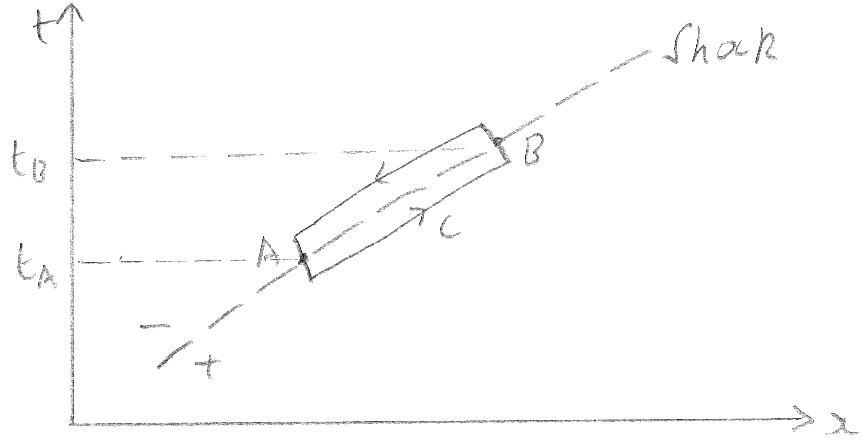
- Since $\underline{P}_t$ and $\underline{Q}_x$ are continuous by assumption and S is arbitrary, it follows that $\underline{u}$ satisfies the PDE (1).

- Conversely, if $\underline{u}$ is a continuously differentiable solution of (1), then it satisfies (2) by Green's theorem.

- Hence, (1) $\equiv$ (2) for continuously differentiable $\underline{u}$. Such a solution is called a <u>classical solution</u>.

Shocks

- Suppose now a weak solution $\underline{u}$ is continuously differentiable except on a shock at $x = s(t)$ across which it is discontinuous.



- Label regions either side of shock by + and −.

- Take A, B to be points on shock and C to be a narrow pill-box contour through them enclosing the shock.

- If we shrink C toward the shock on either side, then in (2) we are left with

$$
\begin{aligned}
\underline{0} &= \oint_C \underline{Q}\,dt - \underline{P}\,dx \\
&= \int_A^B \underline{Q}_+\,dt - \underline{P}_+\,dx + \int_B^A \underline{Q}_-\,dt - \underline{P}_-\,dx \\
&= \int_A^B [\underline{Q}]_-^+\,dt - [\underline{P}]_-^+\,dx \\
&= \int_{t_A}^{t_B} \left([\underline{Q}]_-^+ - \dot{s}[\underline{P}]_-^+\right) dt, \quad \text{since } \frac{dx}{dt} = \dot{s}(t) \text{ on } x = s(t).
\end{aligned}
$$

- Since A and B are arbitrary, we have (assuming the integrand is continuous)

$$V[\underline{P}]_-^+ = [\underline{Q}]_-^+, \qquad (3)$$

where $V = \dot{s}(t)$ is the shock speed.

- (3) are the Rankine-Hugoniot conditions - they can be read off from the conservation form (1) of the PDE.

- NB: We assumed continuity of $\underline{u}_+$ and $\underline{u}_-$ in deriving (3), which is a weaker condition than for the derivation from the integral conservation law.

Example : shallow water equations

(3) $\Rightarrow$ $$V\left[\begin{pmatrix} h \\ hu \end{pmatrix}\right]_-^+ = \left[\begin{pmatrix} hu \\ hu^2 + gh^2/2 \end{pmatrix}\right]_-^+$$

$$\Rightarrow \quad V[h]_-^+ = [hu]_-^+, \quad V[hu]_-^+ = [hu^2 + \tfrac{1}{2}gh^2]_-^+ = 0$$

$$\Rightarrow \quad [h(u-V)]_-^+ = 0, \quad [hu(u-V) + \tfrac{1}{2}gh^2]_-^+ - V[h(u-V)]_-^+ = 0$$

$$\Rightarrow \quad [h(u-V)]_-^+ = 0, \quad [h(u-V)^2 + \tfrac{1}{2}gh^2]_-^+ = 0$$

- These are the same as before as found from first principles.

- Can perform a similar analysis for 1D gas dynamics – see online notes.

Nonuniqueness of conservation laws

- The conservation form ① of the PDE is not unique.

- Thus, to get the 'correct' Rankine-Hugoniot conditions we need to choose the correct physical quantities to conserve.

- There is no mathematical way to tell which is correct.

Example: shallow water equations

- $h_t + u h_x + h u_x = 0$, $u_t + u u_x + g h_x = 0$ $\Rightarrow$ $\underline{P}_t + \underline{Q}_x = \underline{0}$, where

$$\underline{P} = \begin{pmatrix} h \\ u \end{pmatrix}, \quad \underline{Q} = \begin{pmatrix} hu \\ \frac{1}{2}u^2 + gh \end{pmatrix}, \quad \underline{u} = \begin{pmatrix} h \\ u \end{pmatrix}.$$

- These give the Rankine-Hugoniot conditions

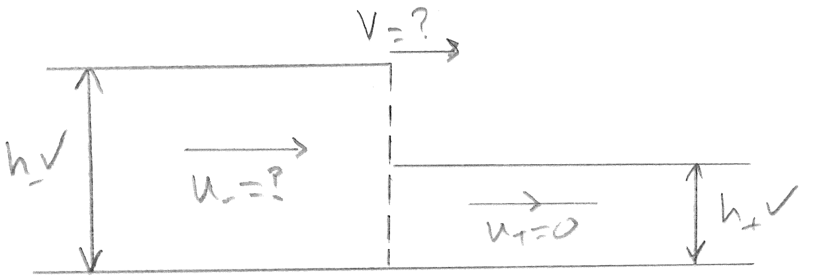
$$V\,[h]_-^+ = [hu]_-^+, \quad V\,[u]_-^+ = \left[\tfrac{1}{2}u^2 + gh\right]_-^+$$

- These are different jump conditions which conserve mass and energy across the shock, rather than mass and momentum.

Non-uniqueness of weak solutions

- Even once $\underline{P}$ and $\underline{Q}$ have been chosen, the weak solution may not be uniquely determined.

- For example, the choice of direction of the moving bore.

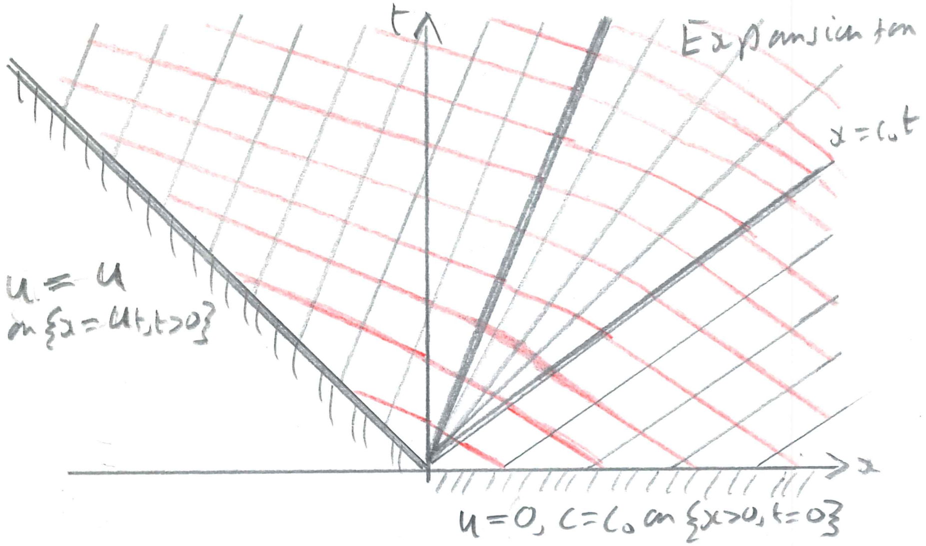


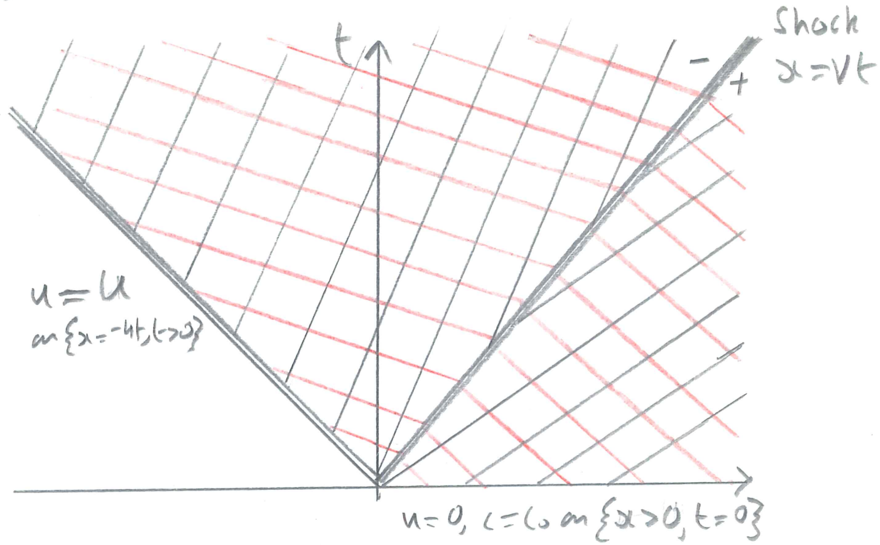
$$\Rightarrow \quad V = \pm \left(\frac{g h_- (h_+ + h_-)}{2 h_+} \right)^{1/2}$$

- This nonuniqueness is resolved by the energy condition for the shallow water equations and by the entropy condition for the equations of 1D gas dynamics.

Example: piston withdrawal in 1D gas dynamics

- Characteristic diagrams for two possible solutions when $U<0$:





- Expansion fan solution valid for $-U \le \frac{2c_0}{\gamma-1}$, while shock solution predicts $V = c_0(a+\sqrt{a^2+1})$, where $a = \frac{\gamma+1}{4}\frac{U}{c_0}$, which gives a possible shock speed when $U<0$.

- But $V^2 - \frac{1}{2}(\gamma+1)UV - c_0^2 = 0 \Rightarrow M_+^2 = \left(\frac{0-V}{c_0}\right)^2 = 1 + \frac{(\gamma+1)}{2}\frac{UV}{c_0^2} < 1$ for $U<0$

$\Rightarrow$ flow changes from subsonic to supersonic as gas crosses shock $\Rightarrow$ entropy decreases $\Rightarrow$ invalid.