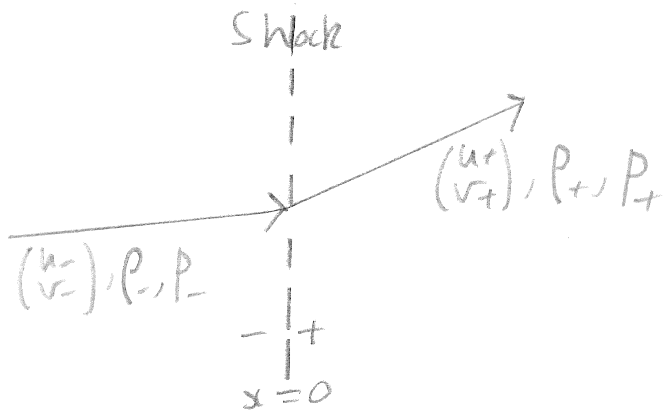
Waves and Compressible Flow

Lecture 16

Two-dimensional steady shocks

- Consider 2D steady gas flow with $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ through a shock at $x = 0$.



- Mass conservation: $\quad [\rho u]_-^+ = 0$

- Momentum conservation: $\quad \left[\rho u \begin{pmatrix} u \\ v \end{pmatrix}\right]_-^+ = \left[-p \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_-^+$

- Energy conservation: $\quad [\rho u e]_-^+ = [-pu]_-^+$, where $e = \frac{1}{2}(u^2 + v^2) + c_v T$

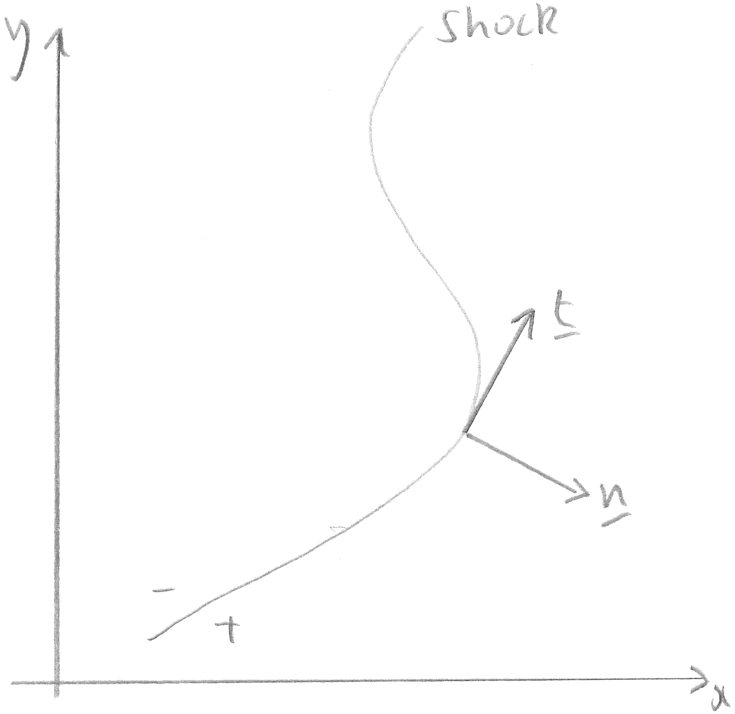
- Combining these jump conditions with $p = (\gamma-1)c_v \rho T$, we deduce that

$$[\rho u]_-^+ = [v]_-^+ = [\rho u^2 + p]_-^+ = \left[\tfrac{1}{2}u^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0.$$

- Hence, the only difference from a 1D shock is that the tangential velocity is continuous across the shock, i.e. 2D shock $\equiv$ 1D shock with superimposed tangential velocity.

- For example, for flow from $-$ to $+$, the entropy condition $\Rightarrow$ $u_- > u_+$ (i.e. supersonic to subsonic), so that the flow is deflected toward the shock, as illustrated.

- For an arbitrary shock orientation work with the normal and tangential velocities $u_n = \underline{u} \cdot \underline{n}$ and $u_t = \underline{u} \cdot \underline{t}$, where $\underline{n}$ and $\underline{t}$ are a unit normal and tangent to the shock.



Rankine-Hugoniot conditions:

$$[\rho u_n]_-^+ = 0$$

$$[u_t]_-^+ = 0$$

$$[\rho u_n^2 + p]_-^+ = 0$$

$$\left[\frac{1}{2}u_n^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_-^+ = 0$$

• Alternatively, we can start from the conservative form

$$\frac{\partial \underline{P}}{\partial x} + \frac{\partial \underline{Q}}{\partial y} = \underline{0}, \quad \underline{u} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \quad \underline{P} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u e + p u \end{bmatrix}, \quad \underline{Q} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v e + p v \end{bmatrix}.$$

• The Rankine-Hugoniot conditions are then $\frac{dy}{dx}[\underline{P}]_-^+ = [\underline{Q}]_-^+$ across a shock with slope $\frac{dy}{dx}$, so that

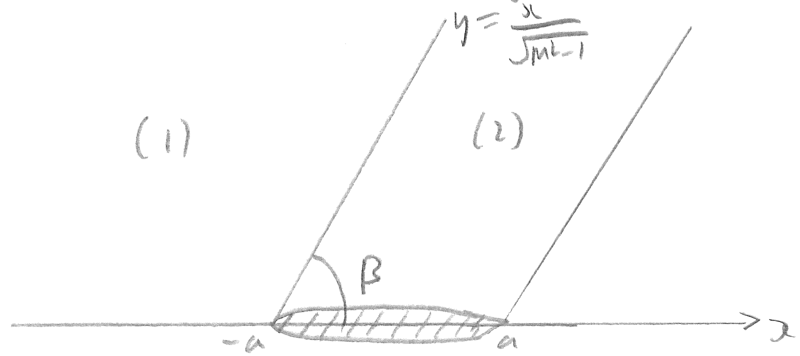
$$\frac{dy}{dx} = \frac{[\rho v]_-^+}{[\rho u]_-^+} = \frac{[\rho u v]_-^+}{[\rho u^2 + p]_-^+} = \frac{[\rho v^2 + p]_-^+}{[\rho u v]_-^+} = \frac{[\rho v e + p v]_-^+}{[\rho u e + p u]_-^+}$$

• Can recover previous formulation in terms of u_n and u_t using the fact that $\tan\theta = \frac{dy}{dx}$ if $\underline{t} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, $\underline{n} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

Example: flow past a wedge

- Recall supersonic flow past a thin wing with Mach number $M = \frac{U_0}{c_0} > 1$

$$\underline{u} = U_0 \underline{i} + \nabla\phi$$
$$|\nabla\phi| \ll U_0$$

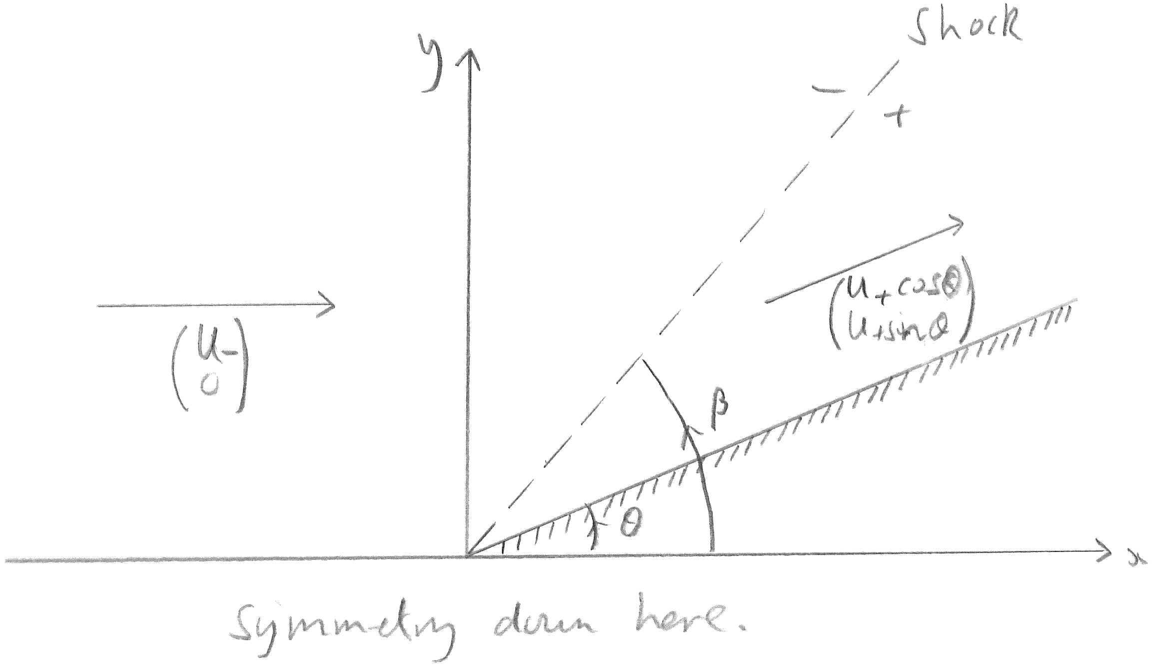


$$\tan\beta = \frac{1}{\sqrt{M^2-1}}$$

- Recall $\phi = 0$ in (1) and $\phi = \frac{-U_0}{\sqrt{M^2-1}} f_+\left(x - y\sqrt{M^2-1}\right)$ in (2) for $f_+(-a\pm) = 0$.

- Velocity jumps from $\underline{u} = U_0\underline{i}$ to $\underline{u} = U_0\underline{i} - \frac{U_0 f_+'(-a)}{\sqrt{M^2-1}}\left(\underline{i} - \sqrt{M^2-1}\,\underline{j}\right)$ across $y = \frac{x}{\sqrt{M^2-1}}$ $(y>0)$, i.e. characteristic $y = \frac{x}{\sqrt{M^2-1}}$ is actually a (weak) shock!

- Consider supersonic flow past a wedge with angle 2θ:



- Insert shock at angle β to deflect flow by an angle θ.

- What is the angle β?

- If we let $\underline{t} = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$, $\underline{n} = \begin{pmatrix} \sin\beta \\ -\cos\beta \end{pmatrix}$, then

$$u_{n-} = \begin{pmatrix} u_- \\ 0 \end{pmatrix} \cdot \underline{n} = u_- \sin\beta \, , \quad u_{t-} = \begin{pmatrix} u_- \\ 0 \end{pmatrix} \cdot \underline{t} = u_- \cos\beta$$

$$u_{n+} = \begin{pmatrix} u_+ \cos\theta \\ u_+ \sin\theta \end{pmatrix} \cdot \underline{n} = u_+ \sin(\beta - \theta), \quad u_{t+} = \begin{pmatrix} u_+ \cos\theta \\ u_+ \sin\theta \end{pmatrix} \cdot \underline{t} = u_+ \cos(\beta - \theta).$$

- Hence, the Rankine-Hugoniot conditions give

$$u_- \cos\beta = u_+ \cos(\beta - \theta)$$

$$\rho_- u_- \sin\beta = \rho_+ u_+ \sin(\beta - \theta)$$

$$p_- + \rho_- u_-^2 \sin^2\beta = p_+ + \rho_+ u_+^2 \sin^2(\beta - \theta)$$

$$\frac{1}{2} u_-^2 \sin^2\beta + \frac{\gamma p_-}{(\gamma - 1)\rho_-} = \frac{1}{2} u_+^2 \sin^2(\beta - \theta) + \frac{\gamma p_+}{(\gamma - 1)\rho_+}$$

- Since U_-, ρ_- and p_- are given upstream, the Rankine – Hugoniot conditions yield 4 equations for the 4 unknowns: U_+, ρ_+, p_+, β.

- Using previous results for a 1D shock applied to $u_- = U_- \sin\beta$ and $u_+ = U_+ \sin(\beta - \theta)$ we can relate the up- and down-stream Mach numbers:

$$M_+^2 \sin^2(\beta - \theta) = \frac{2 + (\gamma - 1) M_-^2 \sin^2\beta}{2\gamma M_-^2 \sin^2\beta - (\gamma - 1)},$$

where $M_\pm = U_\pm / c_0$ and $c_0 = \left(\gamma p_0 / \rho_0\right)^{1/2}$, as well as the density ratio:

$$\frac{\tan\beta}{\tan(\beta - \theta)} = \frac{\rho_+}{\rho_-} = \frac{M_-^2 \sin^2\beta}{M_+^2 \sin^2(\beta - \theta)} \, \frac{1 + \gamma M_+^2 \sin^2(\beta - \theta)}{1 + \gamma M_-^2 \sin^2\beta}$$

- Eliminating $M_+^2 \sin^2(\beta - \theta)$ between these expressions gives a single equation for β:

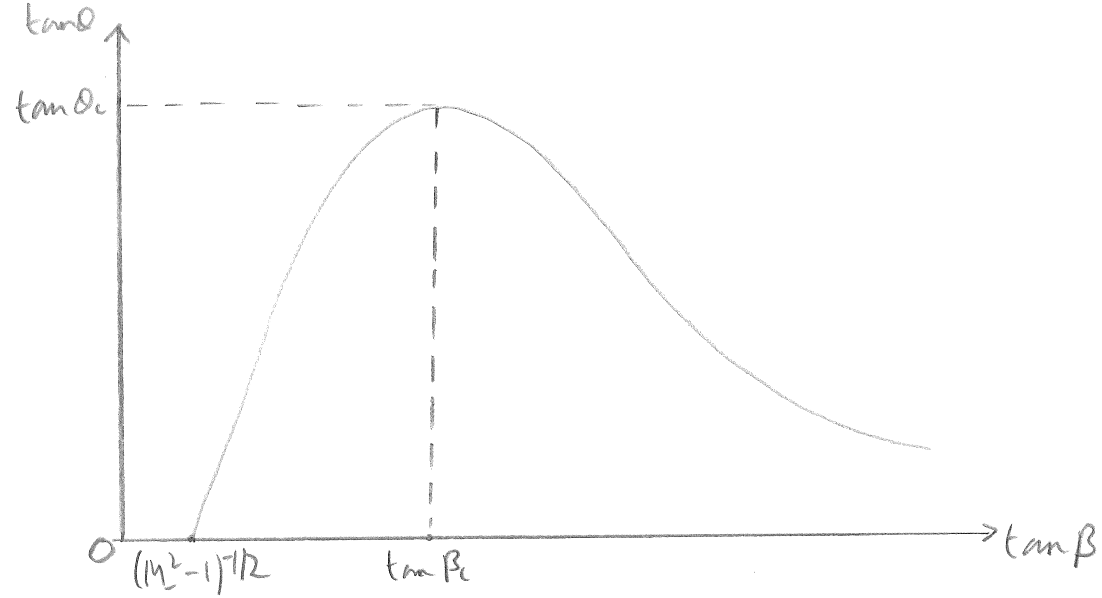
$$\frac{\tan\beta}{\tan(\beta-\theta)} = \frac{(\gamma+1)M_-^2 \sin^2\beta}{2+(\gamma-1)M_-^2\sin^2\beta}$$

- Finally, solving for $\tan\theta$, we obtain a single equation that we can analyse graphically:

$$\tan\theta = \frac{2\left[(M_-^2-1)\tan^2\beta - 1\right]}{\tan\beta\left[(2+(\gamma-1)M_-^2)\tan^2\beta + (2+(\gamma+1)M_-^2)\right]},$$

in which we recall θ and $M_- = \dfrac{u_-}{c_0}$ are prescribed, and β TBD.

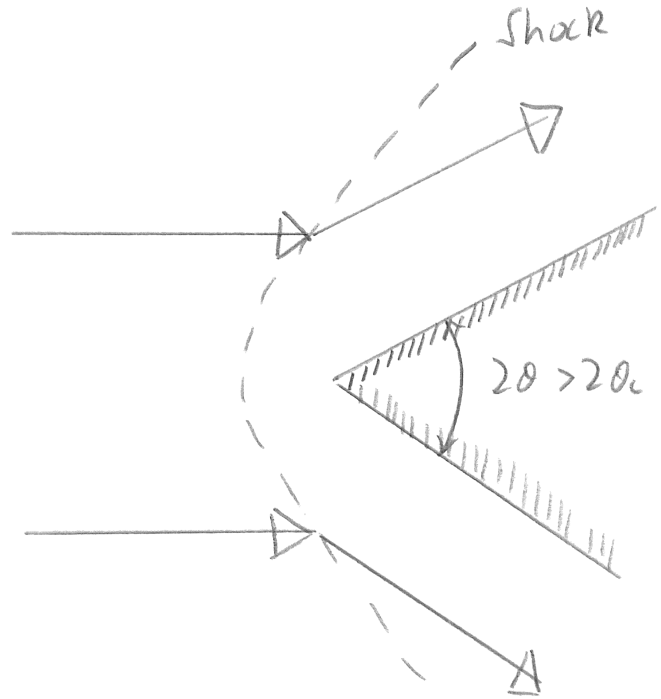
- A typical plot of $\tan\theta$ verses $\tan\beta$ is as shown:



- Note $\tan\beta = (M_-^2-1)^{-1/2}$ when $\tan\theta = 0$ in agreement with linear theory for small θ.

- For $\theta > \theta_c$, no solution of the Rankine-Hugoniot conditions exists for β. What happens?

- For $\theta > \theta_c$ a shock is observed to form <u>ahead</u> of the wedge apex and to be curved!



- This is surprising because the wedge influences the flow upstream of itself – flow is no longer homentropic and equations of gas dynamics must be solved numerically.