Fourier Series & Partial Differential Equations

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1 Introduction

1.1 Fourier series

Fourier's claim

- Fourier (1807): "every" real-valued function defined on a finite interval can be expanded as an infinite series of elementary trigonometric functions — cosines and sines.

- Equivalent claim: given a function $f : \mathbb{R} \to \mathbb{R}$ that is periodic with period 2π, there exist constants $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ s.t. f may be expanded

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx) \big) \quad \text{for } x \in \mathbb{R}. \tag{$\star$}$$

- The infinite trigonometric series in $(\star)$ is the *Fourier series* for f.

- Fourier's claim raises two fundamental questions:

  Question 1: If $(\star)$ is true, can we find a_n and b_n in terms of f?

  Question 2: With these a_n and b_n, when is $(\star)$ true?

Consequences: a mathematical revolution

- The need for rigorous mathematical analysis to address these questions led to a surprisingly large proportion of material covered in prelims, part A and beyond.
- The implications of Fourier's claim for practical applications were no less powerful or far-ranging and continue to be exploited today in numerous fields.
- In this course we introduce fundamental results for pointwise convergence of Fourier series.
- We then follow in Fourier's footsteps by using them to construct solutions to fundamental problems involving the three most ubiquitous PDEs in mathematics, science and engineering: the heat equation, the wave equation and Laplace's equation.
- We begin with a motivational example illustrating the existence of a convergent Fourier series.

Example: existence of a convergent Fourier series

Recall that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $z \in \mathbb{C}$

Let $z = e^{i\theta} = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$

Then $\mathrm{Re}(e^z) = \mathrm{Re}\left(e^{\cos\theta} e^{i\sin\theta}\right) = e^{\cos\theta}\cos(\sin\theta)$

and $\mathrm{Re}(z^n) = \mathrm{Re}\left(e^{in\theta}\right) = \cos n\theta$

Hence, $e^{\cos\theta}\cos(\sin\theta) = \underbrace{\sum_{n=0}^{\infty} \frac{\cos n\theta}{n!}}_{\text{Fourier cosine series}}$ for $\theta \in \mathbb{R}$

$\square$

Qn: How would you derive an example of a convergent Fourier sine series?

Ans: Taking the imaginary part instead of the real part in the example above, gives

$$e^{\cos\theta}\sin(\sin\theta) = \underbrace{\sum_{n=1}^{\infty}\frac{\sin n\theta}{n!}}_{\text{Fourier sine series}} \quad \text{for } \theta\in\mathbb{R}$$

Example: existence of a convergent Fourier series

- Recall from Analysis I:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for} \quad z \in \mathbb{C}.$$

- If $z = \exp(\mathrm{i}\theta) = \cos\theta + \mathrm{i}\sin\theta$, where $\theta \in \mathbb{R}$, then

$$\mathrm{Re}\big(\exp(z)\big) = \mathrm{Re}\big(\exp(\cos\theta)\exp(\mathrm{i}\sin\theta)\big) = \exp(\cos\theta)\cos(\sin\theta),$$

  and

$$\mathrm{Re}\big(z^n\big) = \mathrm{Re}\big(\exp(\mathrm{i}n\theta)\big) = \cos n\theta.$$

- Hence, taking the real part of the power series for $\exp(z)$ gives

$$\exp(\cos\theta)\cos(\sin\theta) = \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!} \quad \text{for} \quad \theta \in \mathbb{R}.$$

- This is an example of a *Fourier cosine series*. ■

- **Question**: How would you generate a convergent *Fourier sine series*?
- **Answer:** Taking instead the imaginary part we obtain

$$\exp(\cos\theta)\sin(\sin\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n!} \quad \text{for} \quad \theta \in \mathbb{R}.$$

- **Remark:** The method is of limited applicability for two reasons:
  1. it can only generate the Fourier series of an infinitely differentiable real-valued function;
  2. how should the complex-valued function be chosen to obtain the Fourier series that converges to a given real-valued function?
- At the heart of this course is a much simpler and more powerful method pioneered by Fourier that allows a much wider class of functions to be represented as convergent Fourier series.

1.2 Ordinary differential equations

- Here we revise essential background concerning ordinary differential equations.

- **Definition:** An *ordinary differential equation* (ODE) is an equation involving a function of one variable and at least one of its derivatives, *i.e.* an ODE for the function $y(x)$ may be written in the form
$$G\left(x, y(x), \frac{\mathrm{d}y}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right) = 0 \qquad (\dagger)$$
for some function G and some positive integer n.

- **Definition:** The function y is called the *dependent variable* and x the *independent variable*.

- **Definition:** The *order* of an ODE is the order of the highest order derivative that it contains, *e.g.* the order of $(\dagger)$ is n.

- **Definition:** An ODE is *linear* if the dependent variable and its derivatives appear in terms with degree at most one. An equation that is not linear is *nonlinear*.

- **Definition:** The most general nth-order linear ODE for $y(x)$ takes the form

$$\mathcal{L}y(x) = f(x),$$

where $f(x)$ is a given *forcing function* and $\mathcal{L}$ is the *linear differential operator* defined by

$$\mathcal{L}y(x) = a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y(x)$$

for some coefficients $a_0(x)$, $a_1(x)$, $\ldots$, $a_n(x)$ with $a_n(x) \neq 0$.

- **Definition:** The ODE $\mathcal{L}y(x) = f$ is called *homogeneous* if the right-hand side f is identically zero and if not then it is called *inhomogeneous*.

- **Remark:** Since differentiation is distributive, the differential operator $\mathcal{L}$ is *linear* in the sense that

$$\mathcal{L}\big[\alpha_1 y_1(x) + \alpha_2 y_2(x)\big] = \alpha_1 \mathcal{L}y_1(x) + \alpha_2 \mathcal{L}y_2(x)$$

for any constants α_1, $\alpha_2 \in \mathbb{R}$ and any (suitably differentiable) functions $y_1(x)$, $y_2(x)$.

- **Definition:** A consequence of the linearity of $\mathcal{L}$ is the *Principle of Superposition* that the linear combination of two or more solutions is also a solution for a linear homogeneous ODE — but not for a linear inhomogeneous ODE nor a nonlinear ODE.

See Introductory Calculus Theorem 3.1

- In Introductory Calculus you studied methods to find the *general solution* of first- and second-order linear ODEs, *e.g.*
    - the integrating factor method for first-order linear inhomogeneous ODEs;
    - reduction of order for second-order linear homogeneous ODEs;
    - methods for second-order linear inhomogeneous ODEs with constant coefficients.
- Linearity of $\mathcal{L}$ played a key role: the general solution of $\mathcal{L}(y) = f$ is the superposition of the n linearly independent solutions of the homogeneous problem, together with any solution of the inhomogeneous problem.
- Exploitation of linearity will play a similar fundamental role in the methods we shall use to solve the heat equation, the wave equation and Laplace's equation.

See Introductory Calculus Theorems 3.2 and 3.4

- You applied your toolbox of ODE methods to solve
  - *initial value problems* (IVPs) in which an nth-order ODE is supplemented by n initial conditions at some point x_0;
  - *boundary value problems* (BVPs) in which an nth-order ODE is supplemented by a total of n boundary conditions at two distinct points between which the ODE pertains.
- In general the method was to determine the general solution of the ODE and then to try to choose the n arbitrary constants that it contains to satisfy the n initial or boundary conditions.
- This does not work in general because a solution may not exist or if a solution exists it may not be unique.

Example: non-existence for an ODE BVP

Suppose $y(x)$: $\frac{d^2y}{dx^2} + y = 0$ for $0 < x < 2\pi$

$y(0) = 1$ and $y(2\pi) = 0$

$y = e^{mx} \Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$

Hence, the general solution is $y = A\cos x + B\sin x$ $(A, B \in \mathbb{R})$

$$\left.\begin{aligned} y(0) = 1 &\Rightarrow A = 1 \\ y(2\pi) = 0 &\Rightarrow A = 0 \end{aligned}\right\}$$

Contradiction, hence no solution!

$\square$

Example: non-existence for an ODE BVP

- Consider the boundary value problem for $y(x)$ given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0 \quad \text{for } 0 < x < 2\pi,$$

  with $y(0) = 1$ and $y(2\pi) = 0$.

- The ODE has general solution

$$y(x) = A\cos x + B\sin x,$$

  where A and B are arbitrary constants.

- The boundary conditions then require

$$A = 1 \quad \text{and} \quad A = 0,$$

  so that the constants A and B cannot be chosen to satisfy them.

- Hence, there is no solution. ■

Example: non-uniqueness for an ODE BVP

Suppose $y(x)$: $\dfrac{d^2y}{dx^2} + y = 0$ for $0 < x < 2\pi$

$$y(0) = 0 \,, \quad y(2\pi) = 0$$

Hence, $y = A\cos x + B\sin x \quad (A, B \in \mathbb{R})$

$$\left.\begin{array}{lll} y(0) = 0 & \Rightarrow & A = 0 \\ y(2\pi) = 0 & \Rightarrow & A = 0 \end{array}\right\} \Rightarrow B \text{ undetermined}$$

Hence, the solution is not unique, being given by $y = B\sin x$ for arb. $B \in \mathbb{R}$. $\square$

Example: non-uniqueness for an ODE BVP

- Consider the boundary value problem for $y(x)$ given by

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0 \quad \text{for } 0 < x < 2\pi,$$

with $y(0) = 0$ and $y(2\pi) = 0$.

- Again the ODE has general solution

$$y(x) = A\cos x + B\sin x,$$

where A and B are arbitrary constants.

- But now the boundary conditions require

$$A = 0 \quad \text{and} \quad A = 0,$$

so that B is left undetermined.

- Hence, the solution is not unique. ■

- Questions of existence and uniqueness of ordinary differential equations will be a central theme in *e.g.* part A Differential Equations 1.
- **Question:** Why discuss here the issues of existence and uniqueness?
- **Answer:** Because we face precisely the same issues when solving a partial differential equation, so we should keep them in mind.

Covered material for Problem Sheet 1 Question 1

1.3 Partial differential equations

Already solved some! See Introductory Calculus Section 4.3

- We now introduce partial differential equations building on the terminology outlined in §1.2.
- **Definition:** A *partial differential equation* (PDE) is an equation for an unknown function of two or more *independent variables* that involves at least one partial derivative of that function. The unknown function is called the *dependent variable*.
- **Definition:** The *order* of a PDE is the order of the highest order partial derivative that it contains.
- **Definition:** A PDE is *linear* if the dependent variable and its partial derivatives appear in terms with degree at most one. An equation that is not linear is *nonlinear*.
- In this course we focus on the case in which there are two independent variables: (x, y) or (x, t), where in applications x and y often represent spatial variables and t often represents time.

Example: general and linear first-order PDEs

- A first-order PDE for $u(x, y)$ may be written in the form

  $$G(x, y, u, u_x, u_y) = 0$$

  for some function G, where here and hereafter we use subscripts as shorthand for partial derivatives, *i.e.* $u_x = \partial u/\partial x$ etc.

- The most general first-order linear PDE for $u(x, y)$ is an equation of the form

  $$a_1 u_x + a_2 u_y + a_3 u = f,$$

  where a_1, a_2, a_3 and f are given functions of (x, y). The PDE is *homogeneous* if $f = 0$ and *inhomogeneous* otherwise. ■

Example: general and linear second-order PDEs

- A second-order PDE for $u(x, y)$ may be written in the form

$$H(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

for some function H.

- The most general second-order linear PDE for $u(x, y)$ is an equation of the form

$$a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u = f,$$

where a_1, a_2, ..., a_6 and f are given functions of (x, y). The PDE is *homogeneous* if $f = 0$ and *inhomogeneous* otherwise. ■

Example: some important PDEs

- There are many important PDEs, *e.g.*

transport equation for $u(x,t)$: $u_t + xtu_x = 0;$

inviscid Burger's equation for $u(x,t)$: $u_t + uu_x = 0;$

heat equation for $u(x,t)$: $u_t = \kappa u_{xx};$

Fisher's equation for $u(x,t)$: $u_t = \kappa u_{xx} + ru(1-u);$

viscous Burger's equation for $u(x,t)$: $u_t + uu_x = \nu u_{xx};$

porous medium equation for $u(x,t)$: $u_t = (u^m u_x)_x;$

thin-film equation for $u(x,t)$: $u_t + (u^m u_{xxx})_x = 0;$

wave equation for $u(x,t)$: $u_{tt} = c^2 u_{xx};$

plate equation for $u(x,t)$: $u_{tt} + \alpha^2 u_{xxxx} = 0;$

Eikonal equation for $u(x,y)$: $u_x^2 + u_y^2 = 1;$

Laplace's equation for $u(x,y)$: $u_{xx} + u_{yy} = 0;$

Poisson's equation for $u(x,y)$: $u_{xx} + u_{yy} = f(x,y).$

- **Exercise:** What is the order of each PDE? Which are linear/nonlinear? ■

Example: some more important PDEs

- There are many more important PDEs studied throughout the mathematics course using a range of mathematical techniques, *e.g.*
    - Euler's equations for inviscid fluid flow (A10);
    - Schrodinger's equation for the wave function in quantum mechanics (A11);
    - Euler-Lagrange equations in the calculus of variations (ASO);
    - Navier-Stokes equations for viscous fluid flow (B5.3);
    - Turing's reaction-diffusion equations for pattern formation (B5.5);
    - Maxwell's equations of electromagnetism (B7.2);
    - Black-Scholes' equation for derivative pricing (B8.3).
- Many of these PDEs may be written concisely using the *vector differential operators*, which are introduced in Multivariable Calculus. ■

The mathematical modelling process

- The PDEs in the last two examples encode a model of a physical or real-world process and arise as part of the *mathematical modelling process*:
  1. Start from a *physical* or *real-world* problem.
  2. Use physical or non-physical principles to translate it into mathematics — this involves developing appropriate mathematical technology.
  3. Use empirical laws to derive a soluble mathematical model.
  4. Solve the mathematical model — again this involves developing mathematical techniques.
  5. Use mathematical results to make predictions about the real system — usually these can only be sensible if there exists a unique solution to the underling mathematical problem.
- In this course we will illustrate the mathematical modelling process by
  - deriving from physical principles the heat equation, the wave equation and Laplace's equation;
  - using Fourier series methods to construct and analyse solutions to physical problems involving them.

- The physical problems we shall consider will often take the form of
    - *initial boundary value problems* (IBVPs) for the heat and wave equations in which suitable *boundary conditions* and *initial conditions* will need to be prescribed;
    - *boundary value problems* (BVPs) for Laplace's equation in which a suitable *boundary condition* will need to be prescribed.
- In each physical problem we will:
    - establish *existence* by constructing explicitly a solution;
    - prove *uniqueness* by showing that the difference between any two solutions much vanish.
- We will demonstrate thereby that we have *correctly specified* the number and form of boundary and/or initial conditions.
- The course will finish with a brief introduction to the notion of *well-posedness* of a PDE problem.
- We wrap up the Introduction with an example of an IBVP for the heat equation that illustrates the connection to Fourier series and the practical need to answer the fundamental questions in §1.1.

Example: IBVP for the heat equation

Suppose $T(x,t)$: ① $T_t = T_{xx}$ for $0 < x < \pi$, $t > 0$;

② $T(0,t) = 0$, $T(\pi,t) = 0$ for $t > 0$;

③ $T(x,0) = e^{\cos x} \sin(\sin x)$ for $0 < x < \pi$.

Let $T(x,t) = \sum_{n=1}^{N} b_n \sin(nx) e^{-n^2 t}$, where $b_1, \dots, b_N \in \mathbb{R}$ and $N \in \mathbb{N} | \{0\}$

$T_t = \sum_{n=1}^{N} -n^2 b_n \sin(nx) e^{-n^2 t} = T_{xx} \Rightarrow$ ① satisfied.

$x = 0, \pi \Rightarrow \sin(nx) = 0 \ \forall n \in \mathbb{N} \Rightarrow$ ② satisfied.

$\underline{Q^n}$: How do we satisfy the IC ③?

$\underline{A^{ns}}$: Recall that $e^{\cos x}\sin(\sin x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n!}$ for $x \in \mathbb{R}$

Hence, the IC ③ can only be satisfied if

$$\sum_{n=1}^{N} b_n \sin(nx) = T(x,0) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n!} \quad \text{for } 0 < x < \pi$$

This suggest that we take $N = \infty$ and $b_n = \frac{1}{n!}$ $\forall n$ to pose a candidate solution

$$T(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!} \sin(nx) e^{-n^2 t}.$$

Example: IBVP for the heat equation

- In a suitably scaled mathematical model for heat conduction along a thin metal wire, the temperature $T(x,t)$ satisfies the heat equation

  $$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < \pi, \; t > 0,$$

  with the boundary conditions

  $$T(0,t) = 0, \quad T(\pi,t) = 0 \quad \text{for} \quad t > 0,$$

  and the initial condition

  $$T(x,0) = \exp(\cos x) \sin(\sin x) \quad \text{for} \quad 0 < x < \pi,$$

  where x measures distance along the centreline of the wire and t measures time.

- We delay a description of the modelling assumptions underlying the mathematical model that is encoded in this IBVP.

- We can verify by substitution that the series solution

$$T(x,t) = \sum_{n=1}^{N} b_n \exp(-n^2 t) \sin(nx)$$

satisfies the heat equation and boundary conditions for $b_1, \ldots, b_n \in \mathbb{R}$ and positive integers N.

- **Question:** how should we pick N and the constants b_n?
- **Answer:** Recalling from §1.1 that

$$\exp(\cos\theta)\sin(\sin\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n!} \quad \text{for} \quad \theta \in \mathbb{R},$$

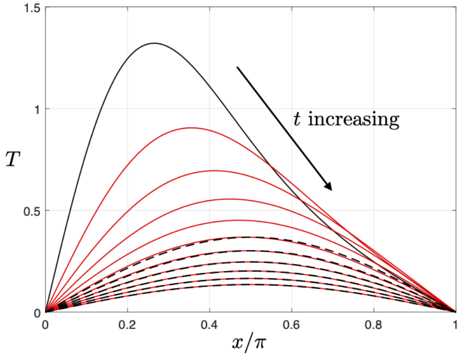
we see that the series solution satisfies the initial condition if $b_n = 1/n!$ and $N = \infty$.

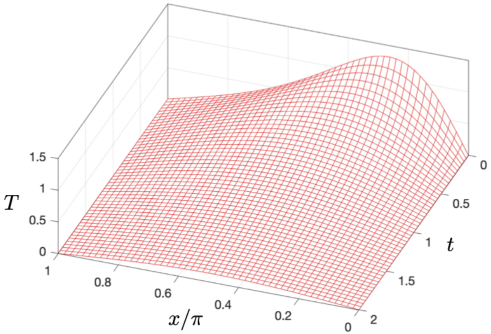
- Hence, a solution of the IBVP would appear to be

$$T(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!} \exp(-n^2 t) \sin(nx).$$

- LHS: initial profile (black line); snap shots of series solution truncated to 5 terms for $t = 0.2, 0.4, \ldots, 2$ (red lines); and leading term of series solution, $\sin(x)\exp(-t)$, for $t = 1, 1.2, \ldots, 2$ (dashed lines).

- RHS: the series solution truncated to 5 terms oriented to give a good view.





- **Question**: But what about other initial conditions?

See movie

■

Notes

(1) Can check directly that the infinite series solution satisfies the boundary and initial conditions.

(2) Comparison methods from Analysis II may be used to show that for $t > 0$ all of the partial derivatives of the infinite series solution exist and may be computed by term-by-term differentiation, so that the infinite series is indeed a solution of the IBVP.

(3) This means that truncating it after a sufficiently large number of terms will result in a good approximation to the solution.

(4) In this course our focus will be on the formal derivation — via Fourier series methods — of infinite series solutions, rather than on addressing the delicate issues concerning their convergence.

Covered material for Problem Sheet 1 Question 2

2 Fourier series

2.1 Periodic, even and odd functions

Fundamental questions

- Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π. We would like an expansion for f of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right) \quad \text{for } x \in \mathbb{R}, \tag{$\star$}$$

  where $a_0,\ a_1,\ \ldots$ and $b_1,\ b_2,\ \ldots$ are constants.

- Recall the two fundamental questions raised in §1.1:

  Question 1: If $(\star)$ is true, can we find a_n and b_n in terms of f?

  Question 2: With these a_n and b_n, when is $(\star)$ true?

Periodic functions

- The building blocks that form the partial sums of a Fourier series are cosines and sines.
- Not only are cosines and sines infinitely differentiable on $\mathbb{R}$, their graphs have important periodicity and symmetry properties: cos is an even periodic function, while sin is an odd periodic function.
- We therefore start with a refresher of what it means for a function to have these properties.
- **Definition:** The function $f : \mathbb{R} \to \mathbb{R}$ is a *periodic function* if there exists $p > 0$ such that

  $$f(x+p) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

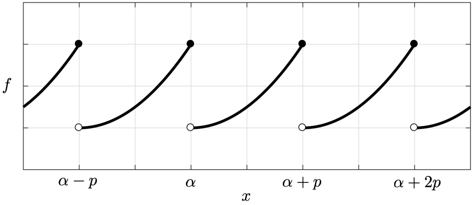
  In this case p is a *period* for f and f is called *p–periodic*. A period is not unique, but if there exists a smallest such p it is called the *prime period*.

Notes

(1) If $f(x) = c$ for $x \in \mathbb{R}$, where c is a real constant, then f is a p-periodic function for each $p > 0$, so does not have a prime period.

(2) Examples of periodic functions are $\cos x$, $\sin x$ with prime period 2π and $\cos(\pi x/L)$, $\sin(\pi x/L)$ with prime period $2L$ for each $L > 0$. Examples of non-periodic functions are x and x^2.

(3) As illustrated in the figure below the graph of a p-periodic function f repeats every p along the x-axis because it is invariant to the translation $(x, y) \mapsto (x + p, y)$.



Periodic extensions

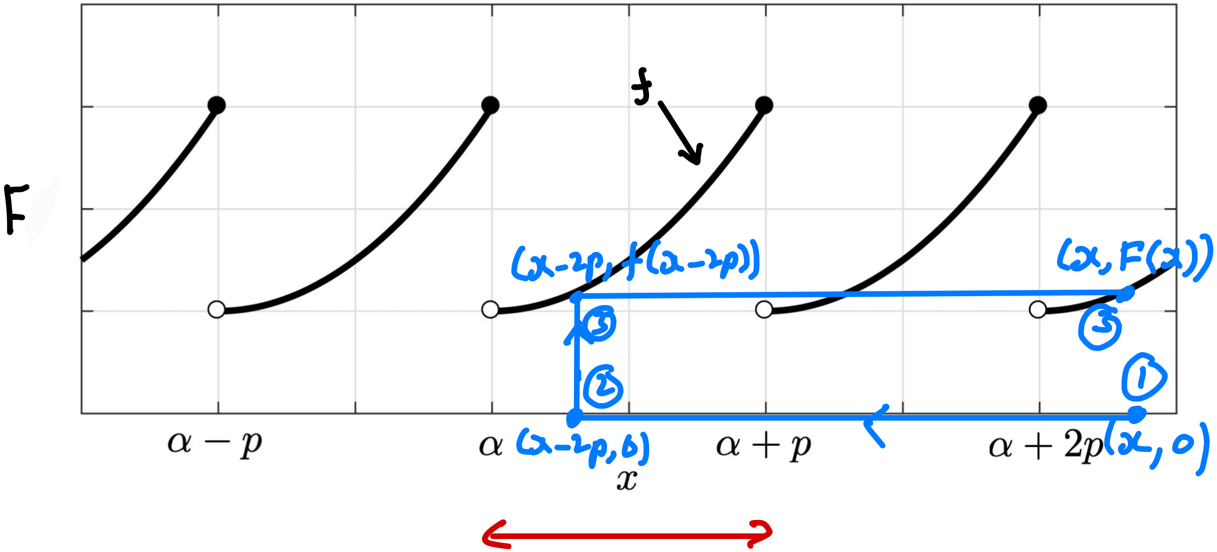
- If a function is defined on a half-open interval of length $p > 0$, *i.e.* on $(\alpha, \alpha + p]$ or $[\alpha, \alpha + p)$ for some $\alpha \in \mathbb{R}$, then we can extend it to a unique periodic function by demanding it to be periodic with period p.
- Formally, we define as follows the periodic extension of such a function.
- **Definition:** The *periodic extension* of the function $f : (\alpha, \alpha + p] \to \mathbb{R}$ is the function $F : \mathbb{R} \to \mathbb{R}$ defined by

  $$F(x) = f\big(x - m(x)p\big) \quad \text{for } x \in \mathbb{R},$$

  where, for each $x \in \mathbb{R}$, $m(x)$ is the unique integer such that $x - m(x)p \in (\alpha, \alpha + p]$.

Periodic extension: geometric interpretation

Suppose that f is defined only for $x \in (\alpha, \alpha+p]$



Suppose we pick e.g. $x \in (\alpha+2p, \alpha+3p]$ ①

Then $x - 2p \in (\alpha, \alpha+p]$, i.e. $m(x) = 2$ ②

Hence $F(x) = f(x-2p)$ ③

Properties of periodic functions

- The following properties follow from the definition of a periodic function and you may find it instructive to interpret them geometrically.
- If f and g are p–periodic, then:
  - (1) f, g are np–periodic for all $n \in \mathbb{N} \setminus \{0\}$;
  - (2) $\alpha f + \beta g$ are p–periodic for all $\alpha, \beta \in \mathbb{R}$;
  - (3) fg is p–periodic;
  - (4) $f(\lambda x)$ is p/λ–periodic for all $\lambda > 0$;
  - (5) $\int\limits_0^p f(x)\,\mathrm{d}x = \int\limits_\alpha^{\alpha+p} f(x)\,\mathrm{d}x$ for all $\alpha \in \mathbb{R}$.
- **Remark:** The prime period can change or cease to exist when multiplying or summing periodic functions. For example, $\cos x$ and $\sin x$ have prime period 2π, while $\cos^2 x$ and $\sin^2 x$ have prime period π and $\cos^2 x + \sin^2 x = 1$ has no prime period.

Proof of properties

(1) $f(x+np) = f(x+np-p) = \overset{\substack{\text{Induction}\\ \downarrow}}{\cdots} = f(x)$ for $x \in \mathbb{R}$

(2) $\alpha f(x+p) + \beta g(x+p) = \alpha f(x) + \beta g(x)$ for $x \in \mathbb{R}$.

(3) $f(x+p) g(x+p) = f(x) g(x)$ for $x \in \mathbb{R}$

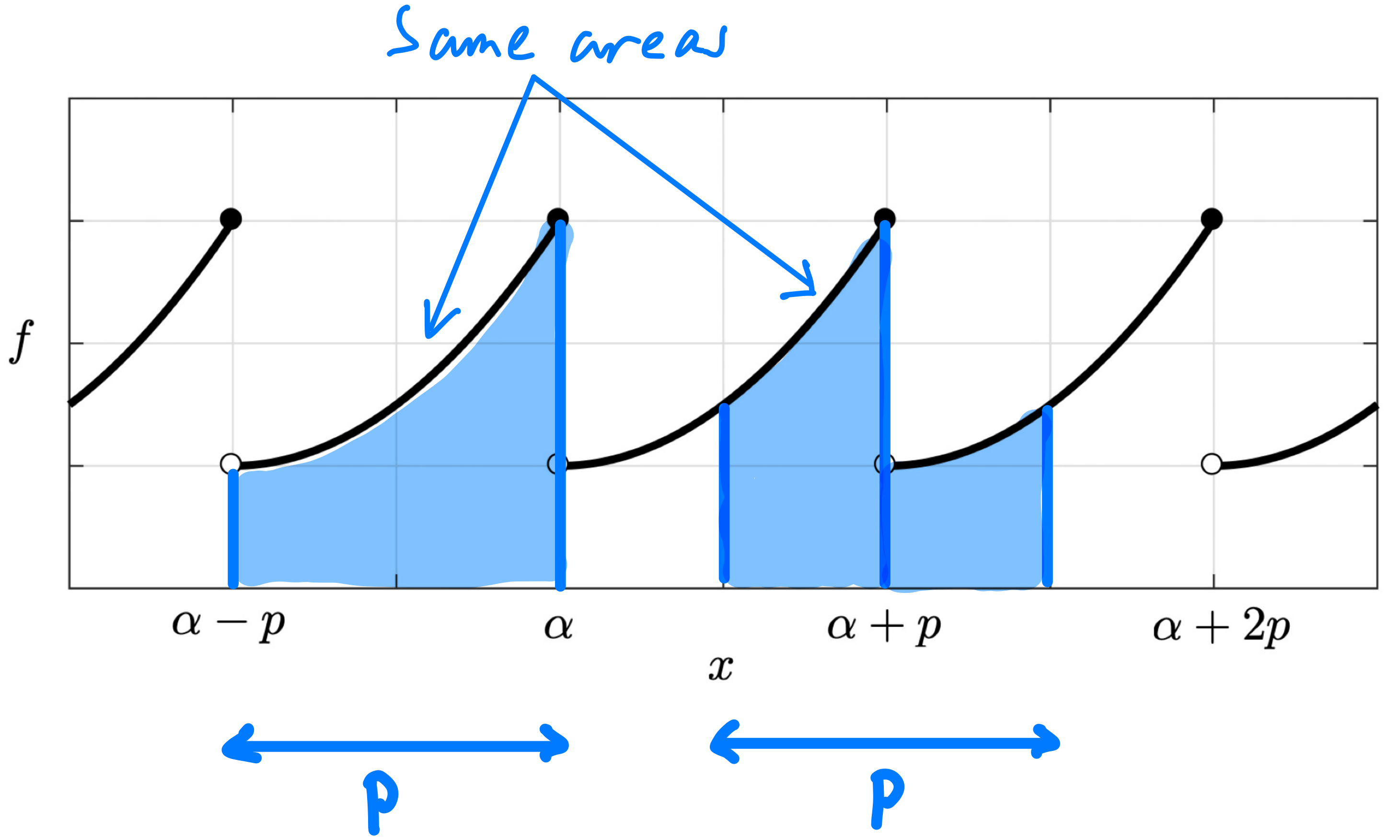
(4) Let $h(x) = f(\lambda x)$ for $x \in \mathbb{R}$. Then $h\left(x + \frac{p}{\lambda}\right) = f\left(\lambda\left(x + \frac{p}{\lambda}\right)\right) = f(\lambda x + p)$
$= f(\lambda x) = h(x)$ for $x \in \mathbb{R}$.

(5) $\displaystyle\int_\alpha^{\alpha+p} f(x)\,dx = \int_\alpha^p f(x)\,dx + \int_p^{\alpha+p} f(x)\,dx = \left(\int_\alpha^p + \int_0^\alpha\right) f(x)\,dx = \int_0^p f(x)\,dx$
for $\alpha \in \mathbb{R}$

$\displaystyle\int_p^{\alpha+p} f(x)\,dx \underset{x = p + \hat{x}}{=} \int_0^\alpha f(p+\hat{x})\,d\hat{x} = \int_0^\alpha f(\hat{x})\,d\hat{x} \;\Big\uparrow$

□

Geometrical interpretation of (5)



Even and odd functions

- **Definition:** The function $g : \mathbb{R} \to \mathbb{R}$ is *even* if

$$g(x) = g(-x) \quad \text{for all } x \in \mathbb{R}.$$

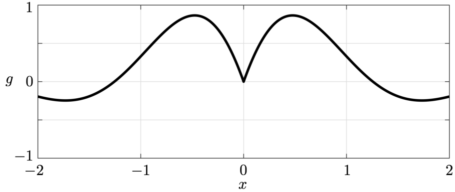
- **Definition:** The function $h : \mathbb{R} \to \mathbb{R}$ is *odd* if

$$h(x) = -h(-x) \quad \text{for all } x \in \mathbb{R}.$$

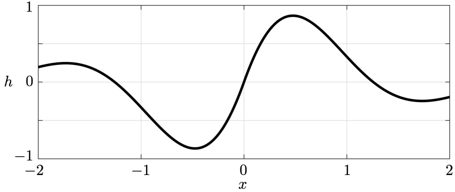
Notes:

(1) Examples of even functions are x^n for positive even integers n (hence the name even function) and $\cos(\lambda x)$ for $\lambda \in \mathbb{R}$. Examples of odd functions are x^n for positive odd integers n (hence the name odd function) and $\sin(\lambda x)$ for $\lambda \in \mathbb{R}$.

(2) The graph of an even function g is symmetric about the y-axis because it is invariant under the transformation $(x, y) \mapsto (-x, y)$.



(3) The graph of an odd function h is unchanged by a rotation by π radians about the origin $(x, y) = (0, 0)$ because it is invariant under the transformation $(x, y) \mapsto (-x, -y)$.



Properties of even/odd functions

- The following properties of even and odd functions follow from their definitions and again you may find it instructive to interpret geometrically.
- If g, g_1 are even and h, h_1 are odd, then:
  (1) gg_1 is even, gh is odd, and hh_1 is even;
  (2) $\int_{-\alpha}^{\alpha} g(x)\,\mathrm{d}x = 2\int_{0}^{\alpha} g(x)\,\mathrm{d}x$ for all $\alpha \in \mathbb{R}$;
  (3) $\int_{-\alpha}^{\alpha} h(x)\,\mathrm{d}x = 0$ for all $\alpha \in \mathbb{R}$;
  (4) $h(0) = 0$.

→ by definitions

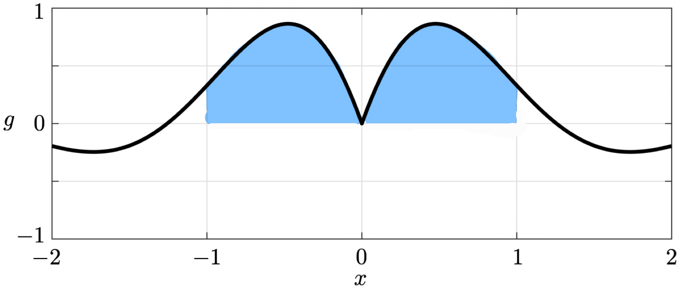
$\left.\right\}$ → problem sheet 1

$\rightarrow h(0) = -h(-0) = -h(0)$

$\Rightarrow 2h(0) = 0 \Rightarrow h(0) = 0$

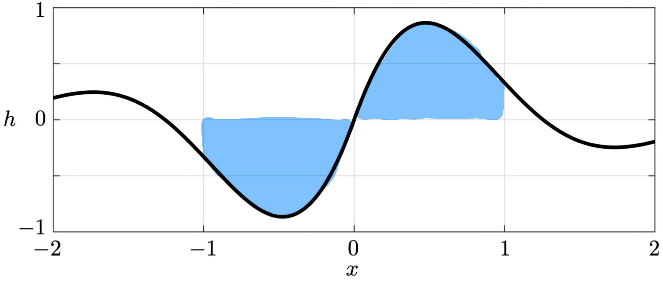
Geometric interpretation (2) & (3)

(2)



$$\int_{-1}^{1} g(x)\,dx = 2\int_{0}^{1} g(x)\,dx$$

(3)



$$\int_{-1}^{1} h(x)\,dx = 0$$

The even and odd part of a function

Proposition : Given a function $f : \mathbb{R} \to \mathbb{R}$ $\exists !$ functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ with g even and h odd s.t. $f = g + h$.

Proof : For existence we note the following functions will do :

$$(\dagger) \quad g(x) = \tfrac{1}{2}(f(x) + f(-x)), \; h(x) = \tfrac{1}{2}(f(x) - f(-x)) \text{ for } x \in \mathbb{R}.$$

For uniqueness suppose $f = g_1 + h_1$ and $f = g_2 + h_2$, with g_1, g_2 even and h_1, h_2 odd. Thus $g_1 - g_2 = h_2 - h_1 = G$ say. G is even and odd. Hence, if $x \in \mathbb{R}$, then $G(x) = G(-x) = -G(x)$ so $2G(x) = 0$, i.e. $G(x) = 0$, $g_1 = g_2$ & $h_1 = h_2$.

$\square$

The even and odd part of a function

- **Proposition:** Given a function $f : \mathbb{R} \to \mathbb{R}$ there exist unique functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ with g even and h odd such that

$$f(x) = g(x) + h(x) \quad \text{for } x \in \mathbb{R}.$$

- **Proof:**
  - To prove existence note that the following functions have the required properties:

$$g(x) = \frac{1}{2}\big(f(x) + f(-x)\big), \ h(x) = \frac{1}{2}\big(f(x) - f(-x)\big) \quad \text{for } x \in \mathbb{R}. \tag{$\dagger$}$$

  - To prove uniqueness suppose that $f = g_1 + h_1$ and $f = g_2 + h_2$, with g_1, g_2 even and h_1, h_2 odd; then $g_1 - g_2 = h_2 - h_1$ is both even and odd, and hence must vanish on $\mathbb{R}$. ■

- **Definition:** The function g in ($\dagger$) is the *even part* of f and h the *odd part* of f.

- **Remark:** The proof of uniqueness illustrates a common theme in the elementary uniqueness proofs in this course, namely that of showing the difference between two solutions must vanish.

Covered material for Problem Sheet 1 Questions 3 and 4