# 2.2 Fourier series for functions of period $2\pi$

**Fundamental questions**

- Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period $2\pi$. We would like an expansion for $f$ of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for } x \in \mathbb{R}, \tag{$\star$}$$

  where $a_0,\ a_1,\ \ldots$ and $b_1,\ b_2,\ \ldots$ are constants.

- Recall the two fundamental questions raised in §1.1:

  **Question 1:** If $(\star)$ is true, can we find $a_n$ and $b_n$ in terms of $f$?

  **Question 2:** With these $a_n$ and $b_n$, when is $(\star)$ true?

- We address the first question in this section and the second in §2.5.

# Context

- Show in Analysis 2 that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$ with $R > 0$,

  then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$, so by induction $a_n = \frac{f^{(n)}(0)}{n!}$.

- Q<sup></sup>

- **Qn**: How do we isolate a Fourier coefficient ($a_n$ or $b_n$)?

- **Ans**: Cannot differentiate out all but one term.

  But we can integrate out all but one term!

  Hence, method is in this sense opposite to that for power series.

## Question 1 : $a_0$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty}\left( \underbrace{\int_{-\pi}^{\pi} a_n \cos nx\,dx}_{\to\, 0} + \underbrace{\int_{-\pi}^{\pi} b_n \sin nx\,dx}_{\to\, 0} \right)$$

assuming $\int\sum = \sum\int$

Hence, $\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

## Question 1

- Suppose $(\star)$ is true and that we can integrate it term-by-term over a period, so that

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \mathrm{d}x + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x \right).$$

- Since, for positive integers $n$,

$$\int_{-\pi}^{\pi} \mathrm{d}x = 2\pi, \quad \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x = 0, \quad \int_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x = 0,$$

  we must have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x,$$

  which determines $a_0$ in terms of $f$.

- **Notes:**

  (1) $f$ is $2\pi$-periodic so could have integrated over any interval of length $2\pi$.

  (2) The leading term $a_0/2$ in the Fourier series for $f$ is equal to the mean of $f$ over a period.

- In order to determine the higher-order coefficients we will need the following Lemma.
- **Lemma:** Let $m$ and $n$ be positive integers. Then we have the *orthogonality relations:*

$$
\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x &= \pi\delta_{mn}, \\
\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x &= 0, \\
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x &= \pi\delta_{mn},
\end{aligned}
$$

  where $\delta_{mn}$ is Kronecker's delta defined by

$$
\delta_{mn} = \begin{cases} 0 & \text{for } m \neq n, \\ 1 & \text{for } m = n. \end{cases}
$$

- **Proof:** see online notes and a problem sheet.

## Question 1 : $a_m$ $(m \in \mathbb{N} \setminus \{0\})$

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{a_0}{2} \cancel{\int_{-\pi}^{\pi} \cos mx \, dx} \xrightarrow{} 0$$

$$+ \sum_{n=1}^{\infty} \left( a_n \cancel{\int_{-\pi}^{\pi} \cos nx \cos mx \, dx} \xrightarrow{} \pi \delta_{mn} \right.$$

$$\left. + b_n \cancel{\int_{-\pi}^{\pi} \sin nx \cos mx \, dx} \xrightarrow{} 0 \right)$$

$$= \sum_{n=1}^{\infty} \pi a_n \delta_{mn}$$

$$= \pi a_m \Rightarrow a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx$$

- Fixing $m \in \mathbb{N} \setminus \{0\}$, multiplying $(\star)$ by $\cos(mx)$ and assuming that the orders of summation and integration may be interchanged, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x =& \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\,\mathrm{d}x \\ &+ \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x \\ &+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x.\end{aligned}$$

- Using the first two of the orthogonality relations, we deduce that

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \frac{1}{2}a_0 \cdot 0 + \sum_{n=1}^{\infty} (a_n \pi \delta_{mn} + b_n \cdot 0) = \pi a_m,$$

so that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x \quad \text{for} \quad m \in \mathbb{N} \setminus \{0\}.$$

- **Question:** How would you derive a similar integral expression for $b_n$?
- **Answer:** By multiplying ($\star$) by $\sin(mx)$, integrating from $x = -\pi$ to $x = \pi$ and assuming that the orders of summation and integration may be interchanged. As shown on a problem sheet, this gives

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, \mathrm{d}x \quad \text{for} \quad m \in \mathbb{N} \setminus \{0\}.$$

- We wrap these formulae into the following definition.
- **Definition:** Let $f : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic and integrable on $[-\pi, \pi]$. Then, regardless of whether or not it converges, the *Fourier series* for $f$ is defined to be the infinite series given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big( a_n \cos(nx) + b_n \sin(nx) \big)$$

for $x \in \mathbb{R}$, where the *Fourier coefficients* of $f$ are the constants $a_n$ and $b_n$ given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x \quad \text{for } n \in \mathbb{N},$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x \quad \text{for } n \in \mathbb{N} \setminus \{0\}.$$

## Notes

(1) The integrability condition ensures the existence of the Fourier coefficients.

(2) We adopt the short-hand notation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

to indicate that the Fourier series for $f$ is given by the RHS of this expression regardless of whether or not it converges.

(3) The factor of $1/2$ in the first term of the Fourier series ensures that the formulae for the Fourier cosine coefficients is the same for all non-negative integers $n$.

(4) It is readily shown that the Fourier series for $f$ may be written in the equivalent complex form

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx},$$

where the complex Fourier coefficients $c_n$ are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x \quad \text{for } n \in \mathbb{Z}.$$

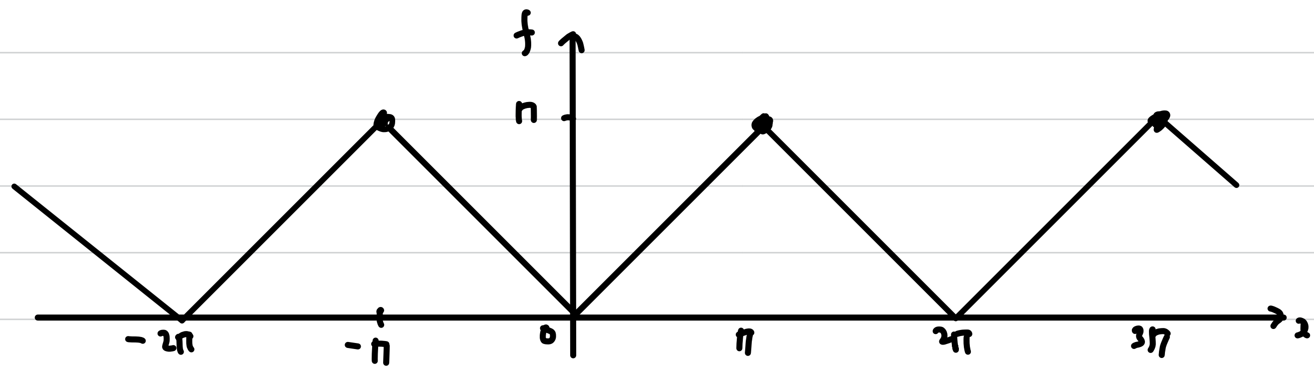
This is an elegant formulation, but the original one is better suited to our PDE applications.

$\boxed{\text{Covered material for Problem Sheet 2 Question 1}}$

## Example 1

Find the Fourier series of the $2\pi$-periodic function $f$ defined by $f(x) = |x|$ for $-\pi < x \leq \pi$.

**Sketch:**




**Fourier coeffs:** $f(x)$ is even $\Rightarrow$ $f(x)\cos(nx)$ is even, $f(x)\sin(nx)$ is odd

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx, \quad b_n = 0.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \left[\frac{x^2}{\pi}\right]_0^{\pi} = \pi$$

For $n \geq 1$, integrate by parts using

$$\int_0^{\pi} (uv)'\,dx = [uv]_0^{\pi} \Rightarrow \int_0^{\pi} uv'\,dx = [uv]_0^{\pi} - \int_0^{\pi} u'v\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{x}_{u}\, \underbrace{\cos(nx)}_{v'}\,dx$$

$$= \frac{2}{\pi}\left[\underbrace{x}_{u}\,\underbrace{\frac{1}{n}\sin(nx)}_{v}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} \underbrace{1}_{u'}\,\underbrace{\frac{1}{n}\sin(nx)}_{v}\,dx$$

$$= 0 + \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

Hence, $a_n$ is zero for $n$ even and non-zero for $n$ odd.

$$a_n = \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\}, \\ -\dfrac{4}{\pi(2m+1)^2} & \text{for } n = 2m+1,\ m \in \mathbb{N}. \end{cases}$$

Since the Fourier series for $f$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

we have

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}$$

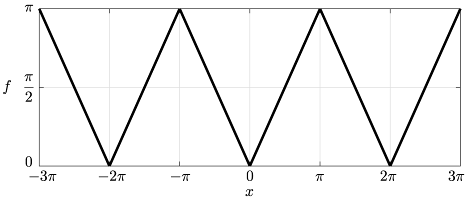
$\square$

## Example 1

- Find the Fourier series for the $2\pi$-periodic function $f$ defined by

$$f(x) = |x| \quad \text{for} \quad -\pi < x \leq \pi.$$

- The plot of the graph of $f$ shows that it has a "sawtooth" profile that is piecewise linear and continuous, with corners at integer multiples of $\pi$.




- Since $f(x)$ is even, $f(x)\cos(nx)$ is even and $f(x)\sin(nx)$ is odd, giving

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = 0.$$

- For $n=0$, direct integration gives

$$a_0 = \frac{2}{\pi} \int\limits_0^\pi x \, \mathrm{d}x = \left[ \frac{2}{\pi} \frac{x^2}{2} \right]_0^\pi = \pi.$$

- For $n \geq 1$, we use integration by parts by taking $u = x$ and $v = \sin(nx)/n$ in the identity

$$\left[ uv \right]_0^\pi = \int_0^\pi (uv)' \, \mathrm{d}x = \int_0^\pi u'v + uv' \, \mathrm{d}x,$$

which gives

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) \, \mathrm{d}x = \frac{2}{\pi} \left( \left[ \frac{x}{n} \sin(nx) \right]_0^\pi - \int\limits_0^\pi 1 \cdot \frac{1}{n} \sin(nx) \, \mathrm{d}x \right).$$

- Hence,

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2}\right]_0^{\pi} \\
&= -\frac{2}{\pi}\frac{[1-(-1)^n]}{n^2} \\
&= \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\}, \\ -\dfrac{4}{\pi(2m+1)^2} & \text{for } n = 2m+1,\ m \in \mathbb{N}. \end{cases}
\end{aligned}
$$

- Thus,

$$
f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2},
$$

the right-hand side being the Fourier series for $f$. ■

**Notes**

(1) The partial sums of the Fourier series for $f$ may be defined for $N \in \mathbb{N}$ by

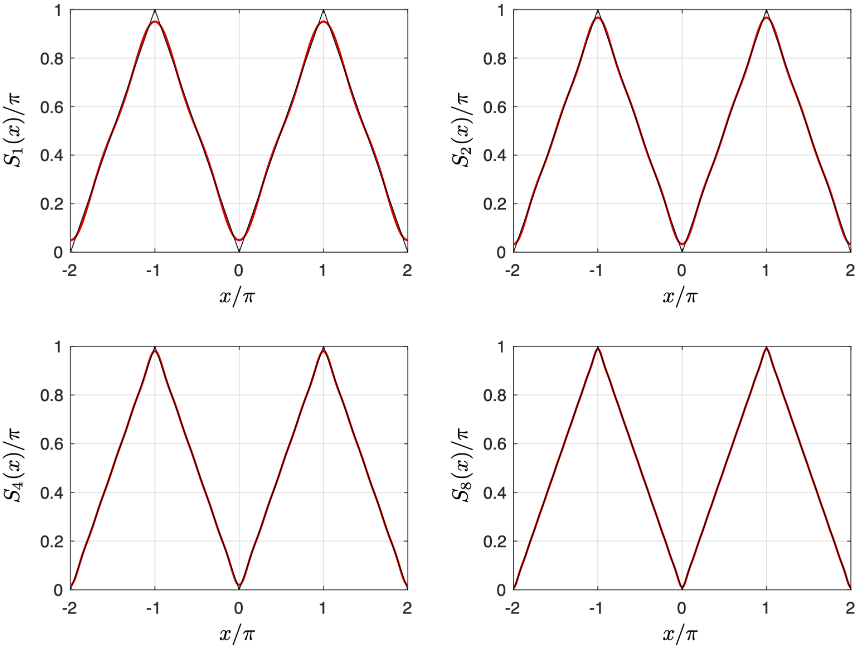
$$S_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{N} \frac{\cos\left((2m+1)x\right)}{(2m+1)^2} \quad \text{for } x \in \mathbb{R}.$$

The plots below show that $S_N$ rapidly approaches $f$ with increasing $N$, suggesting that the Fourier series converges to $f$ on $\mathbb{R}$, *i.e.*

$$\lim_{N\to\infty} S_N(x) = f(x) \quad \text{for } x \in \mathbb{R}.$$

(2) If this is true, then we can pick $x$ to evaluate the sum of a series, *e.g.* $x = 0$ gives

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \quad \Longrightarrow \quad \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$



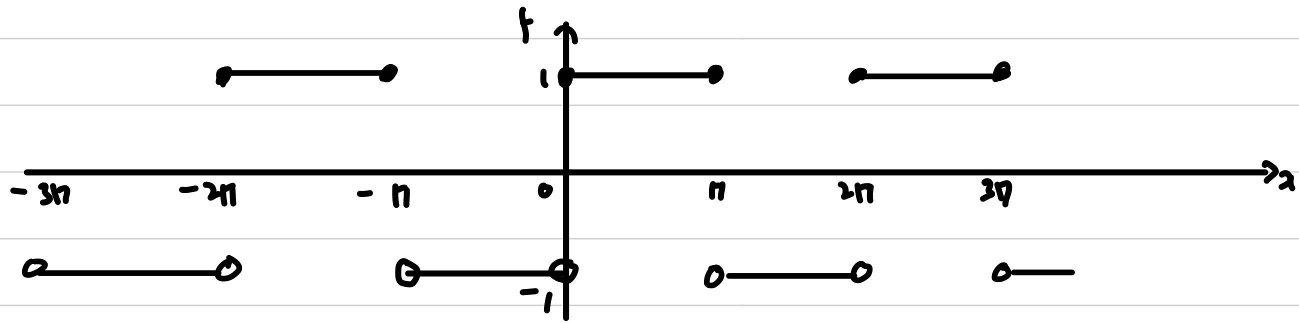
$S_1(x)/\pi$
$S_2(x)/\pi$
$S_4(x)/\pi$
$S_8(x)/\pi$
$x/\pi$
0
0.2
0.4
0.6
0.8
1
-2
-1
0
1
2

## Example 2

Find the Fourier series for the $2\pi$-periodic function $f$ defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ -1 & \text{for } -\pi < x < 0 \end{cases}$$

Sketch:




Fourier coeffs: $f$ is odd for $\frac{x}{\pi} \in \mathbb{R} \setminus \mathbb{Z}$

$\Rightarrow f(x)\cos(nx)$ is odd & $f(x)\sin(nx)$ is even for $\frac{x}{\pi} \in \mathbb{R} \setminus \mathbb{Z}$

$\Rightarrow a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^\pi \sin(nx)\,dx$

$$\Rightarrow \quad b_n = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}$$

$$= -\frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\} \\ \dfrac{4}{\pi(2m+1)} & \text{for } n = 2m+1,\ m \in \mathbb{N}. \end{cases}$$

Since the Fourier series for $f$ is $\sum_{n=1}^{\infty} b_n \sin(nx)$, we deduce that

$$f(x) \sim \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{2m+1}$$

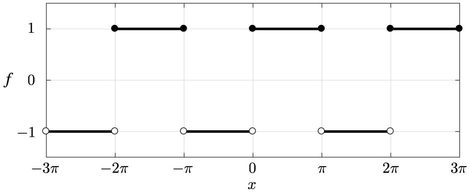
$\square$

## Example 2

- Find the Fourier Series for the $2\pi$-periodic function $f$ defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi, \\ -1 & \text{for } -\pi < x < 0. \end{cases}$$

- The plot of the graph of $f$ shows that it has a "square wave" profile that is piecewise linear with jump discontinuities at integer multiples of $\pi$.




- Since $f(x)$ is odd for $x/\pi \in \mathbb{R}\backslash\mathbb{Z}$, we have $a_n = 0$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

- But $f(x) = 1$ for $0 < x < \pi$, so

$$\begin{aligned} b_n &= \frac{2}{\pi} \int\limits_0^\pi \sin(nx)\, \mathrm{d}x \\ &= \left[ -\frac{2}{\pi} \frac{\cos(nx)}{n} \right]_0^\pi \\ &= \frac{2[1 - (-1)^n]}{\pi n}. \end{aligned}$$

- Hence, setting $n = 2m + 1$ to enumerate the non-zero terms, we obtain

$$f(x) \sim \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x)}{2m+1},$$

the right-hand side being the Fourier series for $f$. ∎

**Notes**

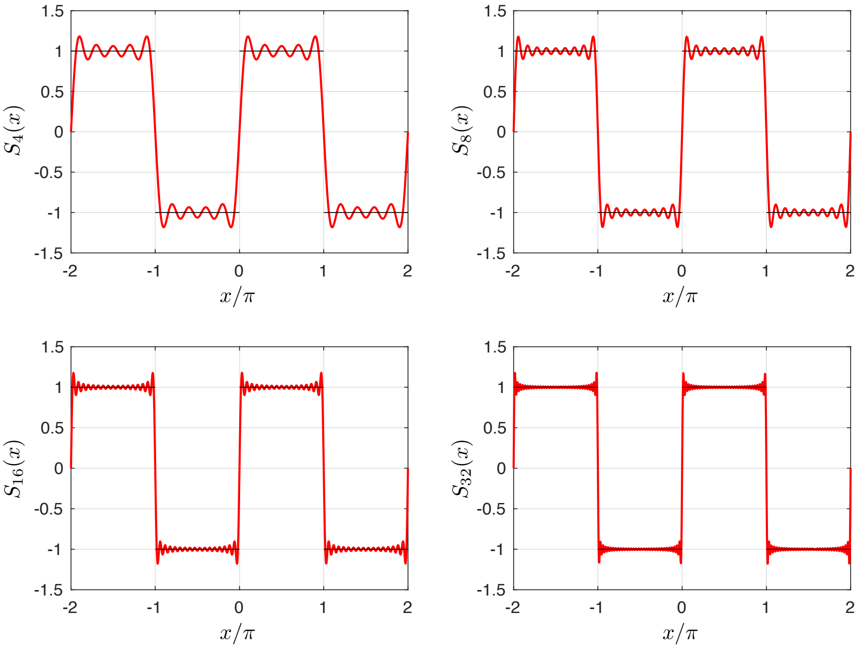
(1) The partial sums of the Fourier series for $f$ may be defined for $N \in \mathbb{N}$ by

$$S_N(x) = \frac{4}{\pi} \sum_{m=0}^{N} \frac{\sin\left((2m+1)x\right)}{2m+1} \quad \text{for } x \in \mathbb{R}.$$

The plots below show that $S_N$ slowly approaches $f$ with increasing $N$ away from the jump discontinuities at which $S_N$ vanishes, suggesting that

$$\lim_{N\to\infty} S_N(x) = \begin{cases} f(x) & \text{for } x/\pi \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{for } x/\pi \in \mathbb{Z}. \end{cases}$$

(2) The convergence is slower than in Example 1 and there is a persistent overshoot near the discontinuities of $f$ — this is called *Gibb's phenomenon*, about which more in §2.7.


$S_4(x)$
$S_8(x)$
$S_{16}(x)$
$S_{32}(x)$
$x/\pi$
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2

## 2.3 Cosine and sine series

- Let $f : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic and integrable on $[-\pi, \pi]$, so that the Fourier coefficients exist.
- In numerous practical applications the relevant function $f$ is even or odd.
- It is for this reason we chose to integrate from $x = -\pi$ to $x = \pi$, rather than over any other interval of length $2\pi$, since we may then exploit immediately the symmetry of $f$, as we shall now describe.
- If $f$ is even, then $f(x)\cos(nx)$ is even and $f(x)\sin(nx)$ is odd, giving

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \quad \text{for } n \in \mathbb{N}, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = 0 \quad \text{for } n \in \mathbb{N}\backslash\{0\}, \end{aligned}$$

so that

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

*i.e.* $f$ has a *Fourier cosine series*.

- If $f$ is odd, then $f(x)\cos(nx)$ is odd and $f(x)\sin(nx)$ is even, giving

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = 0 \quad \text{for } n \in \mathbb{N}, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad \text{for } n \in \mathbb{N}\backslash\{0\}, \end{aligned}$$

  so that

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx),$$

  *i.e.* $f$ has a *Fourier sine series*.

- **Remark:** Since the value of an integral is unchanged if the value of its integrand is modified at a finite number of points, we obtain exactly the same Fourier sine series for $f$ if $f$ is odd on *e.g.* $\mathbb{R}\backslash\{k\pi : k \in \mathbb{Z}\}$, as in Example 2, rather than on the whole of $\mathbb{R}$.

## 2.4 Tips for evaluating the Fourier coefficients

(1) Exploit as early as possible any simplifications afforded by an integrand being even or odd. This will more or less half the work required.

(2) When integrating by parts it is usually safer to write down the identity

$$[uv]_a^b = \int_a^b (uv)' \,\mathrm{d}x = \int_a^b uv' + u'v \,\mathrm{d}x$$

and make appropriate choices for $u$, $v$, $a$ and $b$, rather than doing the calculation in your head.

(3) Similarly, when integrating by parts twice it is usually quicker to write down the identity

$$\big[uv' - u'v\big]_a^b = \int_a^b (uv' - u'v)' \,\mathrm{d}x = \int_a^b uv'' - u''v \,\mathrm{d}x$$

and make appropriate choices for $u$, $v$, $a$ and $b$, rather than undertaking two sequential integration by parts.

(4) If $f$ is a piecewise exponential or trigonometric function, it is usually quicker to evaluate the complex integral expression

$$a_n + \mathrm{i}b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x.$$

(5) Beware of special cases: do not divide by zero. Such special cases sometimes arise for the same reasons that $m = n$ is a special case in the orthogonality relations.

(6) Check that $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$. This is a direct consequence of the *Riemann-Lebesgue Lemma*, which you will prove in Analysis III. Later on in this course we will be more precise about the rate of decay of the Fourier coefficients as $n \to \infty$.

## 2.5 Convergence of Fourier series

## Left- and right-hand limits

- **Definition:** The *RH limit of $f$ at $c$* is $f(c_+) = \lim\limits_{\substack{h\to 0 \\ h>0}} f(c+h)$ if it exists.
- **Definition:** The *LH limit of $f$ at $c$* is $f(c_-) = \lim\limits_{\substack{h\to 0 \\ h<0}} f(c+h)$ if it exists.
- **Notes:**

  (1) $f(c_+)$ can only exist if $f$ is defined on $(c, c+\epsilon)$ for some $\epsilon > 0$.

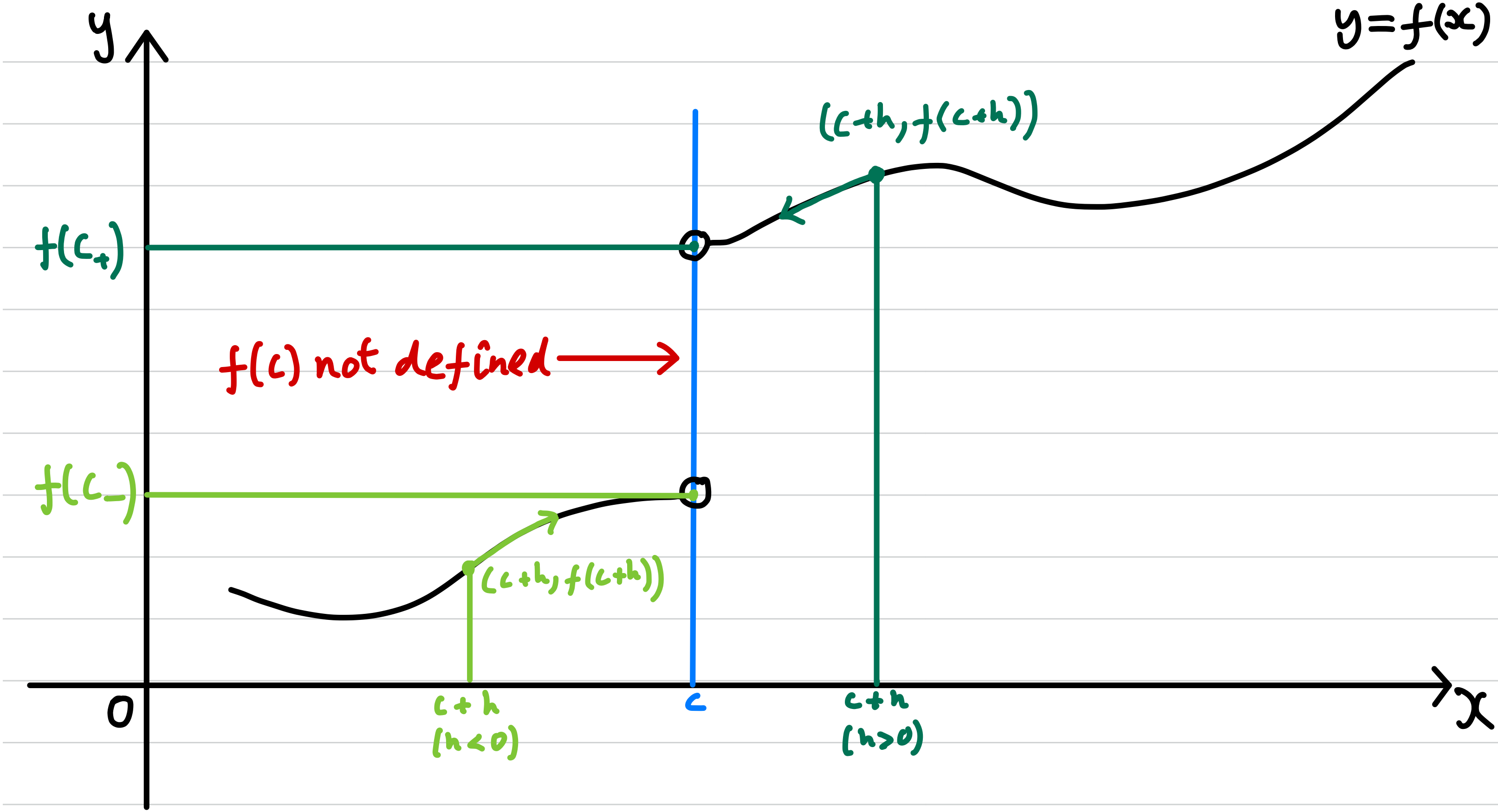
  (2) $f(c_-)$ can only exist if $f$ is defined on $(c-\epsilon, c)$ for some $\epsilon > 0$.

  (3) $f(c)$ need not be defined for $f(c_+)$ or $f(c_-)$ to exist. See sketch

  (4) The existence part is important, *e.g.* if $f(x) = \sin(1/x)$ for $x \neq 0$, then $f(0_\pm)$ do not exist.

  (5) $f$ is continuous at $c$ if and only if $f(c_-) = f(c) = f(c_+)$.

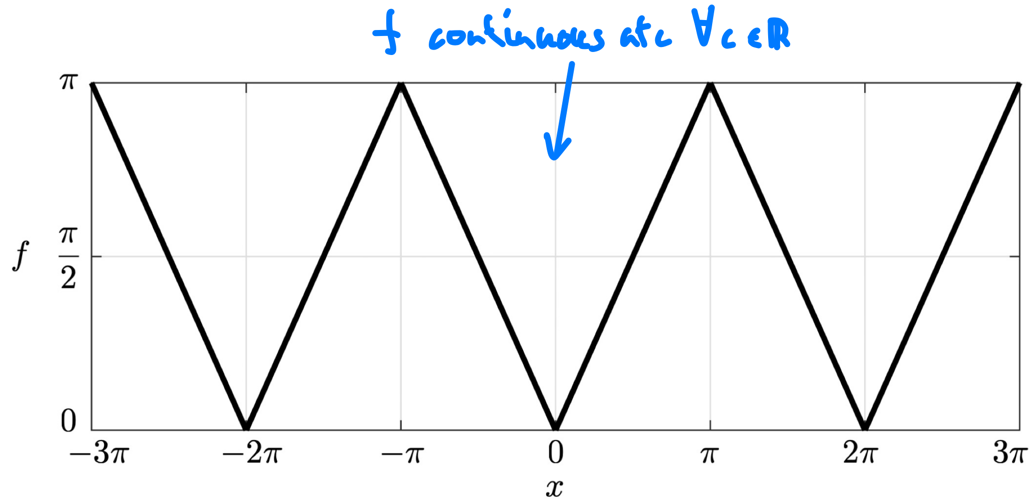
  (6) In Example 2, $f$ is continuous for $x/\pi \in \mathbb{R}\backslash\mathbb{Z}$ with $f(0_\pm) = \pm 1$ and $f(\pi_\pm) = \mp 1$. See sketch



y
y=f(x)
(c+h, f(c+h))
f(c+)
f(c) not defined
f(c−)
(c+h, f(c+h))
0
c+h
(h<0)
c
c+h
(h>0)
x

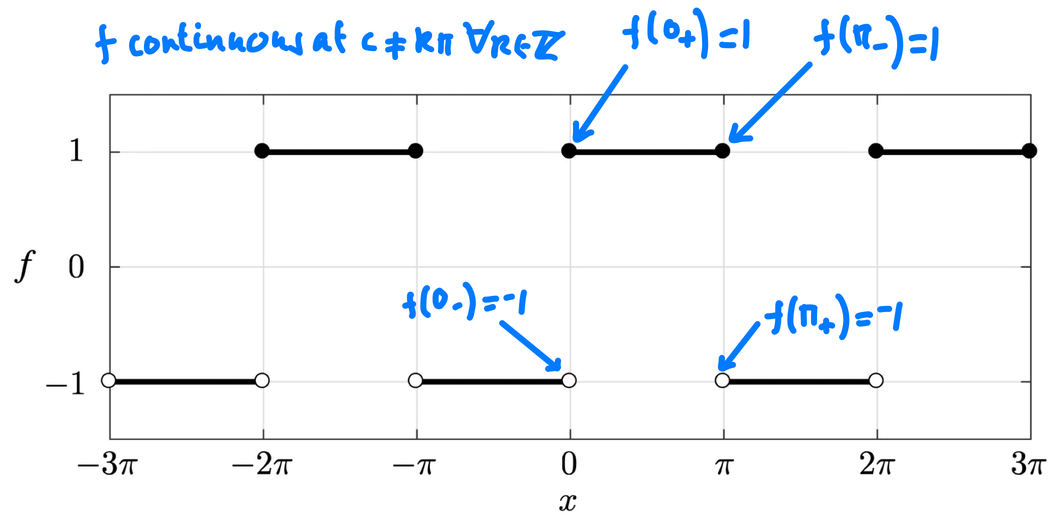
## Example 1:

$$f(x) = |x|$$
for $-\pi < x \leq \pi$




## Example 2:

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \pi \\ -1 & \text{for } -\pi < x < 0 \end{cases}$$



## Piecewise continuity

- **Definition:** $f$ is *piecewise continuous* on $(a, b) \subseteq \mathbb{R}$ if there exists a finite number of points $x_1, \ldots, x_m \in \mathbb{R}$ with $a = x_1 < x_2 < \ldots < x_m = b$ s.t.

  (1) $f$ is defined and continuous on $(x_k, x_{k+1})$ for all $k = 1, \ldots, m-1$;

  (2) $f(x_{k+})$ exists for $k = 1, \ldots, m-1$;

  (3) $f(x_{k-})$ exists for $k = 2, \ldots, m$.

- **Notes:**

  (1) Note that $f$ need not be defined at its exceptional points $x_1, \ldots, x_m$.

  (2) The functions in Examples 1 and 2 are piecewise continuous on any interval $(a, b) \subset \mathbb{R}$.

See sketch

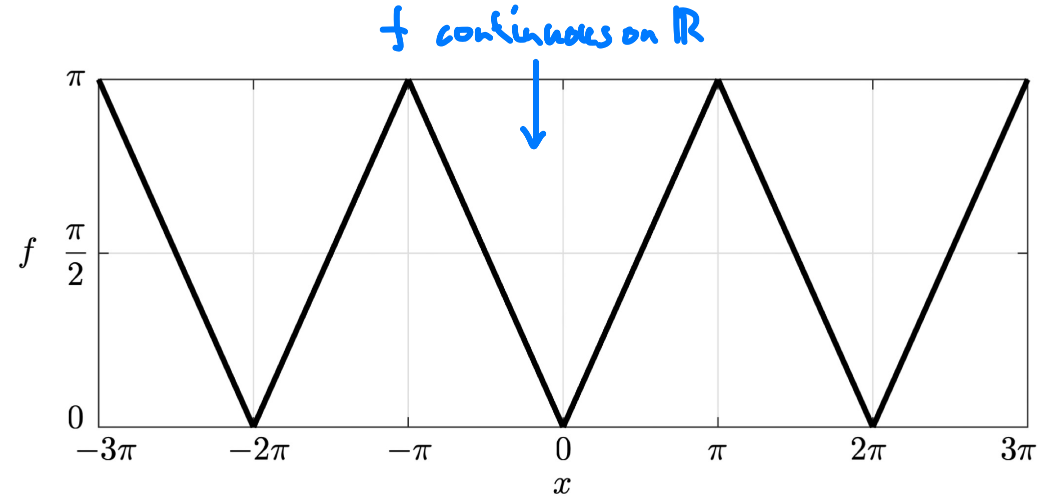
Eg:

## Example 1:

$f(x) = |x|$

for $-\pi < x \leq \pi$

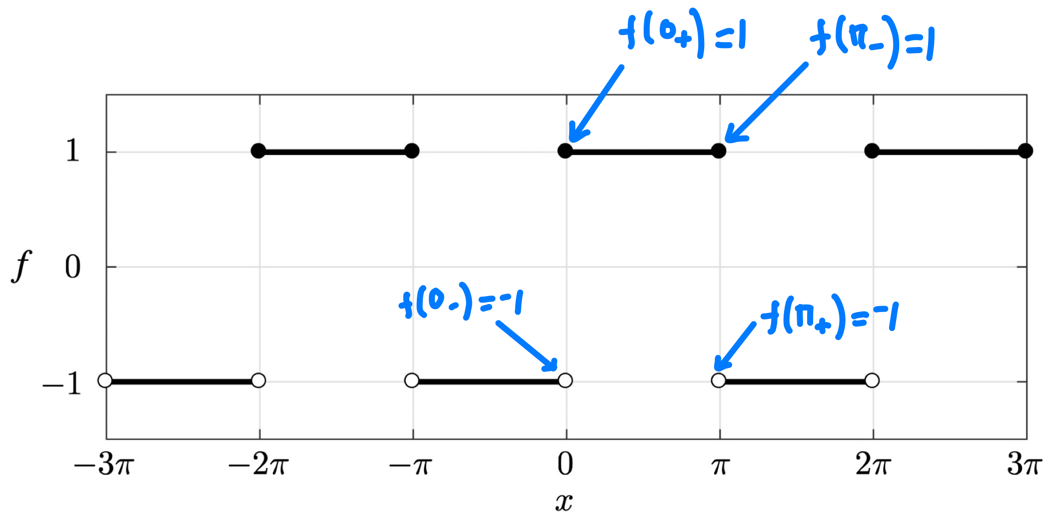
$\boxed{f \text{ piecewise continuous on any } (a,b) \subseteq \mathbb{R}}$




## Example 2:

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \pi \\ -1 & \text{for } -\pi < x < 0 \end{cases}$$

$\boxed{f \text{ piecewise continuous on any } (a,b) \subseteq \mathbb{R}}$



## Fourier Convergence Theorem

- Let $f : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic, with $f$ and $f'$ piecewise continuous on $(-\pi, \pi)$. Then the Fourier series of $f$ at $x$ converges to the value $\frac{1}{2}\big(f(x_+) + f(x_-)\big)$, *i.e.*

$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=0}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for } x \in \mathbb{R},$$

$n = 1$

where the Fourier coefficients $a_n$ and $b_n$ exist and are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, \mathrm{d}x \quad \text{for } n \in \mathbb{N},$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, \mathrm{d}x \quad \text{for } n \in \mathbb{N} \setminus \{0\}$$

Eg:


## Notes on the hypotheses

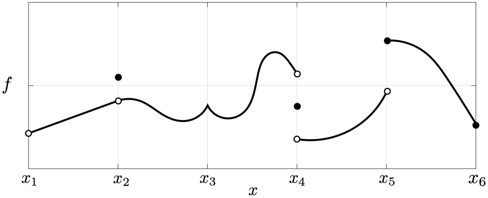
(1) If $f$ and $f'$ are piecewise continuous on $(-\pi, \pi)$, then there exist $x_1, \ldots, x_m \in \mathbb{R}$ with $-\pi = x_1 < x_2 < \ldots < x_m = \pi$ such that

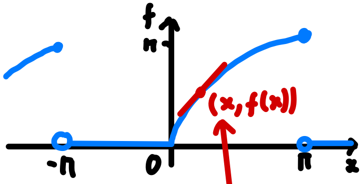
(i) $f$ and $f'$ are continuous on $(x_k, x_{k+1})$ for $k = 1, \ldots, m-1$.

(ii) $f(x_{k+})$ and $f'(x_{k+})$ exist for $k = 1, \ldots, m-1$.

(iii) $f(x_{k-})$ and $f'(x_{k-})$ exist for $k = 2, \ldots, m$.

(2) Thus, in any period $f, f'$ are continuous except possibly at a finite number of points. At each such point $f'$ need not be defined, and one or both of $f$ and $f'$ may have a jump discontinuity, as illustrated for some of the possibilities in the schematic below.






(3) For example, if

$$f(x) = \begin{cases} x^{1/2} & \text{for } 0 \le x \le \pi, \\ 0 & \text{for } -\pi < x < 0, \end{cases}$$

Slope tangent $f'(x) \to \infty$ as $x \to 0+$

then

$$f'(x) = \begin{cases} \frac{1}{2} x^{-1/2} & \text{for } 0 < x < \pi, \\ 0 & \text{for } -\pi < x < 0, \\ \text{undefined} & \text{for } x = 0, \pi. \end{cases}$$

Hence, while $f$ is piecewise continuous on $(-\pi, \pi)$, $f'$ is not because $f'(0_+)$ does not exist.

$\boxed{\text{NB: Examples 1 \& 2 in section 2.2 will be analyzed in detail.}}$

## Notes on the convergence result

(1) The partial sums of the Fourier series are defined for $N \in \mathbb{N}\backslash\{0\}$ by

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right) \quad \text{for } x \in \mathbb{R}.$$

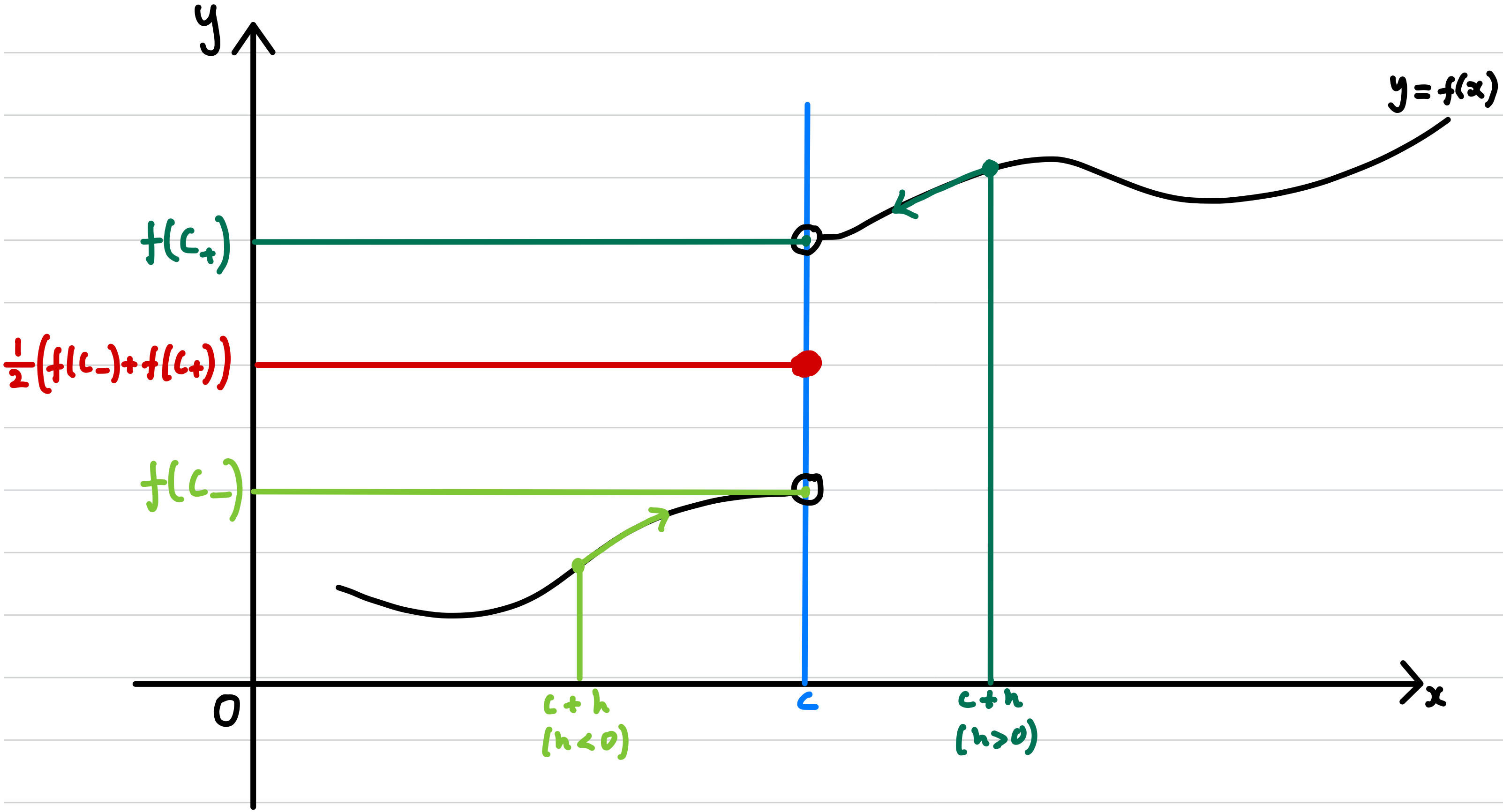
The theorem states that the partial sums converge pointwise in the sense that

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x_+) + f(x_-)\right) \quad \text{for } x \in \mathbb{R}.$$

See sketch

(2) If $f$ has a jump discontinuity at $x$, so that $f(x_+) \neq f(x_-)$, then the Fourier series converges to $\left(f(x_+) + f(x_-)\right)/2$, *i.e.* the average of the left- and right-hand limits of $f$ at $x$.

(3) If $f$ is continuous at $x$, then $f(x_-) = f(x) = f(x_+)$ and the Fourier series converges to $f(x)$.



y
y = f(x)
f(c+)
1/2(f(c−)+f(c+))
f(c−)
0
c + h
(h < 0)
c
c + h
(h > 0)
x

(4) If we redefined $f$ to be equal to the average of its left- and right-hand limits at each of its jump discontinuities, then the Fourier series would converge instead to $f$ on $\mathbb{R}$.

(5) If $f$ is defined only on *e.g.* $(-\pi, \pi]$, then the Fourier Convergence Theorem holds for its $2\pi$-periodic extension.

(6) The Fourier Convergence Theorem implies that

$$\begin{aligned}
\frac{1}{2}\big(g(x_+) + g(x_-)\big) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{for} \quad x \in \mathbb{R}, \\
\frac{1}{2}\big(h(x_+) + h(x_-)\big) &= \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{for} \quad x \in \mathbb{R},
\end{aligned}$$

where $g(x) = \frac{1}{2}\big(f(x) + f(-x)\big)$ is the even part of $f$ and $h(x) = \frac{1}{2}\big(f(x) - f(-x)\big)$ is the odd part of $f$.

Proof ~~not~~ examinable

## Notes on the proof

(1) Use the integral expressions for the Fourier coefficients and properties of periodic, even and odd functions to manipulate the partial sums into the form

$$S_N(x) - \frac{1}{2}\left(f(x_+) + f(x_-)\right) = \int_0^\pi F(x,t)\sin\left[\left(N+\frac{1}{2}\right)t\right]\mathrm{d}t,$$

where

$$F(x,t) = \frac{1}{\pi}\left(\frac{f(x+t)-f(x_+)}{t} + \frac{f(x-t)-f(x_-)}{t}\right)\left(\frac{t}{2\sin(t/2)}\right).$$

(2) Use the *Mean Value Theorem* (of Analysis II) to show that $F(x,t)$ is a piecewise continuous function of $t$ on $(0,\pi)$, and hence deduce from the *Riemann-Lebesgue Lemma* (of Analysis III) that

$$\int_0^\pi F(x,t)\sin\left[\left(N+\frac{1}{2}\right)t\right]\mathrm{d}t \to 0 \quad \text{as } N\to\infty.$$

## Notes on differentiability and integrability

(1) The Fourier series can be integrated termwise under weaker conditions, e.g. if $f$ is $2\pi$-periodic and piecewise continuous on $(-\pi, \pi)$, then the Fourier Convergence Theorem implies

$$\int_0^x f(s)\,\mathrm{d}s = \int_0^x \frac{1}{2} a_0\,\mathrm{d}s + \sum_{n=1}^{\infty} \int_0^x \left(a_n \cos(ns) + b_n \sin(ns)\right) \mathrm{d}s \quad \text{for } x \in \mathbb{R},$$

this function being $2\pi$-periodic if and only if $a_0 = 0$.

(2) However, we need stronger conditions to differentiate termwise, *e.g.* if $f$ is $2\pi$-periodic and continuous on $\mathbb{R}$ with both $f'$ and $f''$ piecewise continuous on $(-\pi, \pi)$, then the Fourier Convergence Theorem implies

$$\frac{1}{2}\left(f'(x_+) + f'(x_-)\right) = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(a_n \cos(nx) + b_n \sin(nx)\right) \quad \text{for } x \in \mathbb{R}.$$

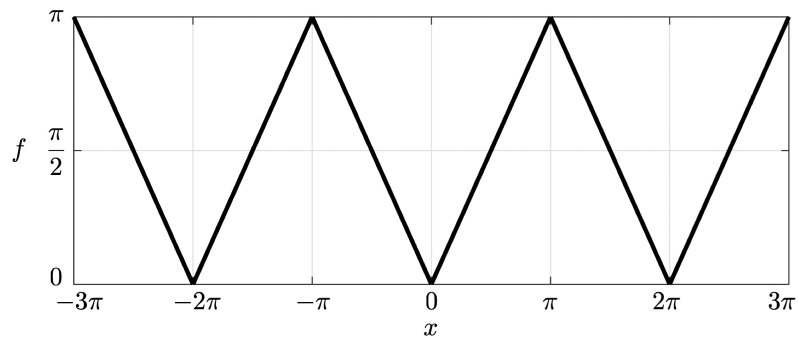
$\boxed{\text{Cf. Analysis I, Theorem 60}}$

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < R$, with $R > 0$,

then $f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(c_n x^n)$ for $|x| < R$.

# Example 1 revisited

Recall that $f$ is $2\pi$-periodic with $f(x) = |x|$ for $-\pi < x \leq \pi$




Recall that $f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}$

Qn: Can we apply the FCT?

Ans: Since $f$ is cts on $(-\pi,\pi)$ & $f(-\pi_+)$, $f(\pi_-)$ exist, $f$ is piecewise cts on $(-\pi,\pi)$.

$$f'(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ -1 & \text{for } -\pi < x < 0 \\ \text{undefined} & \text{for } x = 0, \pi \end{cases}$$

$\Rightarrow$ $f'$ is cts on $(-\pi,0)\cup(0,\pi)$ & $f'(-\pi_+), f'(0_-), f'(0_+), f'(\pi_-)$ exist.

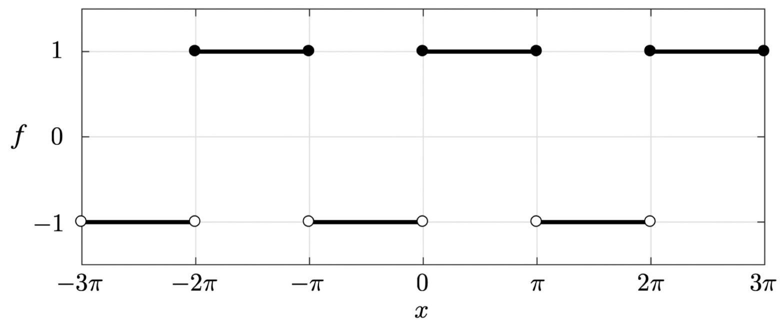
$\Rightarrow$ $f'$ is piecewise cts on $(-\pi,\pi)$.

Hence, the FCT applies and gives

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2} = f(x) \text{ for } x \in \mathbb{R}.$$

# Example 2 revisited

Recall that $f$ is $2\pi$-periodic with $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi, \\ -1 & \text{for } -\pi < x < 0. \end{cases}$




Recall that $f(x) \sim \dfrac{4}{\pi} \displaystyle\sum_{m=0}^{\infty} \dfrac{\sin(2m+1)x}{2m+1}.$

Q$^n$: Can we apply the FCT?

A$^{ns}$: $f$ cts on $[-\pi,0)\cup(0,\pi)$ & $f(-\pi_+)$, $f(0_-)$, $f(0_+)$, $f(\pi_-)$ exist
$\Rightarrow$ $f$ piecewise cts on $(-\pi,\pi)$.

$$f'(x) = \begin{cases} 0 & \text{for } 0<|x|<\pi \\ \text{undefined} & \text{for } x=0,\pi \end{cases}$$

$\Rightarrow f'$ cts on $[-\pi,0)\cup(0,\pi)$ & $f'(-\pi_+)$, $f'(0_-)$, $f'(0_+)$, $f'(\pi_-)$ exist
$\Rightarrow$ $f'$ piecewise cts on $(-\pi,\pi)$.

Hence, the FCT applies and gives

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{2m+1} = \begin{cases} f(x) & \text{for } x/\pi\in\mathbb{R}\setminus\mathbb{Z} \\ 0 & \text{for } \frac{x}{\pi}\in\mathbb{Z} \end{cases}$$

## Examples 1 and 2 revisited

- Recall the $2\pi$-periodic function of Example 1 which we defined by setting

$$f(x) = |x| \quad \text{for } -\pi < x \leq \pi.$$

- We calculate

$$f'(x) = \begin{cases} 1 & \text{for } 0 < x < \pi, \\ -1 & \text{for } -\pi < x < 0, \\ \text{undefined} & \text{for } x = 0,\ \pi. \end{cases}$$

- Since both $f$ and $f'$ are piecewise continuous on $(-\pi, \pi)$, with $f$ continuous on $\mathbb{R}$, the Fourier Convergence Theorem gives

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos\left((2m+1)x\right)}{(2m+1)^2} = f(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{A}$$

- Since $f$ is piecewise continuous on $(-\pi, \pi)$, we can integrate termwise to obtain

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin\left((2m+1)x\right)}{(2m+1)^3} = \int_0^x f(s) - \frac{\pi}{2}\, \mathrm{d}s \quad \text{for} \quad x \in \mathbb{R}. \tag{B}$$

- We calculate

$$f''(x) = \begin{cases} 0 & \text{for } 0 < x < \pi, \\ 0 & \text{for } -\pi < x < 0, \\ \text{undefined} & \text{for } x = 0,\ \pi. \end{cases}$$

- Since $f$ is continuous on $\mathbb{R}$ and both $f'$ and $f''$ are piecewise continuous on $(-\pi, \pi)$, we can differentiate termwise the Fourier series for $f$ to obtain

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin\left((2m+1)x\right)}{2m+1} = \frac{1}{2}\left(f'(x_-) + f'(x_+)\right) = \begin{cases} 1 & \text{for } 0 < x < \pi, \\ -1 & \text{for } -\pi < x < 0, \\ 0 & \text{for } x = 0,\ \pi. \end{cases} \tag{C}$$

- The function to which this Fourier series converges is equal to the function considered in Example 2 for $x/\pi \in \mathbb{R}\backslash\mathbb{Z}$, which deals thereby with the convergence and termwise integration of the Fourier series of that function; it remains to note that, since that function is not continuous on $\mathbb{R}$, its Fourier series cannot be differentiated termwise.

Covered material for Problem Sheet 2 Question 2