2.6 Rate of convergence

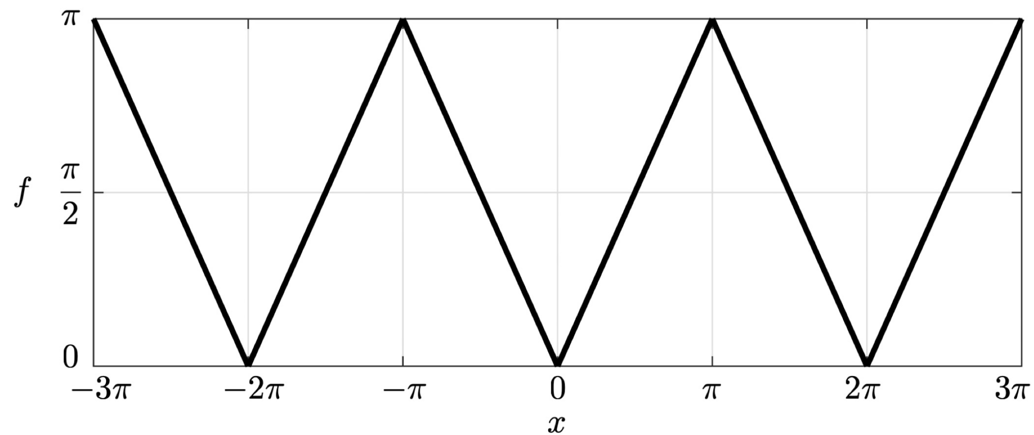
Rate of convergence

- The smoother f, *i.e.* the more continuous derivatives it has, the faster the convergence of the Fourier series for f.
- If the first jump discontinuity is in the p^{th} derivative of f, with the convention that $p = 0$ if there is a jump discontinuity in f, then in general the slowest decaying a_n and b_n decay like $1/n^{p+1}$ as $n \to \infty$.
- More specifically, if the first jump discontinuity is in the p^{th} derivative of the even part of f, then in general a_n decays like $1/n^{p+1}$ as $n \to \infty$; similarly, if the first jump discontinuity is in the p^{th} derivative of the odd part of f, then in general b_n decays like $1/n^{p+1}$ as $n \to \infty$.
- For example, $p = 1$ in (A), $p = 2$ in (B) and $p = 0$ in (C) in the previous example. See sketches

Example 1:

$f(x) = |x|$

for $-\pi < x \leq \pi$



$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2} = f(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{A}$$

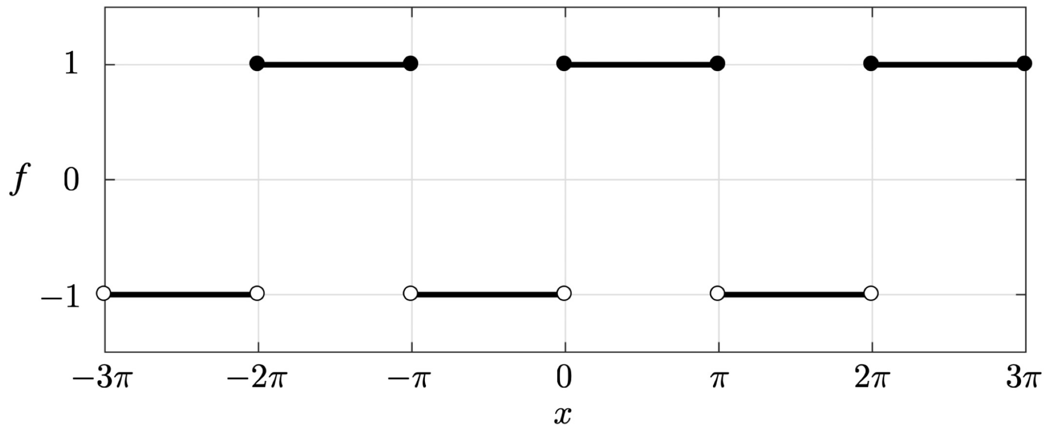
$p = 1$ Jump discontinuity in 1st derivative

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin((2m+1)x)}{(2m+1)^3} = \int_0^x f(s) - \frac{\pi}{2}\,\mathrm{d}s \quad \text{for} \quad x \in \mathbb{R}. \tag{B}$$

$p = 2$ Jump discontinuity in 2nd derivative

Example 2:

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ -1 & \text{for } -\pi < x < 0 \end{cases}$$



$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin\left((2m+1)x\right)}{2m+1} = \begin{cases} 1 & \text{for } 0 < x < \pi, \\ -1 & \text{for } -\pi < x < 0, \\ 0 & \text{for } x = 0,\ \pi. \end{cases} \qquad \text{(C)}$$

$p = 0$

Jump discontinuity in 0th derivative, i.e. in f

- This is an extremely useful result
  - in practice, *e.g.* for approximately 1% accuracy we need 100 terms for $p = 0$, but only 10 terms for $p = 1$;
  - for checking calculations, *e.g.* an erroneous contribution to a Fourier coefficient can be rapidly identified if it does not decay fast enough.
- We can understand the rate of decay as follows.
- Suppose f is such that

  (i) the first jump discontinuity is in the pth-derivative $f^{(p)}(x)$ with jumps at the exceptional points $x_1 < x_2 < \cdots < x_m$, where $x_1 \geq x_0 = -\pi$ and $x_m \leq x_{m+1} = \pi$.

  (ii) $f^{(p+1)}(x)$ is integrable on each of the intervals (x_k, x_{k+1}) for $k = 0, 1, \ldots, m$, which is often the case in practice.
- Then, repeated integration by parts gives . . . .

$$\pi(a_n+ib_n) = \int_{-\pi}^{\pi} f(x)\cos nx\,dx + i\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \int_{-\pi}^{\pi} f(x)e^{inx}\,dx$$

Claim: $$\pi(a_n+ib_n) = \frac{(-1)^p}{(in)^p}\int_{-\pi}^{\pi} f^{(p)}(x)e^{inx}\,dx$$

Pf: True for $p=0$ because $f^{(0)}=f$.

For $p \geq 1$, we use the recursion relation given by

$$\int_{-\pi}^{\pi} \underbrace{f^{(q)}(x)}_{u}\underbrace{e^{inx}}_{v'}\,dx = \Big[\underbrace{f^{(q)}(x)}_{u}\underbrace{\tfrac{1}{in}e^{inx}}_{v}\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \underbrace{f^{(q+1)}(x)}_{u'}\underbrace{\tfrac{1}{in}e^{inx}}_{\cancel{v'}\ \text{TYPO}}\,dx$$

$$= -\frac{1}{in}\int_{-\pi}^{\pi} f^{(q+1)}(x)e^{inx}\,dx$$

for $q = 0, 1, \ldots, p-1$, for which $f^{(q)}$ is 2π-periodic and cts on $\mathbb{R}$.

The identity then follows by recursion. $\square$

$$\pi(a_n + i b_n) = \frac{(-1)^p}{(in)^p}\int_{-\pi}^{\pi} f^{(p)}(x)\, e^{inx}\,dx$$

$$= \frac{(-1)^p}{(in)^p}\sum_{k=0}^{m}\int_{x_k}^{x_{k+1}} \underbrace{f^{(p)}(x)}_{u}\,\underbrace{e^{inx}}_{v'}\,dx$$

$$= \frac{(-1)^p}{(in)^p}\sum_{k=0}^{m}\left\{\left[f^{(p)}(x)\,\frac{1}{in}\,e^{inx}\right]_{(x_k)+}^{(x_{k+1})-} - \int_{x_k}^{x_{k+1}} f^{(p+1)}(x)\,\frac{1}{in}\,e^{inx}\,dx\right\}$$

$$= \underbrace{\frac{(-1)^p}{(in)^{p+1}}\sum_{k=0}^{m}\left[f^{(p)}(x)\,e^{inx}\right]_{(x_k)+}^{(x_{k+1})-}}_{\text{In general, bounded and non-zero as } n\to\infty,} + \underbrace{\frac{(-1)^{p+1}}{(in)^{p+1}}\sum_{k=0}^{m}\int_{x_k}^{x_{k+1}} f^{(p+1)}(x)\,e^{inx}\,dx}_{\to 0 \text{ as } n\to\infty \text{ by the Riemann-Lebesgue Lemma of Analysis III.}}$$

This explains the claimed rates of decay as $n \to \infty$.

$$
\begin{aligned}
\pi\big(a_n + \mathrm{i} b_n\big) &= \int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \\
&= \frac{1}{\mathrm{i}n}\left(\big[f(x)\mathrm{e}^{\mathrm{i}nx}\big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f^{(1)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x\right) \\
&= \frac{-1}{\mathrm{i}n}\int_{-\pi}^{\pi} f^{(1)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \\
&\;\vdots \\
&= \frac{(-1)^p}{(\mathrm{i}n)^p}\int_{-\pi}^{\pi} f^{(p)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \\
&= \frac{(-1)^p}{(\mathrm{i}n)^p}\sum_{k=0}^{m}\int_{x_k}^{x_{k+1}} f^{(p)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \\
&= \frac{(-1)^p}{(\mathrm{i}n)^{p+1}}\sum_{k=0}^{m}\left(\Big[f^{(p)}(x)\mathrm{e}^{\mathrm{i}nx}\Big]_{(x_k)_+}^{(x_{k+1})_-} - \int_{x_k}^{x_{k+1}} f^{(p+1)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x\right)
\end{aligned}
$$

for $p \geq 1$, though final result holds for $p = 0$ by skipping over the second and third equalities.

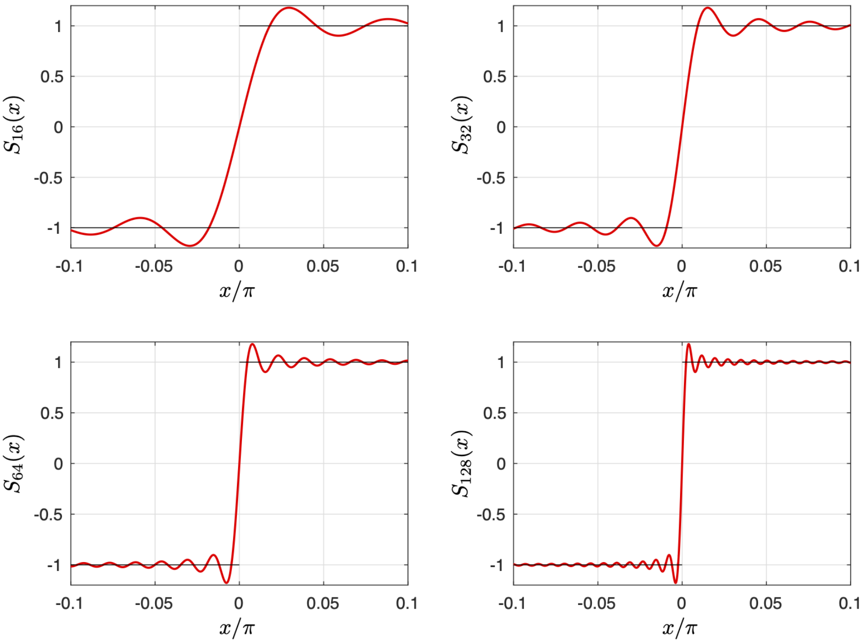
- While the Riemann-Lebesgue Lemma implies that each of the integrals in the sum tend to zero as $n \to \infty$, the pth-derivative $f^{(p)}(x)$ has jump discontinuities at the exceptional points, so in general each of the boundary contributions in the sum is bounded and does not decay as $n \to \infty$. Hence, we recover the claimed rate of decay.

- If the Fourier coefficients decay like $1/n^{p+1}$ as $n \to \infty$ with $p \geq 1$, then the *Weierstrass M-test* of Analysis II may be used to show that the Fourier series for f *converges uniformly* to f on any interval $(a, b) \subset \mathbb{R}$.

- If the Fourier coefficients decay like $1/n$ as $n \to \infty$ (so that $p = 0$), then the partial sums of the Fourier series for f do not converge uniformly on any interval containing a jump discontinuity. Remarkably, the form of the non-uniformity is universal for such functions, being characterized by Gibb's phenomenon, as we shall now describe.

2.7 Gibb's phenomenon

- *Gibb's phenomenon* is the persistent overshoot near a jump discontinuity that we first encountered in Example 2. It happens whenever there is a jump discontinuity.
- In the plots below of the partial sums from Example 2, we have zoomed into near the jump discontinuity at the origin to illustrate the so-called "ringing" nature of the overshoot as the number of terms in the partial sum is increased.



$S_{16}(x)$
$S_{32}(x)$
$S_{64}(x)$
$S_{128}(x)$
x/π
1
0.5
0
-0.5
-1
-0.1
-0.05
0
0.05
0.1

- More generally, as the number of terms in the partial sum tends to infinity:
  - the width of the overshoot region tends to zero by the Fourier Convergence Theorem;
  - it may be shown that the total height of the overshoot region approaches $\gamma|f(x_+) - f(x_-)|$, where

    $$\gamma = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin x}{x}\,\mathrm{d}x \approx 1.18,$$

    i.e. approximately a 9% overshoot top and bottom.
- The plots above illustrate the approach to this value, which is evidently awful for approximation purposes.

- Some geometric insight into the underlying cause of Gibb's phenomenon may be gleamed from the following manipulation of the partial sums of the Fourier series for f, which for positive integers N are defined by

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for } x \in \mathbb{R},$$

where in terms of a dummy variable t, the Fourier coefficients are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, \mathrm{d}t, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, \mathrm{d}t.$$

- Substituting these expressions into the partial sum and interchanging the orders of summation and integration gives

$$\begin{aligned} S_N(x) &= \int_{-\pi}^{\pi} f(t) \left(\frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{N} \big(\cos(nt) \cos(nx) + \sin(nt) \sin(nx)\big) \right) \mathrm{d}t \\ &= \int_{-\pi}^{\pi} f(t) \left(\frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{N} \cos\big(n(t-x)\big) \right) \mathrm{d}t. \end{aligned}$$

- Hence,

$$S_N(x) = \int_{-\pi}^{\pi} f(t) D_N(t-x)\,\mathrm{d}t, \tag{A}$$

where the function $D_N : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_N(t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\cos(nt) \quad \text{for } t \in \mathbb{R}. \tag{B}$$

- The integral in (A) is a *convolution integral* giving the mean of the function $f(t)$ over a period weighted by the *Dirichlet kernel* $D_N(t-x)$. Since D_N does not depend on f it encodes the operation of taking a partial sum of a Fourier series.

- It follows from (B) that D_N is an even 2π-periodic function that is infinitely differentiable on $\mathbb{R}$ and has integral over a period equal to unity, *i.e.*

$$\int_{-\pi}^{\pi} D_N(t)\,\mathrm{d}t = 1. \tag{C}$$

- Using a trigonometric identity we compute

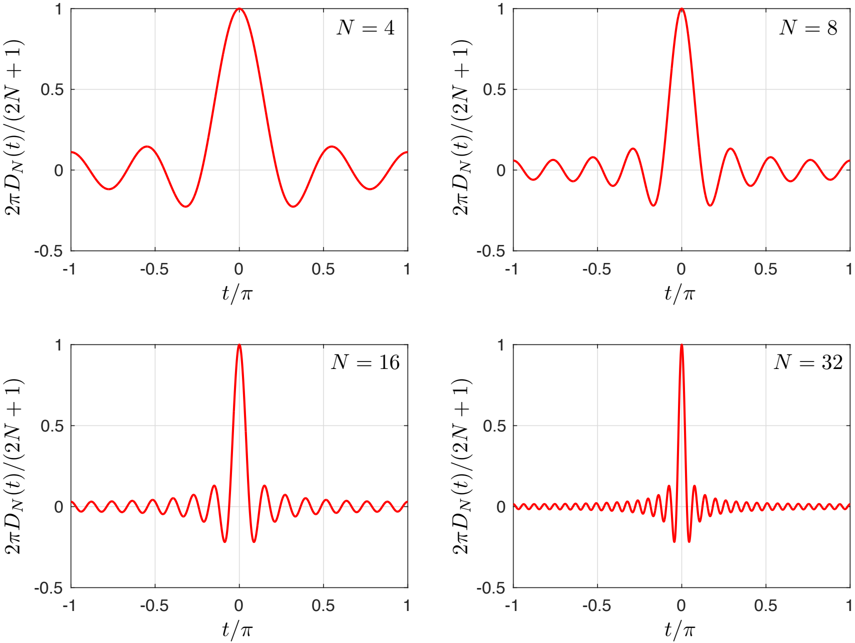
$$\begin{aligned}
2\pi \sin(t/2) D_N(t) &= \sin(t/2) + \sum_{n=1}^{N} 2\cos(nt)\sin(t/2) \\
&= \sin(t/2) + \sum_{n=1}^{N} \Big(\sin\big((n+1/2)t\big) - \sin\big((n-1/2)t\big) \Big) \\
&= \sin\big((N+1/2)t\big),
\end{aligned}$$

  the last equality following from the fact that the preceding sum is telescoping.

- Hence,

$$D_N(t) = \begin{cases} \dfrac{\sin\big((N+1/2)t\big)}{2\pi \sin(t/2)} & \text{for } \dfrac{t}{2\pi} \in \mathbb{R}\backslash\mathbb{Z}. \\ \dfrac{2N+1}{2\pi} & \text{for } \dfrac{t}{2\pi} \in \mathbb{Z}. \end{cases}$$

- We plot below the graph of D_N for $N = 4$, 8, 16 and 32, illustrating that as $N \to \infty$ the main contribution of the integrand in (C) comes from the central lobe that lies above the interval $[-\pi, \pi]/(N+1/2)$.



$N = 4$
$N = 8$
$N = 16$
$N = 32$
$2\pi D_N(t)/(2N+1)$
t/π
1
0.5
0
-0.5
-1
-0.5
0
0.5
1

- When x nears a jump discontinuity of f, it is the interaction of this jump and the rapidly oscillating Dirichlet kernel $D_N(t-x)$ around its dominant central lobe in the convoluton integral

$$S_N(x) = \int_{-\pi}^{\pi} f(t) D_N(t-x)\,\mathrm{d}t$$

that results in Gibb's phenomenon or the so-called "ringing of the partial sums," with the structure of the central lobe causing the 9% overshoot as $N \to \infty$.

- There are ways of mitigating against Gibb's phenomenon, *e.g.* it is eliminated in the *Fejér* series whose Mth-partial sum $F_M(x)$ is equal to the arithmetic mean of the first M partial sums of a Fourier series, *viz.*

$$F_M(x) = \frac{1}{M} \sum_{N=1}^{M} S_N(x) \quad \text{for } x \in \mathbb{R}.$$

However, they are beyond the scope of this course.

2.8 Functions of any period

Suppose $f(x)$ is $2L$-periodic, where $L > 0$.

If we let $X = \frac{\pi x}{L}$, then X increases by 2π when x increases by $2L$.

Hence, if we define ~~the~~ $g : \mathbb{R} \to \mathbb{R}$ by setting $g(X) = f(x)$, then g is 2π-periodic.

TYPO

Pf: Let $X \in \mathbb{R}$, then

$$g(X+2\pi) = f\left(\frac{L}{\pi}(X+2\pi)\right) \quad \left(\text{by defn: } g(X) = f\left(\frac{LX}{\pi}\right)\right)$$

$$= f\left(\frac{LX}{\pi} + 2L\right)$$

$$= f\left(\frac{LX}{\pi}\right) \quad (f\ 2L\text{-periodic})$$

$$= g(X) \quad (\text{by defn}) \quad \square$$

Hence, we can derive the theory of Fourier series for f from that for g!

Suppose $g(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ①

so that the Fourier coefficients exist and are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx \, dx$$ ②

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin nx \, dx$$ ③

Qn: What is the corresponding Fourier series for f?

Ans: Since $g(x) = f(x)$ and $x = \frac{\pi x}{L}$, ① implies

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx \, dx \qquad (\text{by } ②)$$

$$= \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\frac{\pi}{L}\,dx \qquad \left(x = \frac{\pi x}{L},\ g(x) = f(x)\right)$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Similarly, by ③

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

- Suppose now $f : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $2L$, where $L > 0$.
- We want to develop the analogous results for the Fourier series for $f(x)$.
- Since this will involve a series in the trigonometric functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$, where n is a positive integer, we make the transformation

$$x = \frac{LX}{\pi}, \quad f(x) = g(X)$$

  which defines a new function $g : \mathbb{R} \to \mathbb{R}$.
- It follows that, for $X \in \mathbb{R}$,

$$g(X + 2\pi) = f\left(\frac{L}{\pi}(X + 2\pi)\right) = f\left(\frac{LX}{\pi} + 2L\right) = f\left(\frac{LX}{\pi}\right) = g(X),$$

  where we used the fact that $g(X) = f(LX/\pi)$ in the first equality and the fact that f is $2L$-periodic in the third equality.
- Hence, g is periodic with period 2π, and we can therefore use the transformation to derive the Fourier theory for f from that for g.

- In particular, suppose we can write

$$g(X) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nX) + b_n \sin(nX)\big)$$

so that the Fourier coefficients a_n and b_n exist.

- Then

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(X) \cos(nX) \, \mathrm{d}X = \frac{1}{\pi} \int_{-L}^{L} g\left(\frac{\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \frac{\pi}{L} \, \mathrm{d}x = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x,$$

where we used $X = \pi x/L$ in the first equality and $g(\pi x/L) = f(x)$ in the second.

- Similarly,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(X) \sin(nX) \, \mathrm{d}X = \frac{1}{\pi} \int_{-L}^{L} g\left(\frac{\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \frac{\pi}{L} \, \mathrm{d}x = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x.$$

- So if we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

  then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x.$$

- We wrap these formal calculations into the definition of the Fourier series for f.

- **Definition:** Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic and integrable on $[-L, L]$. Then, regardless of whether or not it converges, the *Fourier series* for f is defined to be the infinite series given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

  for $x \in \mathbb{R}$, where the *Fourier coefficients* of f are given by

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x \qquad (n \in \mathbb{N}), \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x \qquad (n \in \mathbb{N} \setminus \{0\}). \end{aligned}$$

- **Remark:** The formulae for the Fourier coefficients may also be derived from the Fourier series for f by assuming that the orders of summation and integration may be interchanged and using the orthogonality relations

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = L\delta_{mn},$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 0,$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = L\delta_{mn},$$

  where $n, m \in \mathbb{N} \setminus \{0\}$ and δ_{mn} is Kronecker's delta.

Fourier Convergence Theorem

- Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic, with f and f' piecewise continuous on $(-L, L)$. Then the Fourier series of f at x converges to the value $\frac{1}{2}\big(f(x_+) + f(x_-)\big)$, *i.e.*

$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \quad \text{for} \quad x \in \mathbb{R},$$

where the Fourier coefficients a_n and b_n exist and are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad \text{for } n \in \mathbb{N},$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad \text{for } n \in \mathbb{N}\setminus\{0\}.$$

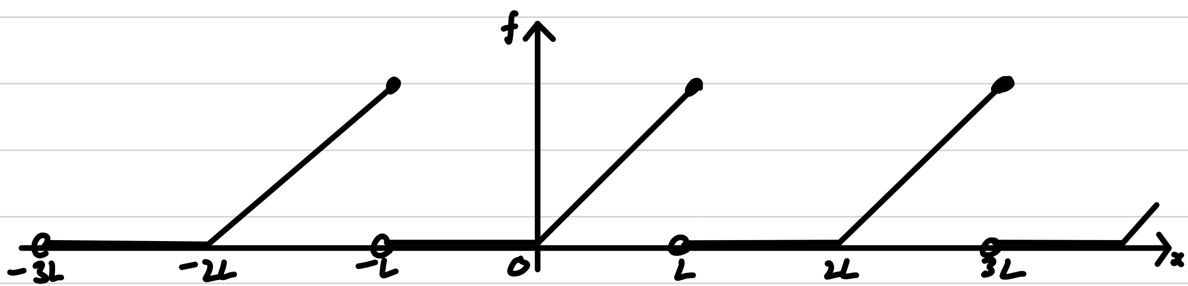
Covered material for Problem Sheet 3 Question 1

Example 3

Let f be the $2L$-periodic function defined by $f(x) = \begin{cases} x & \text{for } 0 \le x \le L, \\ 0 & \text{for } -L < x < 0. \end{cases}$

Find the Fourier series for f and the function to which it converges.

Sketch :



Fourier coefficients : $a_n = \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$, $b_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$

$$a_n + ib_n = \frac{1}{L}\int_0^L \underbrace{x}_{u}\,\underbrace{e^{in\pi x/L}}_{v'}\,dx$$

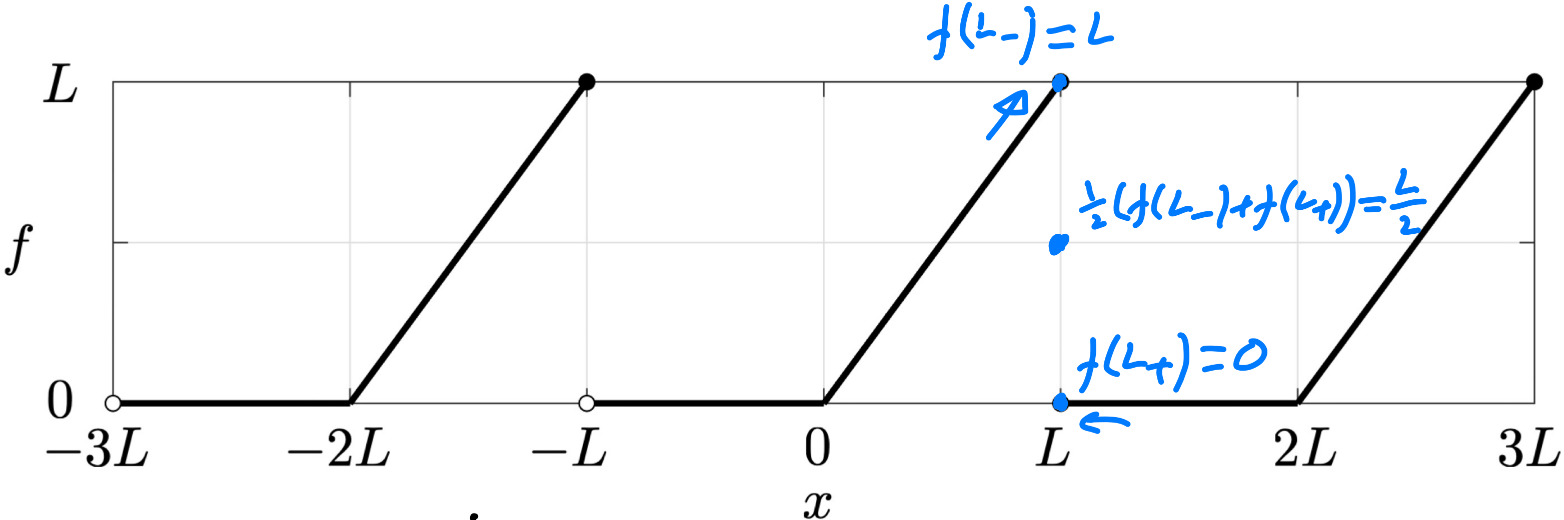
$$= \frac{1}{L}\left[\underbrace{x}_{u}\,\underbrace{\frac{L}{in\pi}e^{in\pi x/L}}_{v}\right]_0^L - \frac{1}{L}\int_0^L \underbrace{1}_{u'}\cdot\underbrace{\frac{L}{in\pi}e^{in\pi x/L}}_{v}\,dx$$

$$= \frac{1}{L}\,\frac{L^2}{in\pi}e^{in\pi} - \frac{1}{L}\left(\frac{L}{in\pi}\right)^2\left[e^{in\pi x/L}\right]_0^L$$

$$= \frac{L}{n^2\pi^2}\left((-1)^n - 1\right) - \frac{iL}{n\pi}(-1)^n \qquad (\text{for } n \geq 1).$$

Also $a_0 = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{L}{2}$.

Hence, $$f(x) \sim \frac{L}{4} + \sum_{n=1}^{\infty}\left\{\frac{L((-1)^n - 1)}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right) + \frac{L(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right\}$$



Since f is piecewise linear, both f and f' are piecewise cts on $(-L, L)$.

Hence, the Fourier convergence theorem applies and gives that the Fourier series for f converges to

- $f(x)$ at points of cty of f, i.e. at $x \neq (2k+1)L$ for $k \in \mathbb{Z}$
- $\frac{L}{2}$ at points of discontinuity, i.e. at $x = (2k+1)L$ for $k \in \mathbb{Z}$.

$\square$

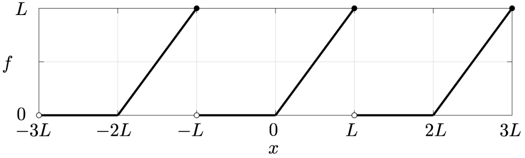
Example 3

- Consider the $2L$-periodic function f defined by

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq L, \\ 0 & \text{for } -L < x < 0. \end{cases}$$

Find the Fourier series for f and the function to which the Fourier series converges.

- The plot of the graph of f shows that it is piecewise linear with corners as $x = 2kL$ for $k \in \mathbb{Z}$ and jump discontinuities at $x = (2k+1)L$ for $k \in \mathbb{Z}$.



- By the definition of f, the Fourier coefficients are given by

$$a_n = \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad b_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

- A direct integration gives $a_0 = L/2$, but for $n \geq 1$ it is a bit quicker to evaluate

$$a_n + ib_n = \frac{1}{L}\int_0^L \underbrace{x}_{u}\underbrace{\exp\left(\frac{in\pi x}{L}\right)}_{v'}\,\mathrm{d}x$$

$$= \left[\frac{1}{L}\underbrace{x}_{u}\underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\right]_0^L - \frac{1}{L}\int_0^L \underbrace{1}_{u'}\underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\,\mathrm{d}x$$

$$= -\left[\frac{1}{L}\left(\frac{L}{in\pi}\right)^2 \exp\left(\frac{in\pi x}{L}\right)\right]_0^L + \frac{L}{in\pi}\exp\left(in\pi\right)$$

$$= \frac{L}{n^2\pi^2}\left((-1)^n - 1\right) + \frac{iL(-1)^{n+1}}{n\pi}.$$

- Hence,

$$f(x) \sim \frac{L}{4} + \sum_{m=1}^{\infty} \left(-\frac{2L}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{L}\right) + \frac{L(-1)^{m+1}}{m\pi} \sin\left(\frac{m\pi x}{L}\right) \right).$$

- Since f and f' are piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f converges to
  - $f(x)$ at points of continuity of f, *i.e.* for $x \neq (2k+1)L$, $k \in \mathbb{Z}$;
  - to the average of the left- and right-hand limits of f at the jump discontinuities, *i.e.* to $\big(f(L_+) + f(L_-)\big)/2 = (0 + L)/2 = L/2$ at $x = L$ and hence at $x = (2k+1)L$, $k \in \mathbb{Z}$ by periodicity. ∎

Notes:

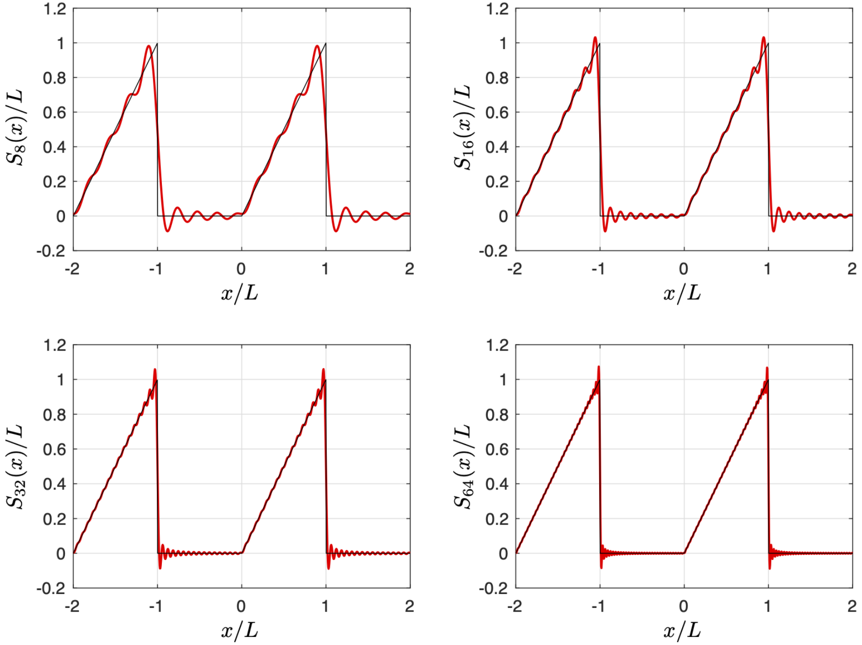
(1) The slowest decaying Fourier coefficients b_n decay as expected like $1/n$ as $n \to \infty$ because f has jump discontinuities so that $p = 0$.

(2) The partial sums of the Fourier series for f may be defined for positive integers N by

$$S_N(x) = \frac{L}{4} + \sum_{m=1}^{N} \left(-\frac{2L}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{L}\right) + \frac{L(-1)^{m+1}}{m\pi} \sin\left(\frac{m\pi x}{L}\right)\right) \quad \text{for } x \in \mathbb{R}.$$

We plot below the partial sums for $N = 8$, 16, 32 and 64, which illustrates that the slow convergence away from the jump discontinuities of f is hindered by Gibb's phenomenon.

$\boxed{\text{Covered material for Problem Sheet 3 Question 2}}$


$S_8(x)/L$
$S_{16}(x)/L$
$S_{32}(x)/L$
$S_{64}(x)/L$
x/L
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-2
-1
0
1
2

2.9 Half-range series

- In many practical applications we wish to express a given function $f : [0, L] \to \mathbb{R}$ in terms of either a Fourier cosine series or a Fourier sine series.
- This may be accomplished by extending f to be even (for only cosine terms) or odd (for only sine terms) on $(-L, 0) \cup (0, L)$ and then extending to a periodic function of period $2L$.
- We wrap these extensions and the corresponding Fourier series into the following definitions.

- **Definition:** The *even* $2L$*-periodic extension* $f_e : \mathbb{R} \to \mathbb{R}$ of $f : [0, L] \to \mathbb{R}$ is defined by

$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L, \\ f(-x) & \text{for } -L < x < 0, \end{cases}$$

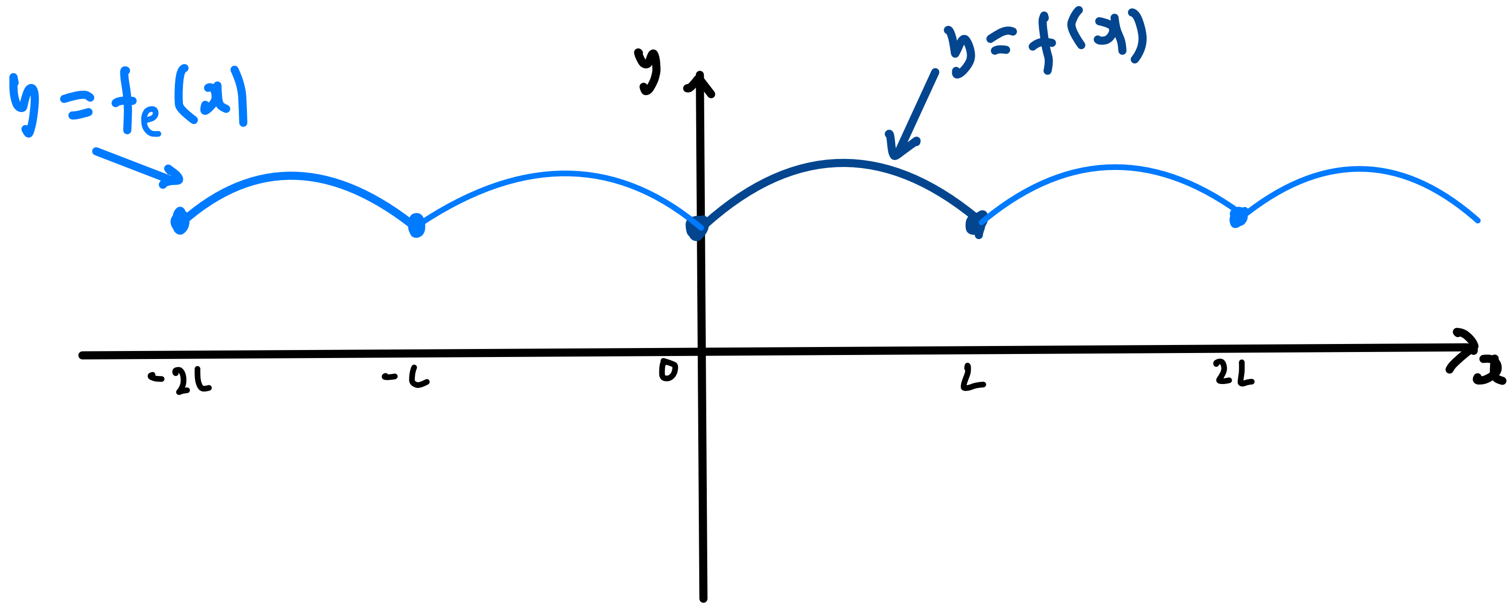
with $f_e(x + 2L) = f_e(x)$ for $x \in \mathbb{R}$. The *Fourier cosine series* for $f : [0, L] \to \mathbb{R}$ is the Fourier series for f_e, *i.e.*

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad \text{for } n \in \mathbb{N}.$$

See sketch



y = f_e(x)
y = f(x)
y
x
-2L
-L
0
L
2L

- **Definition:** The *odd* $2L$*-periodic extension* $f_o : \mathbb{R} \to \mathbb{R}$ of $f : [0, L] \to \mathbb{R}$ is defined by

$$f_o(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L, \\ -f(-x) & \text{for } -L < x < 0, \end{cases}$$

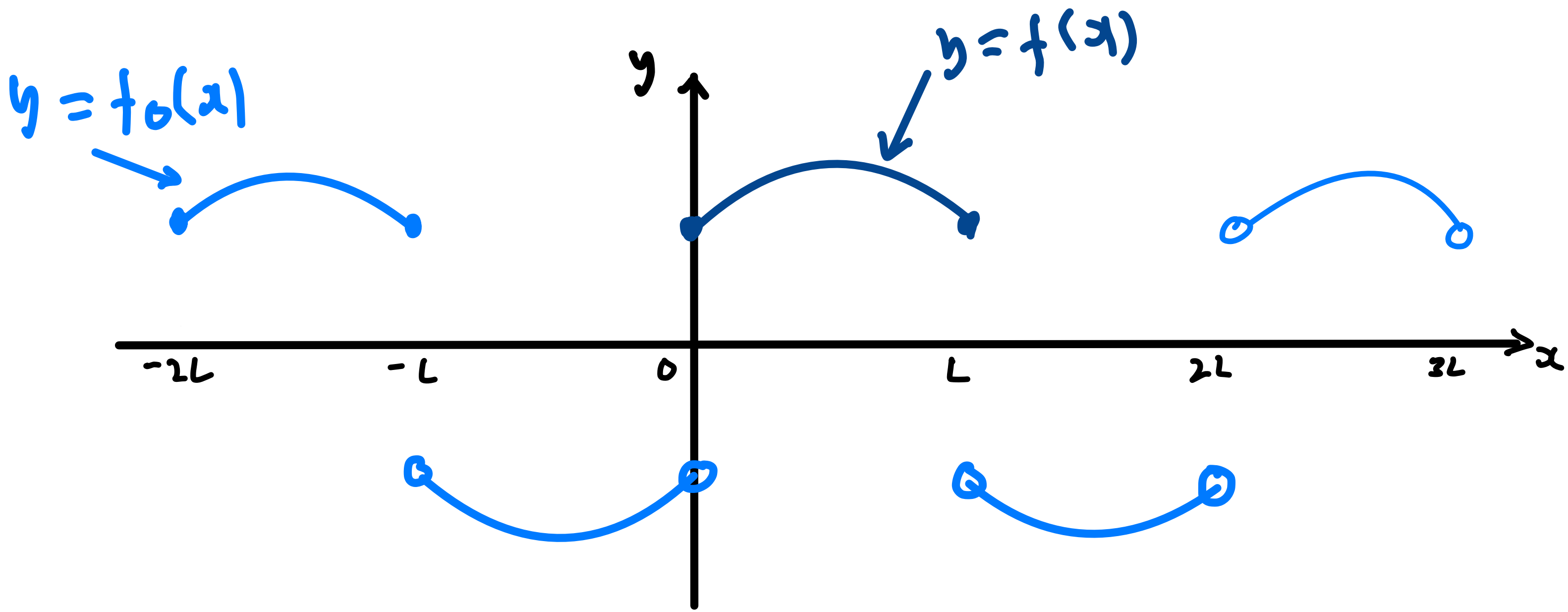
with $f_o(x + 2L) = f_o(x)$ for $x \in \mathbb{R}$. The *Fourier sine series* for $f : [0, L] \to \mathbb{R}$ is the Fourier series for f_o, *i.e.*

$$f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{1}{L} \int_{-L}^{L} f_o(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad \text{for } n \in \mathbb{N}\backslash\{0\}.$$

See sketch



- Odd extension is odd for $x/L \in \mathbb{R} \setminus \mathbb{Z}$
- Odd extension is odd on $\mathbb{R}$ iff $f(0) = f(L) = 0$

Notes:

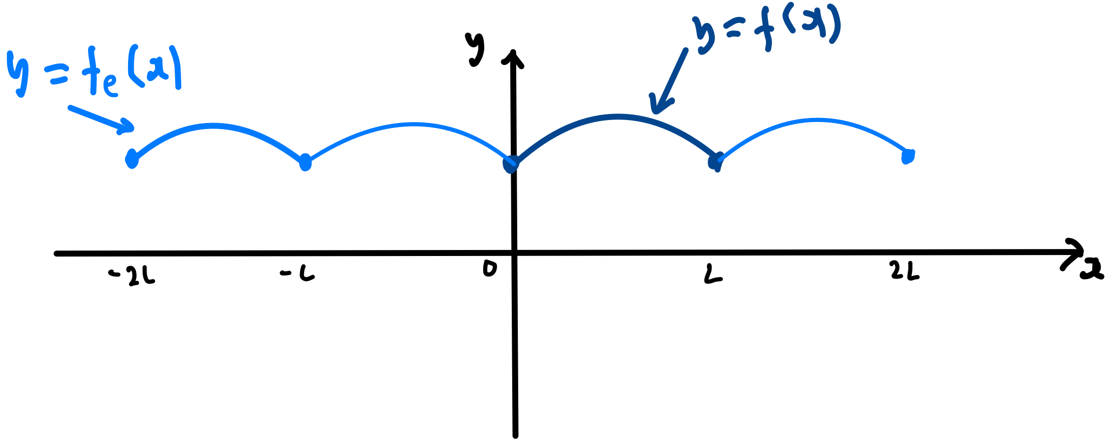
(1) $f_o(x)$ is odd for $x/L \in \mathbb{R}\backslash\mathbb{Z}$ and odd (on $\mathbb{R}$) if and only if $f(0) = f(L) = 0$.

(2) If f is continuous on $[0, L]$ and f' piecewise continuous on $(0, L)$, then the Fourier Convergence Theorem implies that

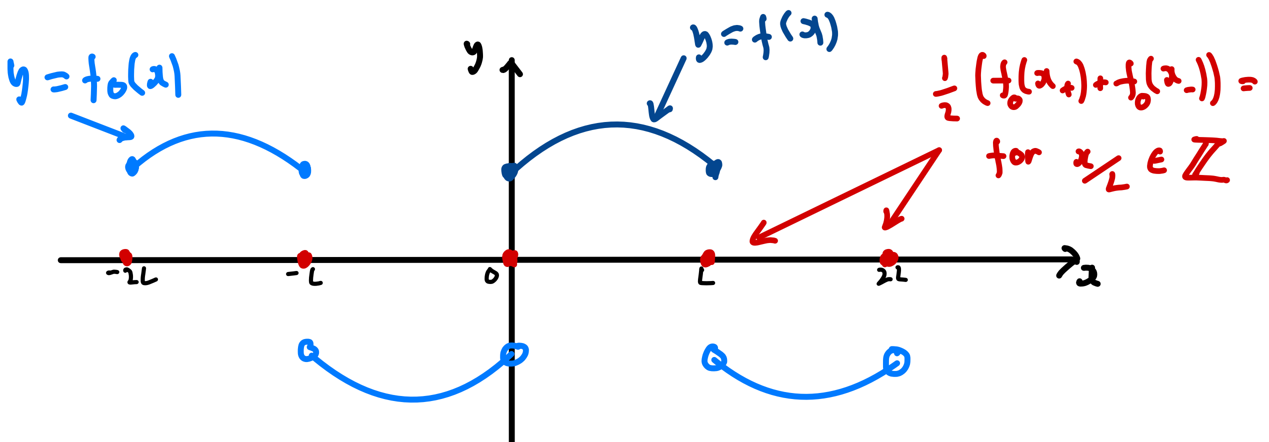
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f_e(x) \text{ for } x \in \mathbb{R},$$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} f_o(x) & \text{for } x/L \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{for } x/L \in \mathbb{R}\backslash\mathbb{Z}. \end{cases}$$

See sketch



$y = f_e(x)$
$y = f(x)$
y
x
$-2L$
$-L$
0
L
$2L$



$y = f_o(x)$
$y = f(x)$
y
x
$-2L$
$-L$
0
L
$2L$
$\frac{1}{2}\left(f_o(x_+) + f_o(x_-)\right) = 0$
for $x/L \in \mathbb{Z}$

Example 4

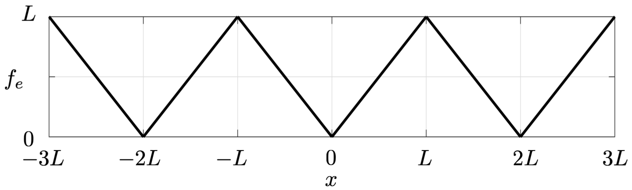
- Consider the function $f : [0, L] \to \mathbb{R}$ defined by $f(x) = x$ for $0 \le x \le L$. Find the Fourier cosine and sine series for f and the functions to which each of them converge on $[0, L]$. Which truncated series gives the best approximation to f on $[0, L]$?

- The even $2L$-periodic extension f_e is defined by

$$f_e(x) = \begin{cases} x & \text{for } 0 \le x \le L, \\ -x & \text{for } -L < x < 0, \end{cases}$$

i.e. $f_e(x) = |x|$ for $-L < x \le L$, with $f_e(x + 2L) = f_e(x)$ for $x \in \mathbb{R}$.

- The plot of the graph of f_e shows that it has a "sawtooth" profile that is piecewise linear and continuous, with corners at integer multiples of L.



- Since f_e is even, we have $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L x \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

- Evaluating this integral as in Example 3 gives the Fourier cosine series

$$f_e(x) \sim \frac{L}{2} - \sum_{m=0}^{\infty} \frac{4L}{(2m+1)^2\pi^2} \cos\left(\frac{(2m+1)\pi x}{L}\right).$$

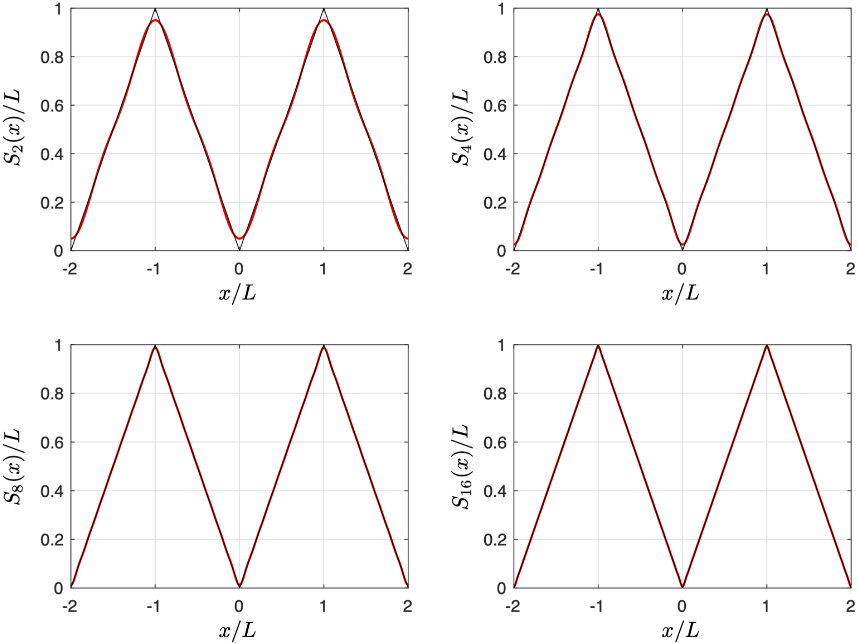
- Since f_e is continuous on $\mathbb{R}$ and f_e' is piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f_e converges to f_e on $\mathbb{R}$.

- Hence the Fourier cosine series for f converges to f on $[0, L]$.

- The partial sums of the Fourier series for f_e may be defined for $N \in \mathbb{N}$ by

$$S_N(x) = \frac{L}{2} - \sum_{m=0}^{N} \frac{4L}{(2m+1)^2\pi^2} \cos\left(\frac{(2m+1)\pi x}{L}\right) \quad \text{for } x \in \mathbb{R}.$$

  We plot below the partial sums for $N = 2$, 4, 8 and 16, which illustrates their rapid convergence to f_e.

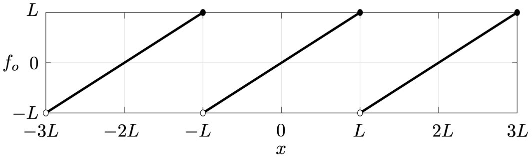

$S_2(x)/L$
x/L
$S_4(x)/L$
x/L
$S_8(x)/L$
x/L
$S_{16}(x)/L$
x/L
1
0.8
0.6
0.4
0.2
0
-2
-1
0
1
2

- Similarly, the odd $2L$-periodic extension f_o is defined by

$$f_o(x) = \begin{cases} x & \text{for } 0 \le x \le L, \\ -(-x) & \text{for } -L < x < 0, \end{cases}$$

i.e. $f_o(x) = x$ for $-L < x \le L$, with $f_o(x + 2L) = f_o(x)$ for $x \in \mathbb{R}$.

- The plot of the graph of f_0 shows that it is piecewise linear with jump discontinuities at $x = (2k+1)L$ for $k \in \mathbb{Z}$.



- Since f_o is odd, we have $a_n = 0$ and

$$b_n = \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

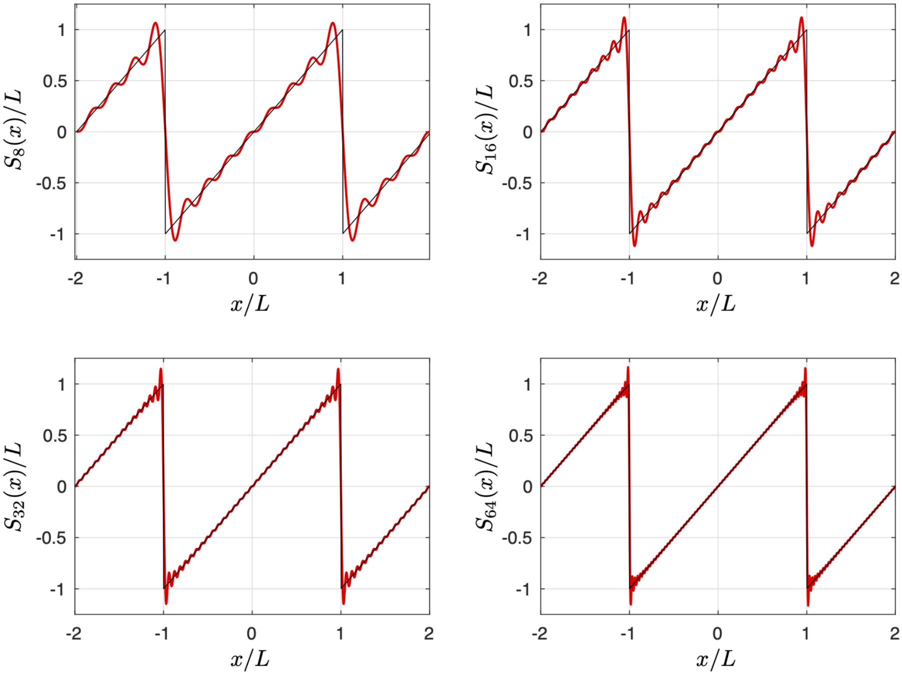
- Evaluating this integral as in Example 3 gives the Fourier sine series

$$f_o(x) \sim \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{L}\right).$$

- Since f_o and f_o' are piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f_o converges to
  - $f_o(x)$ at points of continuity of f_o, *i.e.* for $x \neq (2k+1)L$, $k \in \mathbb{Z}$;
  - the average of the left- and right-hand limits of f_o at its jump discontinuities, *i.e.* to $\big(f(L_+) + f(L_-)\big)/2 = (-L + L)/2 = 0$ for $x = L$ and hence for $x = (2k+1)L$, $k \in \mathbb{Z}$ by periodicity.
- Hence, the Fourier sine series for f converges to $f(x)$ for $0 \leq x < L$, but to 0 for $x = L$.
- The partial sums of the Fourier series for f_o may be defined for positive integers N by

$$S_N(x) = \sum_{n=1}^{N} \frac{2L(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } x \in \mathbb{R}.$$

- We plot below the partial sums for $N = 8$, 16, 32 and 64, which illustrates that the slow convergence away from the jump discontinuities of f_0 is hindered by Gibb's phenomenon.


$S_8(x)/L$
x/L
$S_{16}(x)/L$
x/L
$S_{32}(x)/L$
x/L
$S_{64}(x)/L$
x/L
1
0.5
0
-0.5
-1
-2
-1
0
1
2

- The truncated cosine series gives a better approximation to f on $[0, L]$ than the truncated sine series because
  (1) it converges everywhere on $[0, L]$;
  (2) it converges more rapidly;
  (3) it does not exhibit Gibb's phenomenon. ■

Remark

- Let f_3 denote twice the function in Example 3, so that

$$f_3(x) \sim \frac{L}{2} - \sum_{m=1}^{\infty} \frac{4L}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{L}\right) + \sum_{m=1}^{\infty} \frac{2L(-1)^{m+1}}{m\pi} \sin\left(\frac{m\pi x}{L}\right).$$

- **Question:** Why is the Fourier series for f_3 equal to the sum of the Fourier series for f_e and f_o?
- **Answer:** Because f_e is the even part of f_3 and f_o the odd part of f_3.
- This explains the rate of decay of the Fourier coefficients in Example 3, with $p = 1$ for f_e and $p = 0$ for f_0 in the notation of §2.6.

Covered material for Problem Sheet 3 Questions 3 and 4