3 The heat equation

3.1 Preliminaries

Fundamental Theorem of Calculus

Let $F(x) = \int_{x_0}^{x} f(s)\, ds$, where $x_0 \in \mathbb{R}$. If f is cts in a neighborhood of x, then

$$F'(x) = f(x) \Rightarrow \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x) \Rightarrow \frac{1}{h}\int_{x}^{x+h} f(s)\, ds \to f(x) \text{ as } h \to 0$$

(FTC)

Leibniz' Integral Rule

Q^n: How do we compute $\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\, dx$ for F, a, b suff. smooth?

A^{ns}: Let $G(a,b,t) = \int_a^b F(x,t)\, dx$. If F and F_t are cts, then

$$\frac{\partial G}{\partial b} = F(b,t), \quad \frac{\partial G}{\partial a} = -F(a,t), \quad \frac{\partial G}{\partial t} = \int_a^b F_t(x,t)\, dx.$$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\,dx = \frac{d}{dt}\, G(a(t), b(t), t)$$

$$= \frac{\partial G}{\partial t} + \frac{\partial G}{\partial b}\frac{db}{dt} + \frac{\partial G}{\partial a}\frac{da}{dt}\Big|_{(a(t),\, b(t),\, t)}$$

$$= \int_{a(t)}^{b(t)} F_t(x,t)\,dx + F(b(t),t)\,\dot{b}(t) - F(a(t),t)\,\dot{a}(t)$$

(LIR)

so long as a, b, $\dot{a}$ and $\dot{b}$ are cts.

NB: a, b constant $\Rightarrow \dfrac{d}{dt}\displaystyle\int_a^b F(x,t)\,dx = \int_a^b F_t(x,t)\,dx$ (LIR!)

- **Fundamental Theorem of Calculus:** If $f(x)$ is continuous in a neighbourhood of a, then

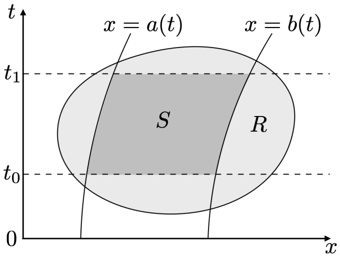
$$\frac{1}{h}\int\limits_a^{a+h} f(x)\,\mathrm{d}x \to f(a) \quad \text{as} \quad h \to 0.$$

- **Leibniz's Integral Rule:** Let $F(x,t)$ and $\partial F/\partial t$ be continuous in both x and t in some region R of the (x,t) plane containing the region $S = \{(x,t) : a(t) \le x \le b(t),\ t_0 \le t \le t_1\}$, where the functions $a(t)$ and $b(t)$ and their derivatives are continuous for $t \in [t_0, t_1]$.

  Then
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} F(x,t)\ \mathrm{d}x = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}(x,t)\,\mathrm{d}x + \dot{b}(t)F(b(t),t) - \dot{a}(t)F(a(t),t).$$

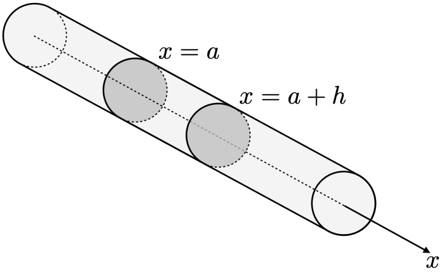
As a result, if $a(t)$ and $b(t)$ are constants, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b F(x,t)\ \mathrm{d}x = \int_a^b \frac{\partial F}{\partial t}(x,t)\,\mathrm{d}x.$$



3.2 Derivation of the one-dimensional heat equation

- Consider a rigid isotropic conducting rod (*e.g.* metal) of constant cross-sectional area A lying along the x-axis.
- We shall consider conservation of thermal or heat energy in the arbitrary section of the rod in $a \leq x \leq a+h$, where a and h are constants, as illustrated below.



- In simplest 1D model we assume that the lateral surfaces of the rod are insulated, so that no thermal energy can be transported through them and the absolute temperature T may be taken to be a function of distance x along an axis of the rod and time t.
- This assumption is applicable if the rod is long and thin, like a wire.

- We denote by ρ the density of the rod and by c_v the specific heat of the rod, and we assume that these material parameters are constant.

- The specific heat c_v of a material is the energy required to heat up a kilogram by one degree kelvin (in SI units, about which more in §3.4), so the thermal energy in the section of the rod in $a \leq x \leq a + h$ is given by
$$A \int_a^{a+h} \rho c_v T(x, t)\, \mathrm{d}x.$$

- We now introduce the heat flux $q(x, t)$ in the positive x-direction, which is the rate at which thermal energy is transported through a cross-section of the rod at station x at time t in the positive x-direction per unit cross-sectional area per unit time.

- By definition, the rate at which thermal energy enters the section through its left-hand cross-section in the plane $x = a$ is $Aq(a, t)$.

- Similarly, the rate at which thermal energy leaves the section through the right-hand cross-section in the plane $x = a + h$ is $Aq(a + h, t)$.

- Hence, with our sign convention on the heat flux, the net rate at which thermal energy enters the section is
$$Aq(a, t) - Aq(a + h, t).$$

- Assuming insulated lateral surfaces and no external heating (*e.g.* due to microwave heating), conservation of energy states that the rate of change of the thermal energy in the section is equal to the net rate at which thermal energy enters the section, so that

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(A\int_{a}^{a+h}\rho c_v T(x,t)\,\mathrm{d}x\right)}_{(1)} = \underbrace{Aq(a,t)}_{(2)} - \underbrace{Aq(a+h,t)}_{(3)},$$

where we have labeled the three terms in order to summarize their physical significance as follows:

(1) is the time rate of change of thermal energy in the section in $a \le x \le a+h$;

(2) is the rate at which thermal energy enters the section through $x = a$;

(3) is the rate at which thermal energy leaves the section through $x = a+h$.

- We note this integral conservation law is also true for $h < 0$ with appropriate reinterpretation of the terms.

- Assuming T_t is continuous, Leibniz's Integral Rule with a and $a+h$ constant gives

$$\frac{\rho c_v}{h} \int_a^{a+h} T_t(x,t)\,\mathrm{d}x + \frac{q(a+h,t) - q(a,t)}{h} = 0,$$

  where we have also rearranged into a form that will enable us to take the limit $h \to 0$.

- To take the limit $h \to 0$,
  - apply the Fundamental Theorem of Calculus assuming T_t is continuous in a neighbourhood of a;
  - use the definition of q_x assuming it to exist at a.
- We obtain thereby the partial differential equation

$$\rho c_v T_t + q_x = 0,$$

  which relates the time rate of change of the temperature and the spatial rate of change of the heat flux.

- To make further progress we must decide how the heat flux q depends on the temperature T.
- This is called a constitutive relation and cannot be deduced, relying instead on some assumptions about the physical properties of the material under consideration.
- An example of a simple constitutive relation is Hooke's law for the extension of a spring — we note that
  - a "thought-experiment" suggests this law is reasonable;
  - it could be confirmed experimentally;
  - it will almost certainly fail under "extreme" conditions.
- To close our model for heat conduction we will adopt *Fourier's Law*, which is the constitutive law given by

$$q = -kT_x,$$

  where k is the thermal conductivity of the rod, which is another material parameter that we take to be constant.

- The minus sign in Fourier's law means that thermal energy flows down the temperature gradient, *i.e.* from high to low temperatures.
- Physical experiments confirm that Fourier's law is an excellent approximation in many practical applications.
- We note that a good conductor of heat (such as silver) will have a higher thermal conductivity than a poor conductor of heat (such as glass).
- Substituting Fourier's law $q = -kT_x$ into the PDE $\rho c_v T_t + q_x = 0$ representing conservation of thermal energy, we arrive at the heat equation

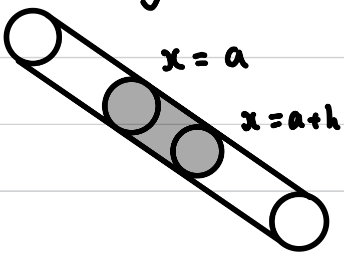
  $$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

  where the thermal diffusivity

  $$\kappa = \frac{k}{\rho c_v}.$$

- The heat equation is a second-order linear partial differential equation.

Summary: derivation heat equation



$$\frac{d}{dt}\left(\underbrace{A\int_a^{a+h}\rho c_v T\,dx}_{\text{thermal or heat energy in section}}\right) = \underbrace{A\,q(a,t)}_{\text{net heat flux in through } x=a} - \underbrace{A\,q(a+h,t)}_{\substack{\text{net heat flux out through } x=a+h\\ (h>0)}}$$

$$\Rightarrow \quad \frac{\rho c_v}{h}\int_a^{a+h} T_t\,dx = -\frac{q(a+h,t)-q(a,t)}{h}$$

(by LIR with a, h constant, OK if T_t is cts)

$$\underset{(h\to 0)}{\Rightarrow} \quad \rho c_v T_t = -q_x$$

(by FTC & defn of q_x, OK if T_t is cts & q_x exists)

$$\Rightarrow \quad \boxed{T_t = \nu T_{xx}, \quad \nu = \frac{k}{\rho c_v}}$$

(by Fourier's law $q = -kT_x$)

3.3 Initial boundary value problems

- There are numerous applications of the heat equation ranging from the diffusive transport of chemical species to the pricing of financial derivatives, the latter being governed by a *backward* heat equation called the *Black-Scholes equation*.

- In this course we focus on the modelling of the evolution of the temperature $T(x, t)$ in a metal rod of finite length L lying along the x-axis in the region $0 \leq x \leq L$.

- Suppose the metal is at room temperature $T_0 \approx 300\,\mathrm{K}$ when some large ice blocks at their melting temperature $T^* \approx 273\,\mathrm{K}$ are held instantaneously against each end of the rod at time $t = 0$.

- We encode this setup into a mathematical model, as follows:

  - the temperature $T(x, t)$ satisfies the heat equation inside the rod, so that

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L, \ t > 0;$$

  - the effect of the ice blocks on the rod are modelled through the boundary conditions

$$T(0, t) = T^*, \quad T(L, t) = T^* \quad \text{for } t > 0;$$

  - the initial state of the temperature in the rod is fed into the initial condition

$$T(x, 0) = T_0 \quad \text{for } 0 < x < L.$$

Notes

(1) The heat equation, boundary conditions and initial condition forms an *initial boundary value problem* (IBVP) for the temperature $T(x, t)$.

(2) The boundary conditions are called *Dirichlet boundary conditions* because they prescribe the value of the dependent variable T. They are *homogeneous* if $T^* = 0$ and *inhomogeneous* otherwise.

(3) While the boundary and initial conditions were motivated on physical grounds, they can only make mathematical sense if the IBVP is *well-posed* in the sense that it has a unique solution that varies continuously with the boundary and initial data (*i.e.* with T^* and T_0) in some suitable sense. We we shall return to the issue of well-posedness in §7.

(4) The total number of boundary conditions is equal to the number of spatial partial derivatives in the heat equation, which is the same count as for a typical ODE BVP. The total number of initial conditions is equal to the number of temporal derivatives in the heat equation, which is the same count as for a typical ODE IVP. These counts are typical for PDE IBVPs.

- **Definition:** The outward normal derivative of T on the boundary is equal to the directional derivative in the direction of the outward pointing unit normal, *i.e.* $-\boldsymbol{i} \cdot \boldsymbol{\nabla} T = -T_x$ on $x = 0$ and $\boldsymbol{i} \cdot \boldsymbol{\nabla} T = T_x$ on $x = L$.

- Other common boundary conditions are:

  - *inhomogeneous Neumann boundary conditions* which prescribe the outward normal derivative of the dependent variable on the boundary (here proportional to the heat flux $q = -kT_x$ by Fourier's law), *e.g.*
  $$-\frac{\partial T}{\partial x}(0,t) = \phi(t), \quad \frac{\partial T}{\partial x}(L,t) = \psi(t) \quad \text{for } t > 0,$$
  where the functions $\phi(t)$ and $\psi(t)$ are given.

  - *inhomogeneous Robin boundary conditions* which prescribe a linear combination of the outward normal derivative and temperature at the boundary, *e.g.*
  $$-\frac{\partial T}{\partial x}(0,t) + \alpha(t)T(0,t) = \phi(t), \quad \frac{\partial T}{\partial x}(L,t) + \beta(t)T(l,T) = \psi(t) \quad \text{for } t > 0,$$
  where the functions $\alpha(t)$, $\phi(t)$, $\beta(t)$ and $\psi(t)$ are given.

3.4 Units and nondimensionalisation

- **Notation:** We denote the dimension of the quantity p by $[p]$ in either fundamental units (M, L, T, Θ *etc*) or SI units (kg, m, s, K *etc*).
- We will work with the latter and recall that kelvin K is the SI unit of temperature, the newton N is the SI derived unit of force ($1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$), while the joule J is the SI derived unit of energy ($1\,\mathrm{J} = 1\,\mathrm{N\,m}$).
- Both sides of an equation modelling a physical process must have the same dimensions, *e.g.* in the integral conservation law,

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(A\int_a^{a+h}\rho c_v T(x,t)\,\mathrm{d}x\right)\right] = [Aq(a,t)] = [Aq(a+h,t)] = \mathrm{J\,s^{-1}},$$

  while in the heat equation,

$$[T_t] = [\kappa T_{xx}] = \mathrm{K\,s^{-1}}.$$

- We can exploit this fact to determine the dimensions of parameters and to check that solutions are dimensionally correct.

- For example, using Fourier's Law we find that

$$[k] = \frac{[q]}{[T_x]} = \frac{\mathrm{J\,m^{-2}\,s^{-1}}}{\mathrm{K\,m^{-1}}} = \mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}},$$

and using the heat equation we find that

$$[\kappa] = \frac{[T_t]}{[T_{xx}]} = \frac{\mathrm{K\,s^{-1}}}{\mathrm{K\,m^{-2}}} = \mathrm{m^2\,s^{-1}}.$$

- We summarize below the dimensions of the variables and parameters involved in the derivation of the one-dimensional heat equation.

Symbol	Quantity	SI units
x	Axial distance	m
t	Time	s
T	Absolute temperature	K
q	Heat flux in positive x-direction	$\mathrm{J\,m^{-2}\,s^{-1}}$
A	Cross-sectional area	$\mathrm{m^2}$
ρ	Rod density	$\mathrm{kg\,m^{-3}}$
c_v	Rod specific heat	$\mathrm{J\,kg^{-1}\,K^{-1}}$
k	Rod thermal conductivity	$\mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}}$
κ	Rod thermal diffusivity	$\mathrm{m^2\,s^{-1}}$

- **Nondimensionalisation:** The method of scaling variables with typical values to derive dimensionless equations. These usually contain dimensionless parameters that characterise the relative importance of the physical mechanisms in the model.
- We illustrate the method with an example.

Example: Nondimensionalization of an IBVP

Suppose $T(x,t)$: ① $T_t = \nu T_{xx}$ for $0 < x < L,\ t > 0,$

② $T(0,t) = T_0,\ T(L,t) = T_1$ for $t > 0,$

③ $T(x,0) = T_2 \frac{x}{L}\left(1 - \frac{x}{L}\right)$ for $0 < x < L.$

①–③ form a dimensional IBVP for $T(x,t)$.

It contains 5 dimensional parameters: $\nu,\ L,\ T_0,\ T_1,\ T_2$

Scale: $x = L\hat{x},\quad t = \underbrace{\tau}_{\text{TBD}}\hat{t},\quad T(x,t) = T_2\hat{T}(\hat{x},\hat{t})$

Chain rule: $\dfrac{\partial T}{\partial t} = T_2 \dfrac{\partial \hat{T}}{\partial \hat{t}}\dfrac{\partial \hat{t}}{\partial t} = \dfrac{T_2}{\tau}\dfrac{\partial \hat{T}}{\partial \hat{t}}$

$$\frac{\partial T}{\partial x} = T_2 \frac{\partial \hat{T}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} = \frac{T_2}{L} \frac{\partial \hat{T}}{\partial \hat{x}} , \quad \frac{\partial^2 T}{\partial x^2} = \frac{T_2}{L^2} \frac{\partial^2 \hat{T}}{\partial \hat{x}^2}$$

(1) $\Rightarrow \dfrac{T_2}{\tau} \dfrac{\partial \hat{T}}{\partial \hat{t}} = \nu \dfrac{T_2}{L^2} \dfrac{\partial^2 \hat{T}}{\partial \hat{x}^2}$

$\Rightarrow \dfrac{\partial \hat{T}}{\partial \hat{t}} = \underbrace{\dfrac{\nu \tau}{L^2}}_{D} \dfrac{\partial^2 \hat{T}}{\partial \hat{x}^2}$ for $0 < \hat{x} < 1$, $\hat{t} > 0$ (1')

(2) $\Rightarrow \hat{T}(0,\hat{t}) = \underbrace{T_0/T_2}_{\alpha_0}$, $\hat{T}(1,\hat{t}) = \underbrace{T_1/T_2}_{\alpha_1}$ for $\hat{t} > 0$ (2')

(3) $\Rightarrow \hat{T}(\hat{x},0) = \hat{x}(1-\hat{x})$ for $0 < \hat{x} < 1$ (3')

(1')-(3') form the dimensionless IBVP for $\hat{T}(\hat{x},\hat{t})$

It contains 3 dimensionless parameters: D, α_0, α_1!

Can reduce to 2 dimensionless parameters by choosing $\tau = \frac{L^2}{\nu}$, i.e. $D = 1$.

Since this balances the terms in the dimensionless heat equation, $\frac{L^2}{\nu}$ is the timescale for heat to conduct a distance L.

NB: If $\hat{T}(\hat{x}, \hat{t}; \alpha_0, \alpha_1)$ is a soln of ① – ③, then the dimensional temperature T defined by

$$T/T_2 = \hat{T}\left(\frac{x}{L}, \frac{\nu t}{L^2}; \frac{T_0}{T_2}, \frac{T_1}{T_2}\right)$$

is a soln ① – ③, i.e.

$$T/T_2 = F^n\left(\frac{x}{L}, \frac{\nu t}{L^2}; \frac{T_0}{T_2}; \frac{T_1}{T_2}\right)$$

Use to check an explicit solution is dimensionally correct.

Q^n : Suppose we solved an IBVP and found the solution

$$T = T^* \sin\left(\frac{\pi x}{L}\right) \exp\left(-\pi^2 \frac{\nu t}{L^2}\right).$$

Is this solution dimensionally correct?

A^ns : Yes provided $[T^*] = \Upsilon_1$, since then

$$T/T^* = \sin\left(\pi \frac{x}{L}\right) \exp\left(-\pi^2 \frac{\nu t}{L^2}\right),$$

where T/T^*, x/L, $\nu t/L^2$ are dimensionless.

Example: nondimensionalisation of an IBVP

- Consider the IBVP for the temperature $T(x, t)$ in a metal rod of length L given by the heat equation
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$
with the inhomogeneous Dirichlet boundary conditions
$$T(0, t) = T_0, \quad T(L, t) = T_1 \quad \text{for} \quad t > 0,$$
and the initial condition
$$T(x, 0) = T_2 \frac{x}{L}\left(1 - \frac{x}{L}\right) \quad \text{for} \quad 0 < x < L,$$
where T_0, T_1 snd T_2 are prescribed constant temperatures.
- **Remark:** There are five dimensional parameters, namely κ, L, T_0, T_1 and T_2.
- We can nondimensionalise by scaling
$$x = L\widehat{x}, \quad t = \tau\widehat{t}, \quad T(x, t) = T_2\widehat{T}(\widehat{x}, \widehat{t}\,),$$
where L, τ and T_2 are a typical lengthscale, timescale and temperature, respectively, so that the quantities $\widehat{x}$, $\widehat{t}$ and $\widehat{T}$ are dimensionless.

- By the chain rule,

$$\frac{\partial T}{\partial t} = T_2 \frac{\partial \widehat{T}}{\partial \widehat{t}} \frac{\mathrm{d}\widehat{t}}{\mathrm{d}t} = \frac{T_2}{\tau} \frac{\partial \widehat{T}}{\partial \widehat{t}}, \qquad \frac{\partial T}{\partial x} = T_2 \frac{\partial \widehat{T}}{\partial \widehat{x}} \frac{\mathrm{d}\widehat{x}}{\mathrm{d}x} = \frac{T_2}{L} \frac{\partial \widehat{T}}{\partial \widehat{x}}, \quad \textit{etc.}$$

- Hence, the dimensional problem for the dimensional temperature $T(x,t)$ implies that the corresponding dimensionless problem for the dimensionless temperature $\widehat{T}(\widehat{x},\widehat{t}\,)$ is given by

$$\frac{\partial \widehat{T}}{\partial \widehat{t}} = D \frac{\partial^2 \widehat{T}}{\partial \widehat{x}^2} \quad \text{for} \quad 0 < \widehat{x} < 1,\ \widehat{t} > 0,$$

with the boundary conditions

$$\widehat{T}(0,\widehat{t}\,) = \alpha_0, \quad \widehat{T}(1,t) = \alpha_1 \quad \text{for} \quad \widehat{t} > 0,$$

and the initial condition

$$\widehat{T}(\widehat{x},0) = \widehat{x}(1-\widehat{x}\,) \quad \text{for} \quad 0 < \widehat{x} < 1,$$

where the three dimensionless parameters D, α_0 and α_1 are defined by

$$D = \frac{\kappa\tau}{L^2}, \quad \alpha_0 = \frac{T_0}{T_2}, \quad \alpha_1 = \frac{T_1}{T_2}.$$

- We can further reduce the number of dimensionless parameters to two by choosing the timescale τ so that $D = 1$, *i.e.* by choosing

$$\tau = L^2/\kappa,$$

  which is the timescale for conductive transport of heat over a distance L because it balances both terms in the heat equation.

- With this choice of timescale, we note that if $\widehat{T}(\widehat{x}, \widehat{t}; \alpha_0, \alpha_1)$ is a solution of the IBVP for $\widehat{T}$, then a solution of the IBVP for T is given by

$$\frac{T}{T_2} = \widehat{T}\left(\frac{x}{L}, \frac{\kappa t}{L^2}; \frac{T_0}{T_2}, \frac{T_1}{T_2}\right),$$

  i.e. T/T_2 must be a function of x/L and $\kappa t/L^2$.

- This means that we can compare heat problems on different scales. For example, two IBVPs that are identical except for L and κ will exhibit the same behaviour on the same timescale if and only if L^2/κ is the same in each problem. ■

3.5 Heat conduction in a finite rod

- Consider the initial boundary value problem for the temperature $T(x,t)$ in a metal rod of length L given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with the homogeneous Dirichlet boundary conditions

$$T(0,t) = 0, \quad T(L,t) = 0 \quad \text{for} \quad t > 0,$$

and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L,$$

where the initial temperature profile $f(x)$ is given.

- We will construct a solution using *Fourier's method*, which consists of three steps.

- **Fourier's method:**

  (I) Use the method of separation of variables to find the countably infinite set of nontrivial separable solutions satisfying the heat equation and boundary conditions, each containing an arbitrary constant.

  (II) Use the principle of superposition — that the sum of any number of solutions of a linear homogeneous problem is also a solution (assuming convergence) — to form the general series solution that is the infinite sum of the separable solutions.

  (III) Use the theory of Fourier series to determine the constants in the general series solution for which it satisfies the initial condition.

- **Notes:**

  (1) Both the partial differential equation and and the boundary conditions are linear and homogeneous, so if T_1 and T_2 satisfy them, then so does $\alpha_1 T_1 + \alpha_2 T_2$ for all $\alpha_1,\, \alpha_2 \in \mathbb{R}$.

  (2) To verify that the resulting infinite series is actually a solution of the heat equation, we need it to converge sufficiently rapidly that T_t and T_{xx} can be computed by termwise differentiation.

Application of Fourier's Method to an IBVP

Suppose $T(x,t)$: ① $T_t = \kappa T_{xx}$ for $0<x<L,\ t>0$

② $T(0,t)=0,\ T(L,t)=0$ for $t>0$

③ $T(x,0) = f(x)$ for $0<x<L$

Apply Fourier's method, as follows.

Step (I)

$$T = F(x)\,G(t) \neq 0 \Rightarrow F(x)\,G'(t) = \kappa F''(x)\,G(t) \Rightarrow \frac{F''(x)}{F(x)} = \frac{G'(t)}{\kappa G(t)}$$

LHS ind. of t & RHS ind. of x, so LHS = RHS ind. of x & t, i.e. equal to a constant, $-\lambda \in \mathbb{R}$ say.

Hence, PDEs $\mapsto$ ODEs, namely $-F'' = \lambda F$ for $0 < x < L$,
$G' = -\lambda \kappa G$ for $t > 0$.

② $\Rightarrow$ $F(0)\,G(t) = 0$, $F(L)\,G(t) = 0$ for $t > 0$

G nontrivial $\Rightarrow$ $F(0) = 0$, $F(L) = 0$

Hence, $\boxed{F'' + \lambda F = 0 \text{ for } 0 < x < L, \text{ with } F(0) = 0, F(L) = 0}$

General solution of ODE different for (i) $\lambda < 0$,
(ii) $\lambda = 0$,
(iii) $\lambda > 0$.

Hence, consider cases...

(i) $\lambda = -\omega^2$ ($\omega > 0$ wlog)

$F'' - \omega^2 F \Rightarrow F(x) = A\cosh \omega x + B\sinh \omega x$ $(A, B \in \mathbb{R})$

$F(0) = F(L) = 0 \Rightarrow A = 0, \; B\sinh \omega L = 0 \Rightarrow A = B = 0 \Rightarrow F = 0$

(ii) $\lambda = 0$

$F'' = 0 \Rightarrow F = Ax + B$ $(A, B \in \mathbb{R})$

$F(0) = F(L) = 0 \Rightarrow B = 0, \; AL = 0 \Rightarrow A = B = 0 \Rightarrow F = 0.$

(iii) $\lambda = w^2$ ($w > 0$ wlog)

$F'' + w^2 F = 0 \Rightarrow F(x) = A\cos wx + B\sin wx \quad (A, B \in \mathbb{R})$

$F(0) = F(L) = 0 \Rightarrow A = 0, \quad B\sin wL = 0$

F non-trivial $\Rightarrow B \neq 0$

$\Rightarrow \sin wL = 0$

$\Rightarrow wL = n\pi \quad , \quad n \in \mathbb{N} \setminus \{0\}$

$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N} \setminus \{0\}$

Hence, the nontrivial solutions for F are given for $n \in \mathbb{N} \setminus \{0\}$ by

$$F(x) = B \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2,$$

where $B \in \mathbb{R}$. Since $G' = -\lambda \kappa G$,

$$G(t) = C \exp(-\lambda \kappa t) = C \exp\left(-\frac{n^2 \pi^2 \kappa t}{L^2}\right),$$

where $C \in \mathbb{R}$.

Combo $\Rightarrow$ non-trivial sep. soln's of ①-② are

$$T_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2 \pi^2 \kappa t}{L^2}\right)$$

for $n \in \mathbb{N} \setminus \{0\}$, where $b_n = BC \in \mathbb{R}$.

Step (II)

① & ③ linear & homogeneous, so assuming convergence we can superimpose them to obtain the general series solution

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 \kappa t}{L^2}\right)$$

Step (III)

The IC ③ $T(x,0) = f(x)$ for $0<x<L$ can only be satisfied by the general series solution if $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = T(x,0) = f(x)$ for $0<x<L$.

The theory of Fourier series $\Rightarrow$ $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ for $n \in \mathbb{N}\setminus\{0\}$.

Hence, we have derived (assuming convergence) an infinite series solution.

Step (I) Find all nontrivial separable solutions of the PDE and BCs

- We begin by seeking a nontrivial separable solution of the form $T = F(x)G(t)$ for which the heat equation $T_t = \kappa T_{xx}$ gives
$$F(x)G'(t) = \kappa F''(x)G(t),$$
with a prime $'$ denoting here and hereafter the derivative with respect to the argument.

- Separating the variables by assuming $F(x)G(t) \neq 0$ therefore gives
$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{\kappa G(t)}.$$

- The LHS of this expression is independent of t, while the RHS is independent of x. Since the LHS is equal to the RHS, they must both be independent of x and t, and therefore equal to a constant, $-\lambda \in \mathbb{R}$ say.

- Hence,
$$F'' + \lambda F = 0 \quad \text{for } 0 < x < L \qquad \text{and} \qquad G' = -\lambda\kappa G \quad \text{for } t > 0.$$

- The boundary condition at $x = 0$ implies that $F(0)G(t) = 0$ for $t > 0$. Since we're seeking solutions T that are nontrivial (*i.e.* not identically equal to zero), there must exist a time $t > 0$ such that $G(t) \neq 0$, and hence we must impose on $F(x)$ the boundary condition $F(0) = 0$. Similarly, the boundary condition at $x = L$ implies that $F(L) = 0$.

- In summary, we have deduced that $F(x)$ satisfies the BVP given by the ODE

$$-F''(x) = \lambda F(x) \quad \text{for} \quad 0 < x < L,$$

with the boundary conditions

$$F(0) = 0, \quad F(L) = 0,$$

where $\lambda \in \mathbb{R}$.

- Now we need to find all $\lambda \in \mathbb{R}$ such that the BVP for $F(x)$ has a nontrivial solution.

- Since the general solution of the ODE is different for (i) $\lambda < 0$, (ii) $\lambda = 0$ and (iii) $\lambda > 0$, there are three cases to consider.

- Case (i): $\lambda = -\omega^2$ ($\omega > 0$ wlog)
  - If $F'' - \omega^2 F = 0$, then $F(x) = A\cosh(\omega x) + B\sinh(\omega x)$, where $A, B \in \mathbb{R}$.
  - The boundary conditions then require $A = 0$, $B\sinh(\omega L) = 0$, so that $F = 0$.
- Case (ii): $\lambda = 0$
  - If $F'' = 0$, then $F(x) = A + Bx$, where $A, B \in \mathbb{R}$.
  - The boundary conditions then require $A = 0$, $BL = 0$, so that $F = 0$.
- Case (iii): $\lambda = \omega^2$ ($\omega > 0$ wlog)
  - If $F'' + \omega^2 F = 0$, then $F(x) = A\cos(\omega x) + B\sin(\omega x)$, where $A, B \in \mathbb{R}$.
  - The boundary conditions then require $A = 0$, $B\sin(\omega L) = 0$.
  - Since $B \neq 0$ for nontrivial F, we must have $\sin \omega L = 0$, *i.e.* $\omega L = n\pi$ for some $n \in \mathbb{N} \setminus \{0\}$.

- Hence, the nontrivial solutions of the BVP for $F(x)$ are given for positive integers n by

$$F(x) = B \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n^2\pi^2}{L^2},$$

where B is an arbitrary constant.

- Since $G(t)$ satisfies the ordinary differential equation $G' = -\lambda\kappa G$, we deduce that

$$G(t) = C \exp\left(-\lambda\kappa t\right),$$

where $C \in \mathbb{R}$.

- Since $T(x,t) = F(x)G(t)$, we conclude that the nontrivial separable solutions of the heat equation that satisfy the boundary conditions are given by

$$T_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

where n is a positive integer, b_n is a constant (equal to BC above) and we have introduced the subscript n on T_n and b_n to enumerate the countably infinite set of such solutions.

Step (II) Apply the principle of superposition

- Since the heat equation and boundary conditions are linear and homogeneous, a formal application of the principle of superposition implies that the general series solution is given by

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right).$$

Step (III) Use the theory of Fourier series to satisfy the IC

- The initial condition can only be satisfied by the general series solution if

$$f(x) = T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad 0 < x < L,$$

  so that we need to find the Fourier sine series for f on $[0, L]$.

- The theory of Fourier series implies that the Fourier coefficients b_n are given by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad \text{for} \quad n \in \mathbb{N} \setminus \{0\}.$$

- Hence, we have derived a solution in the form of an infinite trigonometric series.

Notes:

- The ODE BVP for $F(x)$ and λ is an *eigenvalue problem* in which the unknown parameter λ is called an *eigenvalue* and the corresponding non-trivial solution $F(x)$ an *eigenfunction*.
- The Fourier series expansions for f and T are therefore called *eigenfunction expansions*.
- That there are a discrete countably infinite set of eigenvalues and corresponding eigenfunctions is a property of the BVP that is explained by *Sturm-Liouville theory* of *e.g.* part A DEs 2.
- The integral expressions for the Fourier coefficients may be derived by assuming that the orders of summation and integration may be interchanged and using the orthogonality relations

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{L}{2}\delta_{mn} \quad \text{for } m,\ n \in \mathbb{N}\setminus\{0\}.$$

- Since $T_n(x,t)$ decays exponentially as $n \to \infty$ for $t > 0$, comparison methods from Analysis II may be used to show that if the Fourier coefficients b_n are merely bounded for all n, then the general series solution has partial derivatives of all orders for $t > 0$ that may be computed by term-by-term differentiation.

- It follows from the Fourier convergence theorem that if f and f' are piecewise continuous on $(0, L)$, then the infinite series solution is indeed a solution of the IBVP. Thus, Fourier's method can accommodate even jump discontinuities in the initial temperature profile, the heat equation acting to instantaneously "smooth" them out.

- If the initial temperature profile has a jump discontinuity, then the truncated series solution for $T(x,t)$ will exhibit Gibb's phenomenon at $t = 0$, and hence at sufficiently small times $t \ll L^2/\kappa$ by continuity.

- In principle this deficiency can be avoided at some fixed $t > 0$ by keeping enough terms. In contrast, the exponential decay of $T_n(x,t)$ with $n^2\kappa t/L^2$ means that the solution will be well approximated by the leading-term $T_1(x,t)$ at sufficiently large large times $t \gg L^2/\kappa$.

Example: the smoothing effect of the heat equation

- Consider the IBVP in which the initial temperature profile given by

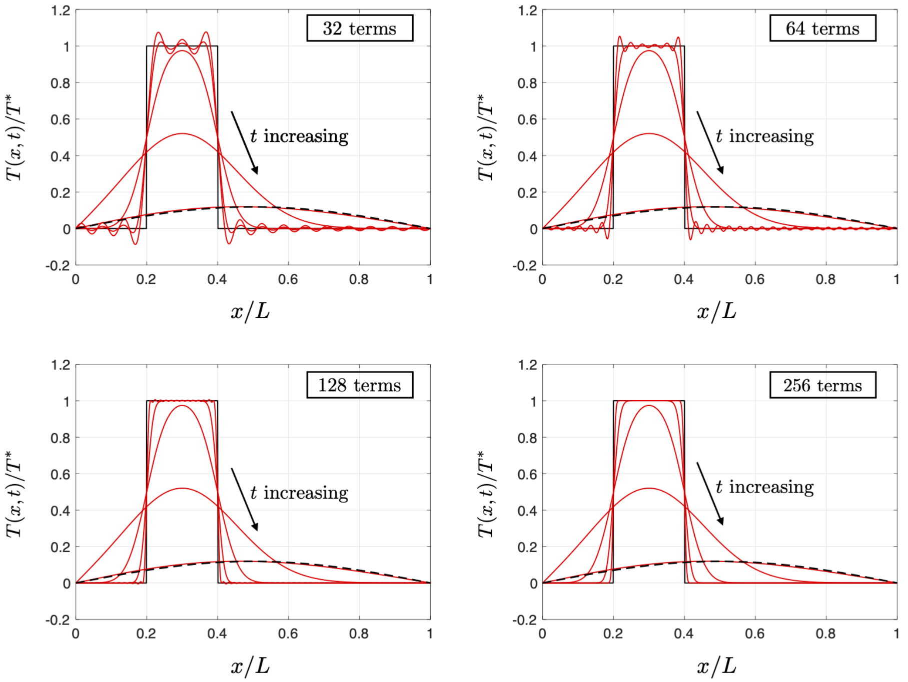
$$f(x) = \begin{cases} T^* & \text{for } L_1 < x < L_2, \\ 0 & \text{otherwise,} \end{cases}$$

where T^*, L_1 and L_2 are constants, so that the Fourier coefficients are given by

$$b_n = \frac{2}{L} \int_{L_1}^{L_2} T^* \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2T^*}{n\pi} \left(\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right) \quad \text{for } n \in \mathbb{N}\backslash\{0\}.$$

- We plot below snapshots of the partial sums of the truncated series solution (red lines) with 32, 64, 128 and 256 terms at times t given by $\kappa t/L^2 = 10^{-5}$, 10^{-4}, 10^{-3}, 10^{-2} and 10^{-1} for $L_1/L = 0.2$, $L_2/L = 0.4$.

- The jump conditions in the initial temperature profile at $L_1/L = 0.2$ and $L_2/L = 0.4$ are indicated by vertical black lines.


32 terms
64 terms
128 terms
256 terms
t increasing
$T(x,t)/T^*$
x/L

- As the number of terms increases we see that Gibb's phenomenon is suppressed more rapidly.
- Any profile that is oscillatory or not positive for $0 < x < L$, $t > 0$ is a poor approximation of the solution, so we see that only the plot with 256 terms is acceptable for the times chosen.
- The final snap shot in each case is close to $T_1(x, t)$ (dashed line) for which $\pi^2 \kappa t/L^2 = \pi^2/10$.
- The early time behaviour is captured much more effectively by the *asymptotic solution*

$$T(x, t) \approx \frac{T^*}{\sqrt{4\pi\kappa t}} \int_{L_1}^{L_2} \exp\left(-\frac{(s-x)^2}{4\kappa t}\right) \mathrm{d}s,$$

  which is valid as $t \to 0+$.
- The asymptotic solution does not exhibit Gibb's phenomenon and tends to the initial profile as $t \to 0+$ except at the jump discontinuities where it tends to $T^*/2$.
- The asymptotic solution is the superposition of *fundamental solutions of the heat equation* and may be derived systematically using the *method of matched asymptotic expansions* — see part A Differential Equations 2 and Integral Transforms. ■