3.6 Uniqueness Theorem

- In the last section we considered the IBVP for the temperature $T(x,t)$ given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \qquad (1)$$

  with the homogeneous Dirichlet boundary conditions

$$T(0,t) = 0, \quad T(L,t) = 0 \quad \text{for} \quad t > 0, \qquad (2)$$

  and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \qquad (3)$$

  where the initial temperature profile $f(x)$ is given.

- We used Fourier's method to construct an infinite series solution, but is it the only solution?

- **Uniqueness Theorem:** The IBVP has at most one solution.

Proof

- Let $W = T - \tilde{T}$ be the difference between two solutions T and $\tilde{T}$, say.

- Linearity of ①–② $\Rightarrow$ W satisfies the IBVP given by...

$$W_t = (T-\tilde{T})_t = T_t - \tilde{T}_t = \kappa T_{xx} - \kappa \tilde{T}_{xx} = \kappa (T-\tilde{T})_{xx} = \kappa W_{xx} \text{ for } 0<x<L,\ t>0 \quad (1')$$

$$W(0,t) = T(0,t) - \tilde{T}(0,t) = 0-0=0,\ W(L,t) = T(L,t) - \tilde{T}(L,t) = 0-0=0 \text{ for } t>0 \quad (2')$$

$$W(x,0) = T(x,0) - \tilde{T}(x,0) = f(x) - f(x) = 0 \text{ for } 0<x<L \quad (3')$$

- Trick: Analyse the integral $I(t) = \int_0^L \frac{1}{2} W(x,t)^2\, dx$ for $t \geq 0$.

- Evidently $I(t) \geq 0$ for $t \geq 0$, and $I(0) = 0$.

- Now show $I(t)$ cannot increase, as follows.

$$\frac{dI}{dt} = \int_0^L \frac{\partial}{\partial t}\left(\tfrac{1}{2} w^2\right) dx \qquad \text{(by LIR with } L \text{ const.)}$$

$$= \int_0^L w\, w_t \, dx$$

$$= \int_0^L \underbrace{w}_{u}\, \underbrace{\nu w_{xx}}_{v'} \, dx \qquad \text{(by (1'))}$$

$$= \Big[\underbrace{w}_{u}\, \underbrace{\nu w_x}_{v} \Big]_{x=0}^{x=L} - \int_0^L \underbrace{w_x}_{v'}\, \underbrace{\nu w_x}_{v} \, dx \qquad \text{(by IBP)}$$

$$= 0 - \nu \int_0^L w_x^{\,2} \, dx \qquad \text{(by (2'))}$$

$$\leq 0 \qquad \text{for } t > 0$$

- Hence, $I(t) \leq I(0) = 0$ for $t \geq 0$.

- But $I(t) \geq 0$ for $t \geq 0$ (by construction), so $I(t) = 0$ for all $t \geq 0$.

- This means $\int_0^L w(x,t)^2 \, dx = 0$ for $t \geq 0$.

- Hence $w(x,t) = 0$ for $0 \leq x \leq L$, $t \geq 0$ (assuming w is cts there).

$\square$

NB: Proof works for linear BCs for which can show $[K w w_x]_{x=0}^{x=L} \leq 0$, e.g. inhomogeneous Neumann & Robin BCs.

Proof:

- Our strategy is to show that the difference between any two solutions much vanish.
- Thus, we suppose that $T(x,t)$ and $\widetilde{T}(x,t)$ are solutions and let $W(x,t) = T(x,t) - \widetilde{T}(x,t)$.
- By linearity, the IBVP for $T(x,t)$ and $\widetilde{T}(x,t)$ imply that $W(x,t)$ satisfies the heat equation

$$\frac{\partial W}{\partial t} = \frac{\partial T}{\partial t} - \frac{\partial \widetilde{T}}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2} - \kappa\frac{\partial^2 \widetilde{T}}{\partial x^2} = \kappa\frac{\partial^2 W}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

  with the boundary conditions

$$W(0,t) = T(0,t) - \widetilde{T}(0,t) = 0, \quad W(L,t) = T(L,t) - \widetilde{T}(L,t) = 0 \quad \text{for} \quad t > 0,$$

  and the initial condition

$$W(x,0) = T(x,0) - \widetilde{T}(x,0) = f(x) - f(x) = 0 \quad \text{for} \quad 0 < x < L.$$

- The trick is to analyse the integral $I(t)$ defined by

$$I(t) = \frac{1}{2}\int_0^L W(x,t)^2 \,\mathrm{d}x.$$

- Evidently $I(t) \geq 0$ for $t \geq 0$ and $I(0) = 0$ by the initial condition.

- But, for $t > 0$,

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \int_0^L W \frac{\partial W}{\partial t}\,\mathrm{d}x && \text{(by Liebniz Integral Rule)} \\
&= \int_0^L W \kappa \frac{\partial^2 W}{\partial x^2}\,\mathrm{d}x && \text{(by the heat equation)} \\
&= \left[\kappa W \frac{\partial W}{\partial x}\right]_{x=0}^{x=L} - \kappa \int_0^L \frac{\partial W}{\partial x}\frac{\partial W}{\partial x}\,\mathrm{d}x && \text{(by integration by parts)} \\
&= -\kappa \int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x && \text{(by the boundary conditions)} \\
&\leq 0
\end{aligned}$$

which means that $I(t)$ cannot increase, so that $I(t) \leq I(0) = 0$ for $t \geq 0$.

- Since $I(t) \geq 0$ and $I(t) \leq 0$ for $t \geq 0$, we deduce that $I(t) = 0$ for $t \geq 0$, and hence that $W(x, t) = 0$ for $0 \leq x \leq L$, $t \geq 0$ (assuming continuity of W there). ■

Notes

(1) Since W is the temperature in a metal rod whose initial temperature is everywhere zero and whose ends are held at zero temperature thereafter, on physical grounds we expect the rod to remain at zero temperature, *i.e.* $W = 0$ for $0 \leq x \leq L$ and $t \geq 0$, which is precisely what we showed to prove uniqueness.

(2) The proof works for any boundary conditions for which it is possible to show that

$$\left[\kappa W \frac{\partial W}{\partial x}\right]_{x=0}^{x=L} \leq 0.$$

Examples include inhomogeneous Dirichlet and Neumann boundary conditions.

3.7 Inhomogeneous Dirichlet boundary conditions

- Suppose $T(x,t)$: ① $T_t = \nu T_{xx}$ for $0 < x < L$, $t > 0$,

  ② $T(0,t) = T_0$, $T(L,t) = T_1$ for $t > 0$,

  ③ $T(x,0) = f(x)$ for $0 < x < L$

  where $T_0, T_1 > 0$ and $f(x)$ is given.

- Let's try Fourier's method : let $T = F(x)G(t) \neq 0$.

- ② $\Rightarrow$ $F(0)\,G(t) = T_0$, $F(L)\,G(t) = T_1$ for $t > 0$

  $\Rightarrow$ $G(t) = \text{constant} = 1$ wlog

  $\Rightarrow$ $T = F(x)$, i.e. ind of time!

- Then ①, ② $\Rightarrow$ $F''(x) = 0$ for $0 < x < L$, with $F(0) = T_0$, $F(0) = T_1$

$$\Rightarrow \quad \boxed{T = F(x) = T_0\left(\frac{x}{L}\right) + T_1\left(1 - \frac{x}{L}\right)} \qquad (\dagger)$$

NB: (4) is a steady-state or time independent solution of ①-②
in which the heat flux $q = -k\frac{dT}{dx} = \frac{k}{L}(T_0 - T_1) > 0$ iff $T_0 > T_1$

Cannot satisfy IC ③ in general, so Fourier's method has failed —
this is because the BCs ② are inhomogeneous.

On physical grounds, we conjecture that

$$T(x,t) \to S(x) := T_0\left(\frac{x}{L}\right) + T_1\left(1 - \frac{x}{L}\right) \text{ as } t \to \infty$$

Suggests we seek a solution of ①-③ of form $T(x,t) = S(x) + u(x,t)$,
where $u(x,t)$ is TBD.

① $\Rightarrow$ $(S+u)_t = \kappa(S+u)_{xx} \Rightarrow u_t = \kappa u_{xx}$ for $0 < x < L, t > 0$ (1′)

② $\Rightarrow$ $\left.\begin{aligned} u(0,t) &= T(0,t) - S(0) = T_0 - T_0 = 0 \\ u(L,t) &= T(L,t) - S(L) = T_1 - T_1 = 0 \end{aligned}\right\}$ for $t > 0$ (2′)

③ $\Rightarrow$ $u(x,0) = T(x,0) - S(x) = \underline{f(x) - S(x)}$ for $0 < x < L$. (3′)

The IBVP (1′)-(3′) for $u(x,t)$ is amenable to Fourier's method because the BCs are homogeneous.

§3.5 $\Rightarrow$ $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right)$, $b_n = \frac{2}{L}\int_0^L (f(x) - S(x)) \sin\left(\frac{n\pi x}{L}\right) dx$

This is the 'method of shifting the data'.

- Consider the initial boundary value problem for the temperature $T(x,t)$ given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L, \; t > 0,$$

  with the inhomogeneous Dirichlet boundary conditions

$$T(0,t) = T_0, \quad T(L,t) = T_1 \quad \text{for} \quad t > 0,$$

  and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L,$$

  where T_0 and T_1 are prescribed constant temperatures, not both zero, and the initial temperature profile $f(x)$ is given.

- Lets try to apply Fourier's method.

- In step (I) we need to find the nontrivial separable solutions $T(x, t) = F(x)G(t)$ of the heat equation and boundary conditions.
- But the latter would require

$$F(0)G(t) = T_0, \quad F(L)G(t) = T_1 \quad \text{for} \quad t > 0,$$

  forcing G to be constant.
- It follows that the only nontrivial separable solution satisfying the boundary conditions is the time-independent or steady-state solution (about which more shortly).
- Since this cannot satisfy the initial condition, Fourier's method fails because the boundary conditions are not homogeneous.
- However, we can transform the problem into one amenable to Fourier's method, as follows.

- On physical grounds, we conjecture that $T(x, t) \to S(x)$ as $t \to \infty$, where $S(x)$ is the aforementioned steady-state solution of the heat equation and boundary conditions, which satisfies

$$0 = \kappa \frac{\mathrm{d}^2 S}{\mathrm{d}x^2} \quad \text{for} \quad 0 < x < L,$$

with $S(0) = T_0$ and $S(L) = T_1$.

- Thus, $S(x)$ has the linear temperature profile given by

$$S(x) = T_0 \left(1 - \frac{x}{L}\right) + T_1 \left(\frac{x}{L}\right).$$

- **Remark:** In steady state thermal energy is conducted along the rod with constant heat flux

$$q = -k \frac{\partial T}{\partial x} = \frac{k(T_0 - T_1)}{L},$$

so that heat flows steadily in the positive x-direction for $T_0 > T_1$.

- We now observe that if we let

$$T(x,t) = S(x) + U(x,t),$$

then by linearity the IBVP for $T(x,t)$ implies that $U(x,t)$ satisfies the IBVP given by the heat equation

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with the homogeneous Dirichlet boundary conditions

$$U(0,t) = 0, \quad U(L,t) = 0 \quad \text{for} \quad t > 0,$$

and the initial condition

$$U(x,0) = f(x) - S(x) \quad \text{for} \quad 0 < x < L.$$

- The IBVP for $U(x,t)$ is amenable to Fourier's method.
- We solved it in §3.4 to find the solution given by

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

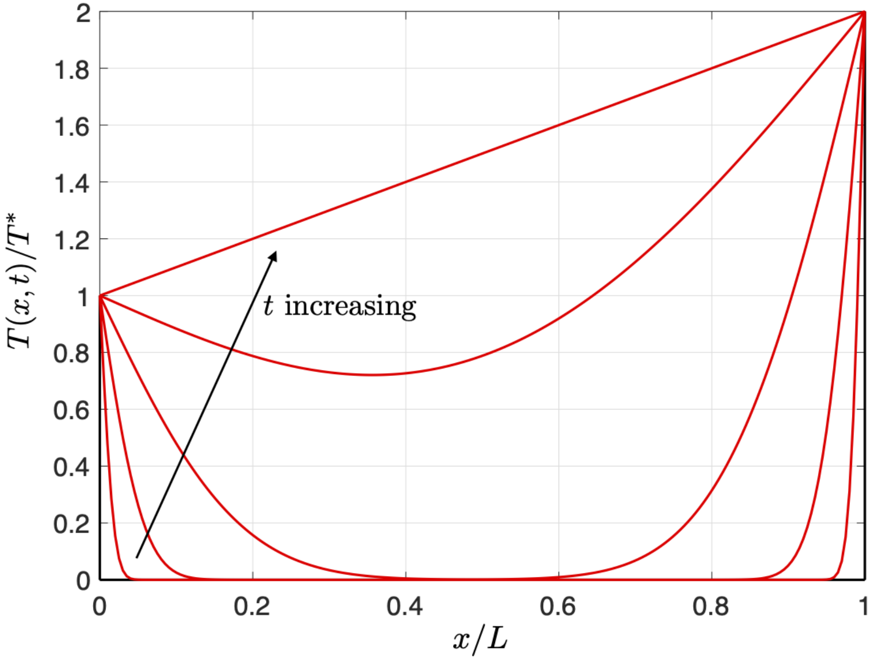
where

$$b_n = \frac{2}{L}\int_0^L \big(f(x) - S(x)\big) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x - \frac{2}{n\pi}(T_0 - (-1)^n T_1).$$

- Since $U(x,t) \to 0$ as $t \to \infty$, we can verify our conjecture that $T(x,t) \to S(x)$ as $t \to \infty$.
- **Remark:** The parameters T_0 and T_1 in the boundary conditions for $T(x,t)$ ended up in the initial condition for $U(x,t)$ — hence the method is sometimes called 'the method of shifting the data.'

Example: infinite speed of propagation

- Consider the IBVP with $f(x) = 0$, $T_0 = T^*$ and $T_1 = 2T^*$.
- We plot below snapshots of the partial sums of the truncated series solution with 128 terms for $\kappa t/L^2 = 0$ (black line) and $\kappa t/L^2 = 10^{-4}$, 10^{-3}, 10^{-2}, 10^{-1}, 1 (red lines).
- The profiles illustrate the manner in which heat conduction rapidly drives the temperature toward the linear steady-state temperature profile.
- Since the temperature is zero for $0 < x < L$ at $t = 0$, but everywhere positive for $t > 0$, the effect of the boundary conditions is felt everywhere instantaneously — the heat equation propagates information with infinite speed.



$T(x,t)/T^*$
x/L
t increasing
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

3.8 Homogeneous Neumann boundary conditions

- Consider the IBVP for the temperature $T(x,t)$ given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L, \; t > 0,$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial T}{\partial x}(0,t) = 0, \quad \frac{\partial T}{\partial x}(L,t) = 0 \quad \text{for} \quad t > 0,$$

and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L.$$

- **Remark:** The ends of the rod are thermally insulated because $q = -k\partial T/\partial x = 0$ there.

- Fourier's method is applied on problem sheet 5 to show that the solution is given by

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

where the constants a_n are the Fourier coefficients of the Fourier cosine series for f given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Notes

(1) The constant separable and steady-state solution $T = a_0/2$ comes from the case in which the separation constant is zero.

(2) The Uniqueness Theorem in §3.6 may be adapted to show the IBVP has at most one solution.

(3) Integrating the heat equation from $x = 0$ to $x = L$ and applying the boundary conditions gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \rho c_v T(x,t)\,\mathrm{d}x = \left[k\frac{\partial T}{\partial x}\right]_{x=0}^{x=L} = 0.$$

This equation represents global conservation of energy: the thermal energy stored in the rod is constant because all of its surfaces are insulated. Integrating and applying the initial condition gives

$$\int_0^L \rho c_v T(x,t)\,\mathrm{d}x = \int_0^L \rho c_v f(x)\,\mathrm{d}x \quad \text{for } t > 0.$$

(4) The exponentially decaying modes in the solution for T imply that the temperature

$$T(x,t) \to \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \quad \text{as } t \to \infty,$$

i.e. the temperature tends to the mean of the initial temperature profile. This is because the rod is insulated so that heat conduction acts to drive the temperature toward the steady-state solution in which T is spatially uniform.

$$\int_0^L \rho c_v T_t \, dx = \int_0^L k T_{xx} \, dx$$

$$\Rightarrow \quad \frac{d}{dt} \int_0^L \rho c_v T \, dx = \left[k T_x \right]_{x=0}^{x=L} = 0$$

$$\Rightarrow \quad \int_0^L \rho c_v T(x,t) \, dx = \int_0^L \rho c_v T(x,0) \, dx = \int_0^L \rho c_v f(x) \, dx \quad \text{for } t \geq 0.$$

Example: trapped heat

- Consider the IBVP in which the initial temperature profile is given by

$$f(x) = T^* \exp\big(\cos(\pi x/L)\big)\cos\big(\sin(\pi x/L)\big) \quad \text{for } 0 < x < L,$$

  where T^* is a positive constant.

- Recalling from §1.1 that

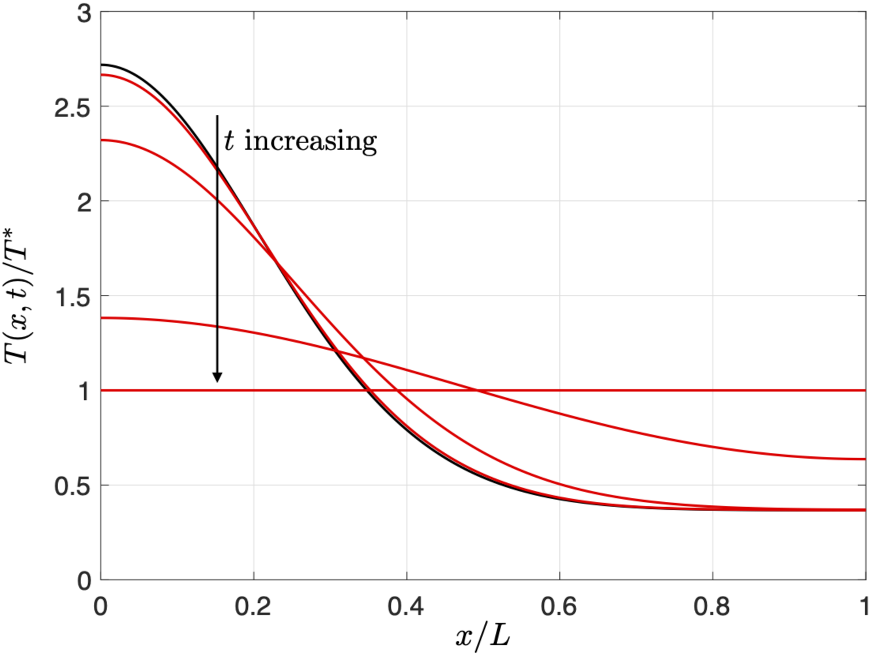
$$\exp(\cos\theta)\cos(\sin\theta) = \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!} \quad \text{for} \quad \theta \in \mathbb{R},$$

  we deduce that $a_0 = 2T^*$ and $a_n = T^*/n!$ for $n \geq 1$, giving the solution

$$T(x,t) = T^* + \sum_{n=1}^{\infty} \frac{T^*}{n!} \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right).$$

- We plot below snapshots of the partial sums of the truncated series solution with 6 terms for $\kappa t/L^2 = 0$ (black line) and $\kappa t/L^2 = 10^{-3}$, 10^{-2}, 10^{-1}, 1 (red lines), illustrating the rapid evolution toward the spatially uniform steady-state in which $T = T^*$.

- Since the thermal energy of the rod is conserved, the area under each curve is the same.



$T(x,t)/T^*$
x/L
t increasing
0
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
0.8
1

3.9 Inhomogeneous heat equation and boundary conditions

- Consider the IBVP for the temperature $T(x,t)$ in a rod of length L given by the inhomogeneous heat equation
$$\rho c_v \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0,$$
with the inhomogeneous Neumann boundary conditions
$$-kT_x(0,t) = q_L(t), \quad -kT_x(L,t) = -q_R(t) \quad \text{for} \quad t > 0,$$
and the initial condition
$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L,$$
where $Q(x,t)$ is the rate of volumetric heating, $q_L(t)$ is the heat flux into the left-hand end, $q_R(t)$ is the heat flux into the right-hand end and $f(x)$ is the initial temperature profile, each of these functions being prescribed.

- **Notes:**

  (1) The Uniqueness Theorem in §3.6 may be adapted to show that the initial boundary value problem has at most one solution.

  (2) Integrating the heat equation across the rod and applying the boundary conditions, we find that

  $$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \rho c_v T(x,t)\,\mathrm{d}x = q_L(t) + q_R(t) + \int_0^L Q(x,t)\,\mathrm{d}x,$$

  which represents global conservation of energy: the thermal energy stored in the rod increases or decreases at the net rate at which thermal energy is supplied to the rod by the heat flux through its ends and by volumetric heating.

- In general Fourier's method cannot be used to solve the IBVP for $T(x,t)$ because the heat equation and boundary conditions are inhomogeneous, *i.e.* $Q(x,t)$, $q_L(t)$ and $q_R(t)$ are non-zero. We now describe a generalization of Fourier's method that works.

- First step is to transform to an IBVP in which the BCs are homogeneous.

- Let $T(x,t) = S(x,t) + u(x,t)$, where S is some fn that satisfies the BCs.

- We can take e.g. $S(x,t) = q_L(t)\frac{(x-L)^2}{2kL} + q_R(t)\frac{x^2}{2kL}$.

- Second step is to find IBVP governing $u(x,t)$.

- $\rho c_v T_t = kT_{xx} + Q \Rightarrow \rho c_v (S+u)_t = k(S+u)_{xx} + Q$

$$\Rightarrow \quad \rho c_v u_t = k u_{xx} + \tilde{Q}(x,t) \text{ where } 0<x<L,\ t>0$$

where $\tilde{Q}(x,t) = Q(x,t) - \rho c_v S_t(x,t) + kS_{xx}(x,t)$

is known in terms of Q, q_L and q_R.

- $-kT_x(0,t) = q_L(t) \Rightarrow -k(S+u)_x(0,t) = q_L(t)$

$$\Rightarrow -k\cancel{S_x(0,t)} - ku_x(0,t) = \cancel{q_L(t)}$$

$$\Rightarrow -ku_x(0,t) = 0 \quad \text{for } t>0.$$

- $-kT_x(L,t) = -q_R(t) \Rightarrow -ku_x(L,t) = 0$ for $t>0$.

- $T(x,0) = f(x) \Rightarrow S(x,0) + u(x,0) = f(x)$

$$\Rightarrow u(x,0) = \tilde{f}(x) \quad \text{for } 0<x<L,$$

where $\tilde{f}(x) = f(x) - S(x,0)$ is known in terms of f, q_L and q_R.

Hence, $u(x,t)$ satisfies the IBVP

① $\rho c_v u_t = K u_{xx} + \tilde{Q}(x,t)$ for $0 < x < L,\ t > 0$

② $u_x(0,t) = 0,\ u_x(L,t) = 0$ for $t > 0$

③ $u(x,0) = \tilde{f}(x)$ for $0 < x < L$

where $\tilde{Q}$ and $\tilde{f}$ are known (in terms of Q, q_L, q_R and f).

Now we need some inspiration to solve for $u(x,t)$!

This is the third and final step.

NB: If $\tilde{Q} \equiv 0$, then Fourier's Method $\Rightarrow$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n e^{-n^2\pi^2 \nu t/L^2}\right)\cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L}\int_0^L \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

This suggests the trick: if $\tilde{Q} \not\equiv 0$, then seek a solution for $u(x,t)$ by expanding it as a Fourier cosine series of the form

$$u(x,t) = \frac{u_0(t)}{2} + \sum_{n=1}^{\infty} u_n(t)\cos\left(\frac{n\pi x}{L}\right),$$

where

$$u_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx.$$

NB: BCs 2 satisfied automatically (assuming suff. rapid convergence).

Q<u>n</u>: How do we determine $u_n(t)$?

Ans: Inspired by the proof of the uniqueness theorem, consider $\frac{du_n}{dt}$.

$$\rho c v \frac{du_n}{dt} = \rho c v \frac{d}{dt} \frac{2}{L}\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L \rho c_v u_t \cos\left(\frac{n\pi x}{L}\right)dx \qquad \text{(by LIR, } L \text{ const.)}$$

$$= \frac{2}{L}\int_0^L \left(k u_{xx} + \tilde{Q}\right)\cos\left(\frac{n\pi x}{L}\right)dx \qquad \text{(by (1))}$$

$$= k\frac{2}{L}\int_0^L u_{xx}\cos\left(\frac{n\pi x}{L}\right)dx + \tilde{Q}_n(t)$$

where $$\tilde{Q}_n(t) = \frac{2}{L}\int_0^L \tilde{Q}(x,t)\cos\left(\frac{n\pi x}{L}\right)dx$$

IBP twice using the identity

$$\int_0^L uv'' - u''v \, dx = \int_0^L (uv' - u'v)' \, dx = \left[uv' - u'v \right]_0^L$$

Let $u = U$ and $v = \cos\left(\frac{n\pi x}{L}\right)$, then

$$\int_0^L U\left(-\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi x}{L}\right)\right) - U_{xx}\cos\left(\frac{n\pi x}{L}\right) dx = \cancel{\left[U\left(-\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\right) - U_x \cos\left(\frac{n\pi x}{L}\right)\right]_0^L}$$

$\rightarrow 0$ by ②

Hence, $\frac{2}{L}\int_0^L U_{xx}\cos\left(\frac{n\pi x}{L}\right) dx = -\frac{n^2\pi^2}{L^2}\,\frac{2}{L}\int_0^L U\cos\left(\frac{n\pi x}{L}\right) dx = -\frac{n^2\pi^2}{L^2} U_n.$

Combo $\Rightarrow$ $\boxed{\rho c_v \frac{dU_n}{dt} + \frac{n^2\pi^2 k}{L^2} U_n = \tilde{Q}_n(t) \quad \text{for } t > 0}$

IC ③ $U(x,0) = f(x)$ for $0 < x < L$ $\Rightarrow$ $\boxed{U_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx}$

- We deal first with the boundary conditions: if we let $T(x,t) = S(x,t) + U(x,t)$, where

$$S(x,t) = q_L(t)\frac{(x-L)^2}{2kL} + q_R(t)\frac{x^2}{2kL},$$

say, is chosen to satisfy the boundary conditions.

- By linearity the IBVP for $T(x,t)$ implies that the IBVP for $U(x,t)$ is given by

$$\rho c_v \frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \widetilde{Q}(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with the homogeneous Neumann boundary conditions

$$U_x(0,t) = 0, \quad U_x(L,t) = 0 \quad \text{for} \quad t > 0,$$

and the initial condition

$$U(x,0) = \widetilde{f}(x) \quad \text{for} \quad 0 < x < L,$$

where the functions

$$\widetilde{Q}(x,t) = Q(x,t) + k\frac{\partial^2 S}{\partial x^2} - \rho c_v \frac{\partial S}{\partial t}, \quad \widetilde{f}(x) = f(x) - S(x,0)$$

are known in terms of $Q(x,t)$, $q_L(t)$, $q_R(t)$ and $f(x)$.

- Thus, the boundary conditions have been rendered homogeneous by 'shifting the data' in the sense that both $q_L(t)$ and $q_R(t)$ have moved from the boundary conditions for $T(x,t)$ into the heat equation and initial conditions for $U(x,t)$.

- If $\widetilde{Q} \equiv 0$, then we can solve the IBVP for $U(x,t)$ using Fourier's method as in §3.8 to obtain

$$U(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 kt}{\rho c_v L^2}\right), \qquad a_n = \frac{2}{L}\int_0^L \widetilde{f}(x) \cos\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x,$$

where the Fourier coefficients a_n have been chosen to satisfy the initial condition.

- The series solution for $U(x,t)$ suggests that if $\widetilde{Q}(x,t)$ is not identically zero, then we should seek a solution for $U(x,t)$ in the form of the Fourier cosine series

$$U(x,t) = \frac{U_0(t)}{2} + \sum_{n=1}^{\infty} U_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

in which the Fourier coefficients

$$U_n(t) = \frac{2}{L}\int_0^L U(x,t) \cos\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x$$

are to be determined.

- **Question:** How do we derive an equation for $U_n(t)$?
- **Answer:** Inspired by the proof of the uniqueness theorem in §3.6, we proceed as follows
- We differentiate $U_n(t)$ with respect to t to obtain

$$\rho c_v \frac{\mathrm{d}U_n}{\mathrm{d}t} = \frac{2}{L}\int_0^L \rho c_v \frac{\partial U}{\partial t}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L \left(k\frac{\partial^2 U}{\partial x^2} + \widetilde{Q}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

where we used Leibniz's Integral Rule in the first equality and the heat equation in the second.

- Integrating by parts using the identity

$$\int_0^L uv'' - u''v\,\mathrm{d}x = \int_0^L (uv' - u'v)'\,\mathrm{d}x = \big[uv' - u'v\big]_0^L$$

with $u = U$ and $v = \cos(n\pi x/L)$ gives

$$\int_0^L U\left(-\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi x}{L}\right)\right) - U_{xx}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \left[U\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) - U_x\cos\left(\frac{n\pi x}{L}\right)\right]_0^L = 0$$

by the boundary conditions, so that

$$\frac{2}{L}\int_0^L U_{xx}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{n^2\pi^2}{L^2}\frac{2}{L}\int_0^L U\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{n^2\pi^2}{L^2}U_n.$$

- We deduce that $U_n(t)$ is governed by the ODE

$$\rho c_v \frac{\mathrm{d}U_n}{\mathrm{d}t} + \frac{kn^2\pi^2}{L^2} U_n = \widetilde{Q}_n(t) \quad \text{for} \quad t > 0,$$

where the coefficients of the Fourier cosine series for $\widetilde{Q}(x,t)$ are defined by

$$\widetilde{Q}_n(t) = \frac{2}{L}\int_0^L \widetilde{Q}(x,t)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L \left(Q(x,t) + k\frac{\partial^2 S}{\partial x^2} - \rho c_v \frac{\partial S}{\partial t}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

- The initial condition for $U(x,t)$ implies that the initial condition for $U_n(t)$ is given by

$$U_n(0) = \frac{2}{L}\int_0^L \widetilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L \left(f(x) - S(x,0)\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

- Using an integrating factor, we find that the solution for $U_n(t)$ may be written in the form

$$U_n(t) = \left(\frac{1}{\rho c_v}\int_0^t \widetilde{Q}_n(s)\mathrm{e}^{\kappa_n s}\,\mathrm{d}s + U_n(0)\right)\mathrm{e}^{-\kappa_n t},$$

where $\kappa_n = n^2\pi^2\kappa/L^2$ in terms of the thermal diffusivity $\kappa = k/(\rho c_v)$.

- In summary, we have been able to solve analytically the IBVP for $T(x,t)$: the solution is given by

$$T(x,t) = S(x,t) + \frac{U_0(t)}{2} + \sum_{n=1}^{\infty} U_n(t)\cos\left(\frac{n\pi x}{L}\right),$$

where

$$\begin{aligned} S(x,t) &= q_L(t)\frac{(x-L)^2}{2kL} + q_R(t)\frac{x^2}{2kL}, \\ U_n(t) &= \left(\frac{1}{\rho c_v}\int_0^t \widetilde{Q}_n(s)\mathrm{e}^{\kappa_n s}\,\mathrm{d}s + U_n(0)\right)\mathrm{e}^{-\kappa_n t}, \end{aligned}$$

with $\kappa_n = n^2\pi^2\kappa/L^2$, $\kappa = k/(\rho c_v)$ and

$$\begin{aligned} \widetilde{Q}_n(t) &= \frac{2}{L}\int_0^L \left(Q(x,t) + k\frac{\partial^2 S}{\partial x^2} - \rho c_v \frac{\partial S}{\partial t}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \\ U_n(0) &= \frac{2}{L}\int_0^L \big(f(x) - S(x,0)\big)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \end{aligned}$$

Notes

(1) If $\widetilde{Q}(x,t)=0$, then $\widetilde{Q}_n(t)=0$ and we recover the solution for $U_n(t)$ obtained by Fourier's method.

(2) The ODE for $U_0(t)$ is equivalent to the expression representing global conservation of energy.

(3) The derivation of the ODE for $U_n(t)$ may also be accomplished by multiplying the heat equation by $\cos(n\pi x/L)$ and integrating from $x=0$ to $x=L$ to obtain

$$\int_0^L \left(\rho c_v \frac{\partial U}{\partial t} - k\frac{\partial^2 U}{\partial x^2} - \widetilde{Q}(x,t)\right)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = 0;$$

the ODE then follows upon applying Leibniz's integral rule to the U_t term and integrating by parts the U_{xx} term.

(4) **Question:** What are the advantages of expanding U as a Fourier cosine series rather than T?

Answer: Expanding T as a Fourier cosine series is equivalent to expanding S as a Fourier cosine series, which cannot improve the accuracy of the approximate solution that would be obtained by truncation. In general the method of shifting the data (to render homogeneous the boundary conditions) results in a solution that converges more rapidly, especially if Gibb's phenomenon can be avoided by doing so.

3.9 Inhomogeneous heat equation and boundary conditions

Example: sinusoidal forcing

- Consider the IBVP in which

$$q_L(t) = q^* \sin(\omega t), \quad q_R(t) = 0, \quad Q(x,t) = 0, \quad f(x) = 0,$$

where q^* and ω are positive constants, as if the left-hand end of the rod were radiated sinusoidally.

- We obtain,

$$\widetilde{Q}_n(s) = \begin{cases} \dfrac{2q^*}{L}\sin(\omega s) - \dfrac{\omega L q^*}{3\kappa}\cos(\omega s) & \text{for } n = 0, \\ -\dfrac{2\omega L q^*}{\kappa n^2 \pi^2}\cos(\omega s) & \text{for } n \geq 1, \end{cases}$$

with $U_n(0) = 0$ for $n \geq 0$.

- Hence, the solution for $U_n(t)$ gives

$$U_n(t) = \begin{cases} \dfrac{2\kappa T^*}{\omega L^2}\big(1 - \cos(\omega t)\big) - \dfrac{T^*}{3}\sin(\omega t) & \text{for } n = 0, \\ \dfrac{2\omega T^*}{n^2\pi^2(\kappa_n^2 + \omega^2)}\Big(\kappa_n \cos(\omega t) + \omega \sin(\omega t) - \kappa_n \exp(-\kappa_n t)\Big) & \text{for } n \geq 1, \end{cases}$$

where we defined the temperature $T^* = Lq^*/k$.

- It follows that the solution may be written in the form

$$T(x,t) = T_\infty(x,t) + V(x,t),$$

where

$$\begin{aligned} T_\infty(x,t) \quad &= \quad T^* \sin(\omega t) \frac{(x-L)^2}{2L^2} + \frac{\kappa T^*}{\omega L^2}\big(1 - \cos(\omega t)\big) - \frac{T^*}{6} \sin(\omega t) \\ &\qquad + \sum_{n=1}^{\infty} \frac{2\omega T^*}{n^2\pi^2(\kappa_n^2 + \omega^2)} \Big(\kappa_n \cos(\omega t) + \omega \sin(\omega t)\Big) \cos\left(\frac{n\pi x}{L}\right) \end{aligned}$$

and

$$V(x,t) = -\sum_{n=1}^{\infty} \frac{2\kappa_n \omega T^*}{n^2\pi^2(\kappa_n^2 + \omega^2)} \exp(-\kappa_n t) \cos\left(\frac{n\pi x}{L}\right).$$

- Since $V(x,t)$ decays exponentially with t, the solution settles down rapidly to a periodic solution $T_\infty(x,t)$ with frequency ω.

An alternative route to the periodic solution

Suppose $T_p(x,t)$ is a periodic solution with frequency ω of

① $\dfrac{\partial T_p}{\partial t} = \kappa \dfrac{\partial^2 T_p}{\partial x^2}$ for $0 < x < L$,

② $-k \dfrac{\partial T_p}{\partial x}(0,t) = q^* \sin(\omega t)$, $\dfrac{\partial T_p}{\partial x}(L,t) = 0$.

Trick: Seek a solution $T_p = \operatorname{Im}\left(e^{i\omega t} F(x)\right)$, with $-k F'(0) = q^*$ and $F'(L) = 0$.

NB: since taking the imaginary part commutes with partial differentiation, the BCs ② hold & we can in fact work with $T_p = e^{i\omega t} F(x)$, and take the imaginary part at the end.

① $\Rightarrow \quad i\omega e^{i\omega t} F(x) = \kappa e^{i\omega t} F''(x)$

$\Rightarrow \quad F'' = \dfrac{i\omega}{\kappa} F$ for $0 < x < L$

$F = e^{\lambda x} \quad \Rightarrow \quad \lambda^2 = \dfrac{i\omega}{\kappa}$

$\Rightarrow \quad \lambda = \pm\sqrt{\dfrac{\omega}{\kappa}}\, e^{i\pi/4} = \pm\sqrt{\dfrac{\omega}{2\kappa}}\,(1+i)$

$\Rightarrow \quad F = A e^{\nu(1+i)x/L} + B e^{-\nu(1+i)x/L} \quad (A, B \in \mathbb{C})$

where $\nu = L\sqrt{\dfrac{\omega}{2\kappa}}$.

$F'(L) = 0 \quad \Rightarrow \quad F(x) = C \cosh\left[\nu(1+i)\left(\dfrac{x-L}{L}\right)\right] \quad (C \in \mathbb{C})$

$$-kf'(0) = q^* \Rightarrow -k\left(\nu(1+i)\frac{1}{L}\sinh\left[\nu(1+i)\left(\frac{0-L}{L}\right)\right]\right) = q^*$$

Hence,

$$T_p = \mathrm{Im}\left[\frac{T^*\cosh\left[\nu(1+i)\left(\frac{x-L}{L}\right)\right]e^{i\omega t}}{\nu(1+i)\sinh\left[\nu(1+i)\right]}\right]$$

where $T^* = \dfrac{Lq^*}{k}$. $\square$

- Since $V(x,t)$ satisfies the homogeneous versions of the heat equation and boundary conditions, the long-time solution $T_\infty(x,t)$ satisfies the same heat equation, but the inhomogeneous boundary conditions.

- We now show that these properties of the long-time solution can be used to construct it directly.

- The trick is to seek a complex-valued separable solution $\mathrm{e}^{\mathrm{i}\omega t}F(x)$ with frequency ω.

- Substituting this *ansatz* into the homogeneous heat equation, we find that

$$\kappa F^{''} = \mathrm{i}\omega F \quad \text{for } 0 < x < L.$$

- Seeking an exponential solution $F(x) = \mathrm{e}^{\lambda x}$ gives the auxiliary equation $\lambda^2 = \mathrm{i}\omega/\kappa$, so that

$$\lambda = \pm\sqrt{\frac{\omega}{\kappa}}\mathrm{e}^{\mathrm{i}\pi/4} = \pm\sqrt{\frac{\omega}{2\kappa}}(1+\mathrm{i}),$$

  giving the general solution

$$F(x) = A\mathrm{e}^{\nu(1+\mathrm{i})x/L} + B\mathrm{e}^{-\nu(1+\mathrm{i})x/L},$$

  where A and B are arbitrary complex constants and $\nu = L\sqrt{\omega/2\kappa}$ is a dimensionless parameter.

- We now observe that if we impose on F the boundary conditions $-kF'(0) = q^*$ and $F'(L) = 0$, then

$$T_p(x,t) = \mathrm{Im}\big(\mathrm{e}^{\mathrm{i}\omega t} F(x)\big)$$

satisfies both the homogeneous heat equation and sinusoidally-forced boundary conditions because taking the imaginary part commutes with partial differentiation.

- The resulting solution for $F(x)$ may then be written in the form

$$F(x) = \frac{T^* \cosh\big(\nu(1+\mathrm{i})(1-x/L)\big)}{\nu(1+\mathrm{i})\sinh\big(\nu(1+\mathrm{i})\big)},$$

so that

$$T_p(x,t) = \mathrm{Im}\left(\frac{T^* \cosh\big(\nu(1+\mathrm{i})(1-x/L)\big)}{\nu(1+\mathrm{i})\sinh\big(\nu(1+\mathrm{i})\big)}\mathrm{e}^{i\omega t}\right).$$

- **Question:** How are the solutions $T_\infty(x,t)$ and $T_p(x,t)$ related?
- **Answer:** Having found a particular solution $T_p(x,t)$ satisfying the homogeneous heat equation and sinusoidally-forced boundary conditions, we see that we could also solve the IBVP for $T(x,t)$ by setting

$$T(x,t) = T_p(x,t) + W(x,t),$$

since then $W(x,t)$ satisfies

$$\rho c_v \frac{\partial W}{\partial t} = k \frac{\partial^2 W}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with

$$W_x(0,t) = 0, \quad W_x(L,t) = 0 \quad \text{for} \quad t > 0,$$

and

$$W(x,0) = -T_p(x,0) \quad \text{for} \quad 0 < x < L,$$

i.e. the boundary conditions are rendered homogeneous by the *ansatz* for $W(x,t)$ while retaining the homogeneity of the heat equation, in contrast to the *ansatz* for $U(x,t)$ which results in homogeneous boundary conditions but at the expense of a forced heat equation.

- The IBVP for $W(x,t)$ may be solved using Fourier's method as in §3.8, giving the solution

$$W(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 kt}{\rho c_v L^2}\right), \qquad c_n = -\frac{2}{L}\int_0^L T_p(x,0)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

so that

$$W(x,t) \to \frac{c_0}{2} = -\frac{1}{L}\int_0^L T_p(x,0)\,\mathrm{d}x = \frac{\kappa T^*}{\omega L^2} \quad \text{as } t \to \infty.$$

- We can now invoke uniqueness of the IBVP for $T(x,t)$ to deduce that

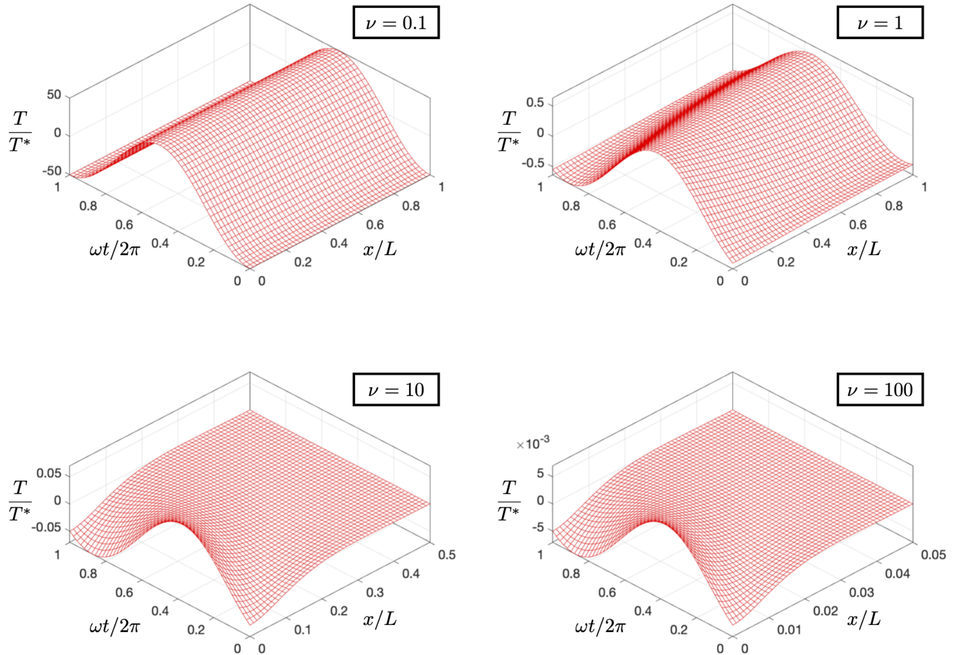
$$T_\infty(x,t) + V(x,t) = T_p(x,t) + W(x,t) \quad \text{for } 0 \le x \le L,\ t \ge 0.$$

But $V(x,t) \to 0$ and $W(x,t) \to \kappa T^*/\omega L^2$ as $t \to \infty$, which can only be the case if

$$T_\infty(x,t) = T_p(x,t) + \frac{\kappa T^*}{\omega L^2} \quad \text{for } 0 \le x \le L,\ t \ge 0 \tag{†}$$

because both $T_\infty(x,t)$ and $T_p(x,t)$ are periodic in t with frequency ω.

- **Remark:** This argument saves us from the unwieldy algebraic manipulations that would otherwise be required to establish the relationship (†), *e.g.* by showing that the Fourier cosine coefficients of each side are identical at say $t = 0$.
- The plots below show a period of oscillation of $T_p(x, t)$ for $\nu = 0.1$, 1, 10 and 100.
- The plots illustrate that the heat flux imposed at $x = 0$ generates a temperature profile that is almost spatially uniform for small ν, but penetrates only partially and inside a thin boundary layer of thickness of order L/ν for large ν.
- This is in accordance with the physical interpretation of $\nu = L/\sqrt{2\kappa/\omega}$ as the ratio of the length of the rod L to the typical distance thermal energy conducts in a period of oscillation (since there is a balance in the heat equation when x and t are scaled with $\sqrt{\kappa/\omega}$ and $1/\omega$, respectively).
- That the shape of the profiles for $\nu = 10$ and $\nu = 100$ are almost identical is because the response in the thin boundary layer is as if the rod were semi-infinite.



$\nu = 0.1$
$\nu = 1$
$\nu = 10$
$\nu = 100$
$\frac{T}{T^*}$
$\omega t/2\pi$
x/L
$\times 10^{-3}$