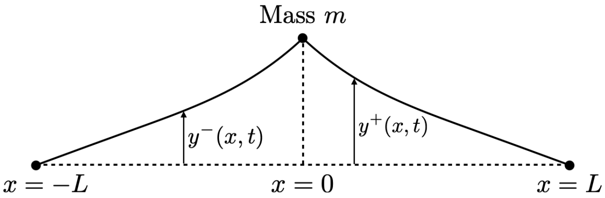
4.8 Normal modes for a weighted string

- An elastic string of length $2L$ has its ends fixed at $(x, y) = (\pm L, 0)$ and a point particle of mass m is attached to the mid-point, as illustrated in the schematic below.



- We seek here the normal modes of vibration.

- Since the transverse displacements are small and the tension T constant, the horizontal components of the forces exerted by the string on the point particle will balance to a first approximation.

- Hence, we need only consider the transverse displacement of the point particle, $Y(t)$ say.

- We let $y^-(x, t)$ and $y^+(x, t)$ denote the small transverse displacements for $-L \leq x < 0$ and $0 < x \leq L$, respectively.

- Then y^- and y^+ must satisfy the wave equations

$$\frac{\partial^2 y^-}{\partial t^2} = c^2 \frac{\partial^2 y^-}{\partial x^2} \quad \text{for} \quad -L < x < 0,$$

$$\frac{\partial^2 y^+}{\partial t^2} = c^2 \frac{\partial^2 y^+}{\partial x^2} \quad \text{for} \quad 0 < x < L,$$

and the boundary conditions

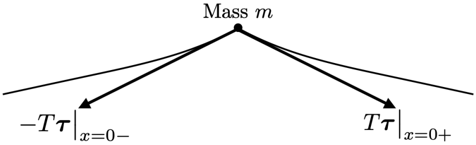
$$y^-(-L, t) = 0,$$

$$y^+(L, t) = 0.$$

- **Question:** What conditions hold at $x = 0$?
- **Answer:** There are two.
- Firstly, since the point particle is attached to the string, we require

$$y^-(0_-, t) = Y(t) = y^+(0_+, t).$$

- Secondly, the string exerts on the point particle the forces illustrated below.



- Here $\boldsymbol{\tau}$ is the right-pointing unit tangent vector to the string given by

$$\boldsymbol{\tau} = \frac{\boldsymbol{i} + y_x \boldsymbol{j}}{(1 + y_x^2)^{1/2}},$$

where $y = y^-$ for $-L < x < 0$ and $y = y^+$ for $0 < x < L$.

- Hence, applying Newton's Second Law to the point particle in the y-direction gives

$$m\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} = \big(T\boldsymbol{\tau}(0_+, t) - T\boldsymbol{\tau}(0_-, t)\big) \cdot \mathbf{j}.$$

- Since

$$\left(1 + y_x^2\right)^{1/2} = 1 + \frac{1}{2}(y_x)^2 + \cdots \quad \text{for} \quad |y_x| \ll 1,$$

we deduce that to a first approximation

$$m\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} = T y_x^+(0_+, t) - T y_x^-(0_-, t).$$

- To find the normal modes we seek nontrivial separable solutions of the form

$$y^{\pm} = F_{\pm}(x)G(t),$$

since we need the same time dependence in $y_{\pm}(x,t)$ if they are to satisfy the BCs at $x = 0$.

- In the usual manner we may deduce from the wave equations that there is a constant λ such that

$$\frac{F''_{\pm}(x)}{F_{\pm}(x)} = \frac{G''(t)}{c^2 G(t)} = -\lambda,$$

- Since we're seeking nontrivial solutions, it follows from the boundary conditions at $x = \pm L$ that

$$F_{-}(-L) = 0, \;\; F_{+}(L) = 0.$$

- Similarly, the boundary conditions at $x = 0$ give

$$F_{-}(0_{-}) = F_{+}(0_{+}),$$

and

$$mF_{\pm}(0)G''(t) = T(F'_{+}(0_{+}) - F'_{-}(0_{-}))G(t).$$

- Using $G''(t) + \lambda c^2 G(t) = 0$ and $c^2 = T/\rho$, we deduce that

$$-\lambda m F_{\pm}(0) = \rho\big(F'_{+}(0_{+}) - F'_{-}(0_{-})\big).$$

- Since we are seeking non-trivial oscillatory solutions, we now focus on the case in which λ is positive by setting $\lambda = \omega^2$, where $\omega > 0$ without loss of generality.
- Then $G''(t) + \lambda c^2 G(t) = 0$ gives $G(t) = C\cos(\omega c t + \epsilon)$, where ϵ is an arbitrary constant and we may take $C = 1$ without loss of generality.
- Moreover, $F_{\pm}(x)$ satisfy

$$\begin{aligned} F_-'' + \omega^2 F_- &= 0 \text{ for } -L < x < 0, \\ F_+'' + \omega^2 F_+ &= 0 \text{ for } 0 < x < L, \end{aligned}$$

with $F_-(-L) = 0$ and $F_+(L) = 0$, so that

$$\begin{aligned} F_-(x) &= A \sin\big(\omega(L + x)\big), \\ F_+(x) &= B \sin\big(\omega(L - x)\big), \end{aligned}$$

where A and B are arbitrary real constants.

- Substituting into the boundary conditions relating $F_{\pm}(x)$ at $x = 0$, we obtain

$$\underbrace{\begin{bmatrix} \sin \omega L & -\sin \omega L \\ \rho \cos \omega L - m\omega \sin \omega L & \rho \cos \omega L \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- For nontrivial solutions $F_{\pm}(x)$, we need

$$\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and hence for the matrix M to be singular: setting $\det(M) = 0$, we deduce that ω must satisfy

$$\sin \omega L \, (2\rho \cos \omega L - m\omega \sin \omega L) = 0.$$

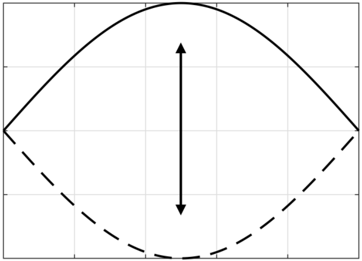
- Hence, there are two cases: either (i) $\sin \omega L = 0$ or (ii) $2\rho \cos \omega L - m\omega \sin \omega L = 0$.

Case (i) $\sin \omega L = 0$

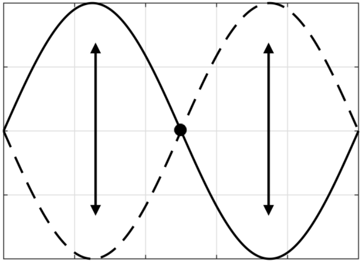
- We deduce immediately that $\omega = n\pi/L$, where n is a positive integer.
- Then the matrix equation fo $[A, B]^T$ gives $B = -A$, so that the normal modes are given by

$$
\begin{aligned}
y_-(x,t) &= A \sin\big(\omega(L+x)\big)\cos(\omega c t + \epsilon), \\
y_+(x,t) &= -A \sin\big(\omega(L-x)\big)\cos(\omega c t + \epsilon).
\end{aligned}
$$

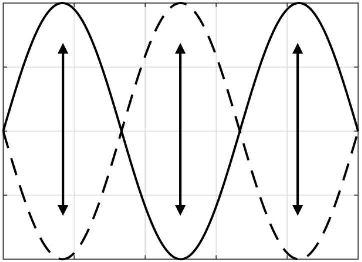
- This means that the normal modes are the same as for a string of length $2L$ with a node at $x = 0$, *i.e.* the point particle is stationary and remains at the origin, as illustrated for the first few such modes in the schematic below.



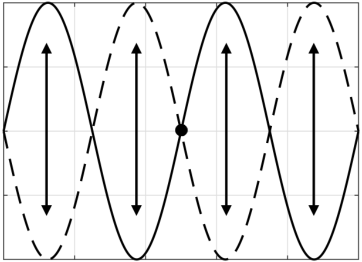
Not admissible



Admissible ($n = 1$)



Not admissible



Admissible ($n = 2$)

Case (ii) $2\rho\cos\omega L - m\omega\sin\omega L = 0$

- If we scale $\omega = \theta/L$, then θ satisfies the transcendental equation

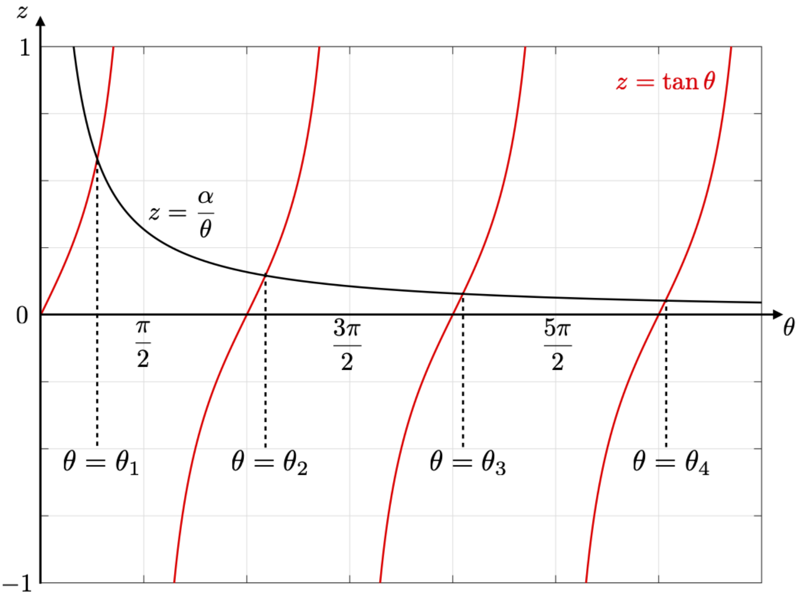
$$\tan\theta = \frac{\alpha}{\theta},$$

where the dimensionless parameter $\alpha = 2L\rho/m$ is the ratio of the mass of the string to that of the point particle.

- By plotting the graphs of $z = \tan\theta$ and $z = \alpha/\theta$, as illustrated below for $\alpha = 1$, we can convince ourselves that there are countably many roots

$$\theta_1 < \theta_2 < \theta_3 < \cdots,$$

with $(n-1)\pi < \theta_n < (n-1/2)\pi$ and $\theta_n/(n-1) \to \pi+$ as $n \to \infty$.



z
1
0
−1
$z = \tan\theta$
$z = \dfrac{\alpha}{\theta}$
$\dfrac{\pi}{2}$
$\dfrac{3\pi}{2}$
$\dfrac{5\pi}{2}$
θ
$\theta = \theta_1$
$\theta = \theta_2$
$\theta = \theta_3$
$\theta = \theta_4$

- Hence, there are countably many natural frequencies

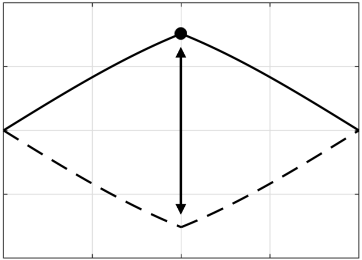
$$\omega c = \theta_n c/L,$$

  where n is a positive integer.

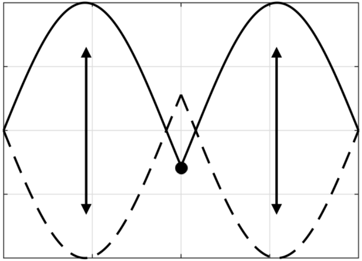
- Now the matrix equation fo $[A, B]^T$ gives $B = A$, so that the normal modes are given by

$$\begin{aligned} y_-(x,t) &= A\sin\big(\omega(L+x)\big)\cos(\omega ct+\epsilon), \\ y_+(x,t) &= A\sin\big(\omega(L-x)\big)\cos(\omega ct+\epsilon). \end{aligned}$$

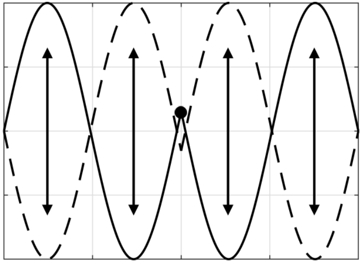
- This means that the string is symmetric about $x = 0$, as illustrated for the first few such modes in the schematic below.



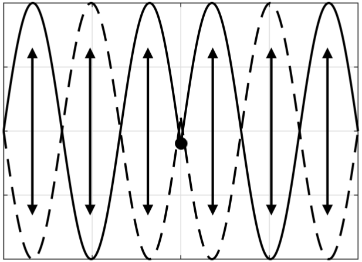
$n = 1$



$n = 2$



$n = 3$



$n = 4$

4.9 General solution to the wave equation

- It is a remarkable fact that it is possible to write down all solutions of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where we recall that the parameter $c > 0$ is the wave speed.

- To verify this fact we introduce new independent variables

$$\xi = x - ct, \quad \eta = x + ct,$$

and seek a solution in which

$$y(x,t) = Y(\xi, \eta).$$

- The chain rule implies

$$y_x = Y_\xi \xi_x + Y_\eta \eta_x = Y_\xi + Y_\eta,$$

$$y_t = Y_\xi \xi_t + Y_\eta \eta_t = -cY_\xi + cY_\eta.$$

- Then, assuming $Y_{\xi\eta} = Y_{\eta\xi}$,

$$y_{xx} = (Y_\xi + Y_\eta)_\xi \xi_x + (Y_\xi + Y_\eta)_\eta \eta_x = Y_{\xi\xi} + 2Y_{\xi\eta} + Y_{\eta\eta},$$

$$y_{tt} = (-cY_\xi + cY_\eta)_\xi \xi_t + (-cY_\xi + cY_\eta)_\eta \eta_t = c^2(Y_{\xi\xi} - 2Y_{\xi\eta} + Y_{\eta\eta}).$$

- We deduce that

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = -4c^2 \frac{\partial^2 Y}{\partial \xi \partial \eta}.$$

- Hence, in the new variables (ξ, η) the wave equation becomes

$$\frac{\partial^2 Y}{\partial \xi \partial \eta} = 0, \quad i.e. \quad \frac{\partial}{\partial \xi}\left(\frac{\partial Y}{\partial \eta}\right) = 0.$$

- Thus, $\partial Y/\partial \eta$ is independent of ξ and is a function of η only, say $G'(\eta)$, *i.e.*

$$\frac{\partial Y}{\partial \eta} = G'(\eta),$$

  and so

$$\frac{\partial}{\partial \eta}\left[Y - G(\eta)\right] = 0.$$

- Thus, $Y - G(\eta)$ is a function of ξ only, say $F(\xi)$, and therefore

$$Y - G(\eta) = F(\xi),$$

  giving

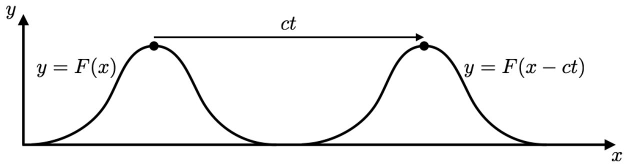
$$y(x,t) = F(x - ct) + G(x + ct),$$

  where F and G are arbitrary twice continuously differentiable functions.

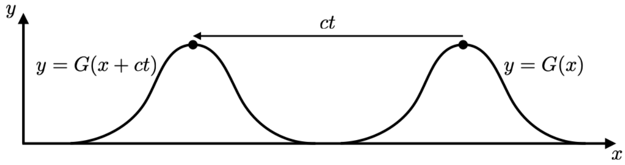
Notes

(1) It is straightforward to use the chain rule to verify that $y(x, t) = F(x - ct) + G(x + ct)$ is a solution of the wave quation. We have shown that all solutions must be of this form.

(2) We note that $F(x - ct)$ is a travelling wave of constant shape moving in the positive x-direction with speed c, as illustrated in the sketch below in which the initial profile $y = F(x)$ at $t = 0$ is translated a distance ct to the right at time t.



(3) We note that $G(x+ct)$ is a travelling wave of constant shape moving in the negative x-direction with speed c, as illustrated in the sketch below in which the initial profile $y = G(x)$ at $t = 0$ is translated a distance ct to the left at time t.



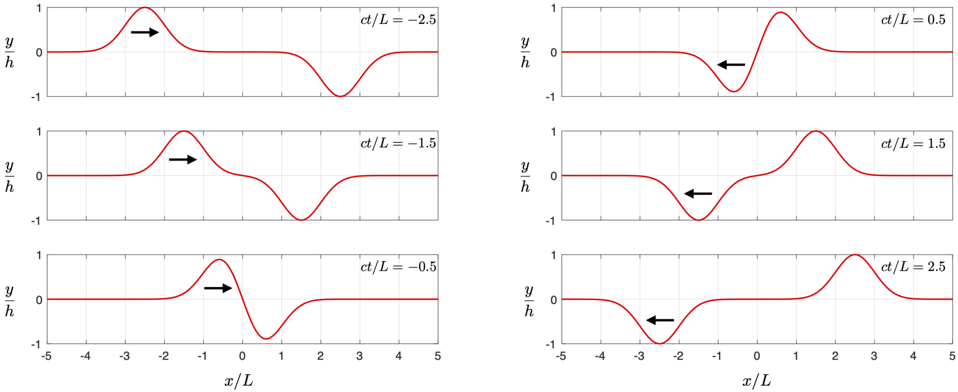
(4) The general solution is therefore the superposition of left- and right-travelling waves each moving with speed c, which is the reason the parameter c is called the wave speed. It follows that the wave equation propagates information at constant speed c in contrast to solutions of the heat equation in which information propagates at infinite speed.

Example: wave reflection

- A string occupies $-\infty < x \leq 0$ and is fixed at $x = 0$. A wave $y(x,t) = f(x - ct)$ is incident from $x < 0$. Find the reflected wave.
- In $y(x,t) = F(x - ct) + G(x + ct)$, we take $F = f$ and G to be found.
- The boundary condition $y(0,t) = 0$ is to be satisfied for all t. Hence, $f(-ct) + G(ct) = 0$ for all t, and so $G(\theta) = -f(-\theta)$ for all θ. Thus,

$$y(x,t) = \underbrace{f(x - ct)}_{\text{incident wave}} - \underbrace{f(-x - ct)}_{\text{reflected wave}}.$$

- The snapshots below illustrates the reflection of an incident wave for $f(x) = h\exp(-x^2/L^2)$, where h and L are positive constants. The arrows indicated the direction of travel with speed c of the incident and reflected waves. Focussing on $x \leq 0$, we see that the reflected wave has the same shape and speed as the incident wave, but the opposite sign and direction of travel.



$ct/L = -2.5$
$ct/L = -1.5$
$ct/L = -0.5$
$ct/L = 0.5$
$ct/L = 1.5$
$ct/L = 2.5$
$\frac{y}{h}$
x/L

4.10 Waves on an infinite string: D'Alembert's formula

- Suppose $y(x,t)$ s.t. ① $y_{tt} = c^2 y_{xx}$ for $-\infty < x < \infty$, $t > 0$,

  ② $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

  where f and g are given.

- ① has the general solution $y(x,t) = F(x-ct) + G(x+ct)$, so it remains to determine the function F and G for which it satisfies the ICs ②.

- ② $\Rightarrow$ $F(x) + G(x) = f(x)$ ⓐ and $-cF'(x) + cG'(x) = g(x)$

$$\Downarrow$$

$$-F(x) + G(x) = a + \frac{1}{c}\int_0^x g(s)\,ds \quad (a \in \mathbb{R}) \quad ⓑ$$

- ⓐ − ⓑ $\Rightarrow$ $F(x) = \frac{1}{2}\left(f(x) - a - \frac{1}{c}\int_0^x g(s)\,ds\right)$

- ⓐ + ⓑ $\Rightarrow$ $G(x) = \frac{1}{2}\left(f(x) + a + \frac{1}{c}\int_0^x g(s)\,ds\right)$

- Hence, $y(x,t) = \frac{1}{2}\left(f(x-ct) - a - \frac{1}{c}\int_0^{x-ct} g(s)\,ds\right) + \frac{1}{2}\left(f(x+ct) + a + \frac{1}{c}\int_0^{x+ct} g(s)\,ds\right)$

$$= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\left(\int_{x-ct}^{0} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right)$$

giving

$$y(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

D'Alembert's Formula

- NB: argument $\Rightarrow \exists!$ solution to IVP ①–②!

- NB: can also prove uniqueness via energy method as for a finite string (assuming y decays suff. rapidly as $x \to \pm\infty$ that the energy exists).

- Consider the initial value problem for the small transverse displacement $y(x, t)$ of an elastic string given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty, \ t > 0,$$

  with the initial conditions

$$y(x, 0) = f(x), \quad \frac{\partial y}{\partial t}(x, 0) = g(x) \quad \text{for} \quad -\infty < x < \infty,$$

  where the initial transverse displacement $f(x)$ and the initial transverse velocity $g(x)$ are given.

- The general solution of the wave equation is $y(x, t) = F(x - ct) + G(x + ct)$, so it remains to determine the functions F and G for which it satisfies the initial conditions.

- Substituting gives

$$F(x) + G(x) = f(x), \qquad -cF'(x) + cG'(x) = g(x).$$

- Integrating the second expression gives the system

$$F(x) + G(x) = f(x), \qquad -F(x) + G(x) = \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + a,$$

where a is an arbitrary constant.

- Subtracting and adding, we deduce that F and G are given by

$$F(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s - a\right), \qquad G(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + a\right).$$

- Hence,

$$\begin{aligned} y(x,t) &= \frac{1}{2}\left(f(x-ct) - \frac{1}{c}\int_0^{x-ct} g(s)\,\mathrm{d}s - a\right) + \frac{1}{2}\left(f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(s)\,\mathrm{d}s + a\right) \\ &= \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\left(\int_{x-ct}^0 g(s)\,\mathrm{d}s + \int_0^{x+ct} g(s)\,\mathrm{d}s\right), \end{aligned}$$

giving *D'Alembert's Formula*,

$$y(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

Notes:

(1) The argument shows that, for given f and g, the initial value problem has one and only one solution, *i.e.* existence and uniqueness.

(2) Uniqueness may also be proved by energy conservation under the additional assumption that y_t, $y_x \to 0$ sufficiently rapidly as $x \to \pm\infty$ that we can ensure the existence of the energy

$$E(t) = \int_{-\infty}^{\infty} \frac{\rho}{2} y_t^2 + \frac{T}{2} y_x^2 \, \mathrm{d}x.$$

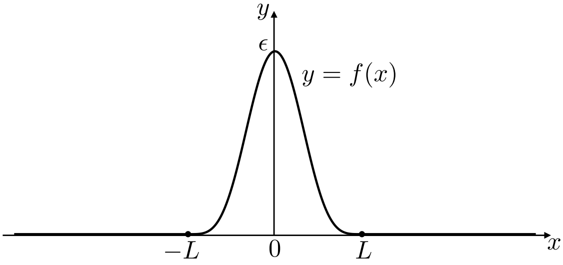
Example 1

- Suppose that f and g are given by

$$f(x) = \begin{cases} \epsilon \cos^4\left(\dfrac{\pi x}{2L}\right) & \text{for } |x| \leq L, \\ 0 & \text{otherwise,} \end{cases} \qquad g(x) = 0,$$

where ϵ and L are positive constants.

- **Remark:** As illustrated in the sketch below, f, f', f'' and f''' are continuous on $\mathbb{R}$ and f is *compactly supported* because it vanishes outside of a closed bounded interval.



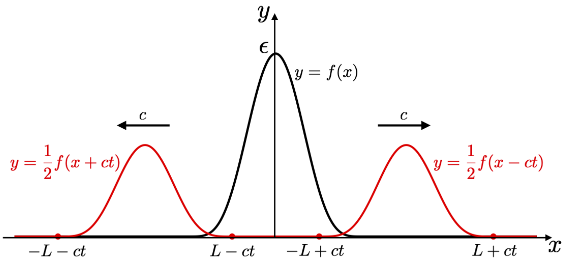
- By D'Alembert's formula the solution is given by

$$y(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big),$$

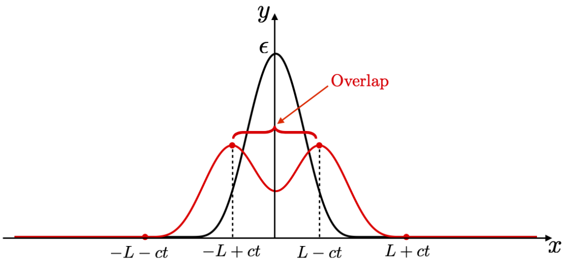
- **Remark:** The solution is a classical solution because it is twice continuously differentiable with respect to x and t and satisfies the IBVP.

- We can sketch the solution $y(x,t)$ at a fixed time $t > 0$ using the geometrical properties of its travelling wave components.

- For $\underline{ct > L}$, the supports of $f(x-ct)$ and $f(x+ct)$ do not overlap, as illustrated below.



- For $\underline{0 < ct < L}$, the supports of $f(x - ct)$ and $f(x + ct)$ overlap, as illustrated below.



- The derivation of explicit formulae for the solution therefore requires some careful bookkeeping for which it is easier to think geometrically rather than algebraically. ■

4.11 Characteristic diagrams

- Consider the initial value problem for the small transverse displacement $y(x,t)$ of an elastic string given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty, \; t > 0,$$

with the initial conditions

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty,$$

where the initial transverse displacement $f(x)$ and the initial transverse velocity $g(x)$ are given.

- In the last section we showed that the solution is given by D'Alembert's Formula:

$$y(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

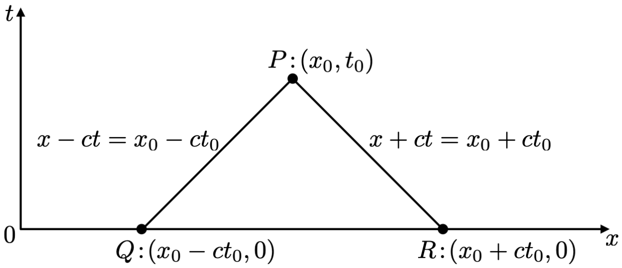
- Let us ask how the solution at a point $P:(x_0, t_0)$ in the upper half of the (x,t)-plane depends upon the data f and g.
- By D'Alembert's Formula, we have

$$y(x_0, t_0) = \frac{1}{2}[f(x_0 - ct_0) + f(x_0 + ct_0)] + \frac{1}{2c}\int_{x_0 - ct_0}^{x_0 + ct_0} g(x)\,\mathrm{d}x,$$

which may be written in the form

$$y(P) = \frac{1}{2}\big(f(Q) + f(R)\big) + \frac{1}{2c}\int_Q^R g(s)\,\mathrm{d}s,$$

where Q and R are the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$, respectively.



- We note the deliberate abuse of notation to aid the geometric interpretation of D'Alembert's formula.
- **Definition:** The lines $x \pm ct = x_0 \pm ct_0$ are the *characteristic lines* through $P\colon(x_0, t_0)$.
- $y(P)$ depends only on
  (i) f though the values f takes at Q and R;
  (ii) g though the values g takes on the x-axis between Q and R.

  This motivates the following definition.
- **Definition:** The interval $[x_0 - ct_0, x_0 + ct_0]$ of the x-axis between Q and R is called the *domain of dependence* of $P\colon(x_0, t_0)$
- If f or g are modified outside the domain of dependence of P, then $y(P)$ is unchanged.

- We can exploit the geometric interpretation of D'Alembert's formula to construct explicit formulae for the solution: the contribution to $y(P)$ from f and g changes at points on the x-axis where f and g change their analytic behaviour.
- Hence, given a particular f and g, the first task is to identify such points on the x-axis and sketch the characteristic lines $x \pm ct =$ constant through each of them — this is the *characterisrtic diagram*.
- The characteristic diagram divides the (x, t)-plane into regions in which the contributions from f and g may be different: the second task is to evaluate $y(P)$ for P in each of these regions.

Example 1 revisited

- Since g vanishes in this case, D'Alembert's formula becomes

$$y(P) = \frac{1}{2}\big(f(Q) + f(R)\big),$$

where Q and R are the left- and right-hand intersections with the x-axis of the characteristic lines though P.

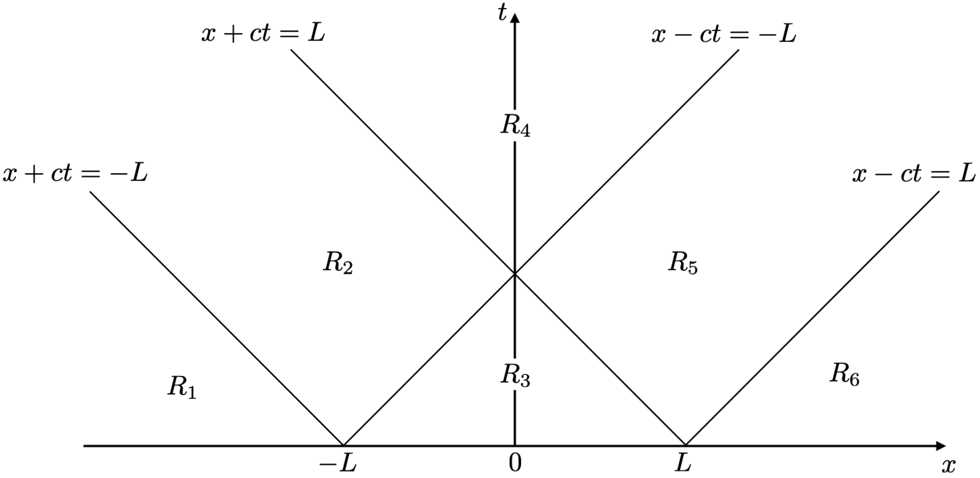
- Recall that f is given by

$$f(x) = \begin{cases} \epsilon \cos^4\left(\frac{\pi x}{2L}\right) & \text{for } |x| \leq L, \\ 0 & \text{otherwise,} \end{cases}$$

so that it is compactly supported with support $(-L, L)$, and therefore changes its analytic behaviour at the points $(-L, 0)$ and $(L, 0)$ on the x-axis in the (x, t)-plane.

- The characteristics through these points are $x \pm ct = -L$ and $x \pm ct = L$ and they divide the upper-half of the (x, t)-plane into six regions $R_1, \ldots, R_6$, forming the characteristic diagram illustrated below.

Characteristic diagram:



- In particular, we let

$$R_1 = \{(x,t) : t > 0,\ x + ct < -L\},$$

$$R_2 = \{(x,t) : t > 0,\ x \leq 0,\ -L \leq x + ct \leq L,\ x - ct \leq -L\},$$

$$R_3 = \{(x,t) : t > 0,\ x - ct > -L,\ x + ct < L\},$$

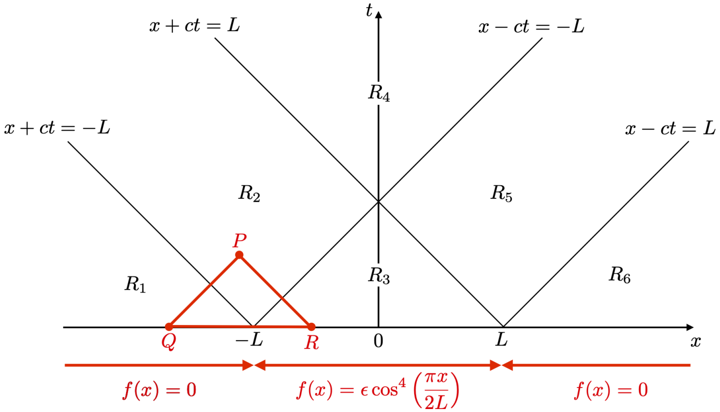
$$R_4 = \{(x,t) : t > 0,\ x - ct < -L,\ x + ct > L\},$$

$$R_5 = \{(x,t) : t > 0,\ x > 0,\ -L \leq x - ct \leq L,\ x + ct \geq L\},$$

$$R_6 = \{(x,t) : t > 0,\ x - ct > L\}.$$

- **Notes:**
  - By including the dividing characteristics in regions R_2 and R_5 (except where they cross at $(0, L/c)$), we have ensured that each point (x, t) in the upper half plane belongs to one and only one region.
  - The choice to have regions R_2 and R_5 contain their bounding characteristics (except for the point $(0, L/c)$) is arbitrary if the solution is everywhere continuous, as it is in this example.

- Since PQ is parallel to the characteristics $x - ct = \pm L$, while PR is parallel to the characteristics $x + ct = \pm L$, we may construct the solution with the aid of the characteristic diagram by drawing on it the triangle PQR for P in each of the different regions.



$P : (x, t)$

$Q : (x - ct, 0)$

$R : (x + ct, 0)$

$$y(P) = \frac{1}{2}\left(f(Q) + f(R)\right)$$

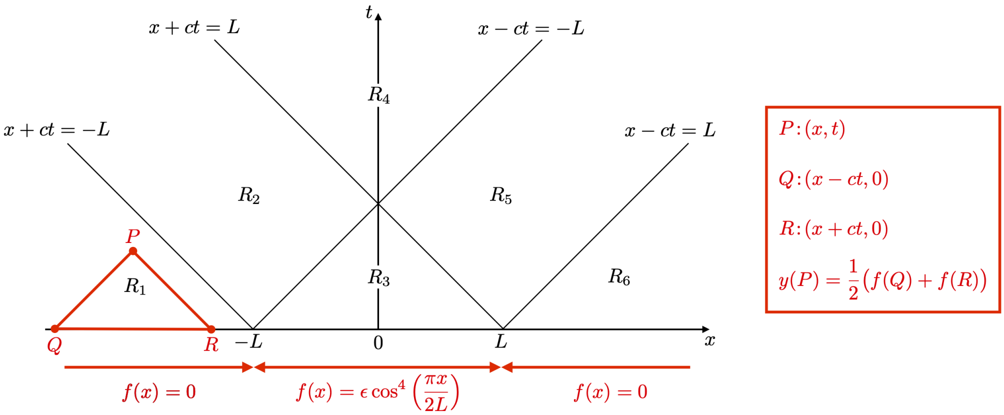
- Thus, the locations of Q and R on the x-axis dictate their contributions, as follows.

- If $P \in R_1$, then Q and R lie to the left of $(-L, 0)$, so

$$f(Q) = f(R) = 0,$$

giving

$$y(x,t) = 0 \quad \text{for } (x,t) \in R_1.$$

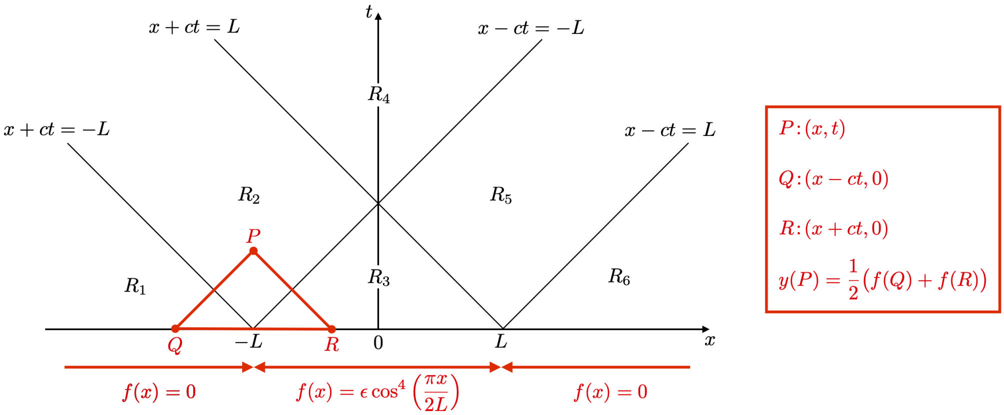


- If $P \in R_2$, then Q lies at or to the left of $(-L, 0)$, while R lies at or between $(-L, 0)$ and $(L, 0)$, so

$$f(Q) = 0, \quad f(R) = f(x+ct) = \epsilon \cos^4\left(\frac{\pi}{2L}(x+ct)\right),$$

giving

$$y(x,t) = \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x+ct)\right) \quad \text{for } (x,t) \in R_2.$$

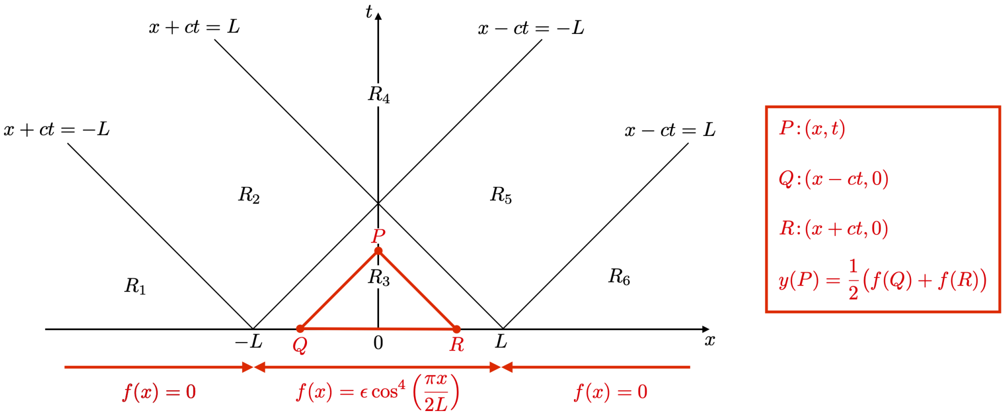


- If $P \in R_3$, then Q and R lie between $(-L, 0)$ and $(L, 0)$, so

$$f(Q) = f(x - ct) = \epsilon \cos^4(\pi(x - ct)/2L), \quad f(R) = f(x + ct) = \epsilon \cos^4\left(\frac{\pi}{2L}(x + ct)\right),$$

giving

$$y(x, t) = \frac{\epsilon}{2} \cos^4\left(\frac{\pi}{2L}(x - ct)\right) + \frac{\epsilon}{2} \cos^4\left(\frac{\pi}{2L}(x + ct)\right) \quad \text{for } (x, t) \in R_3.$$

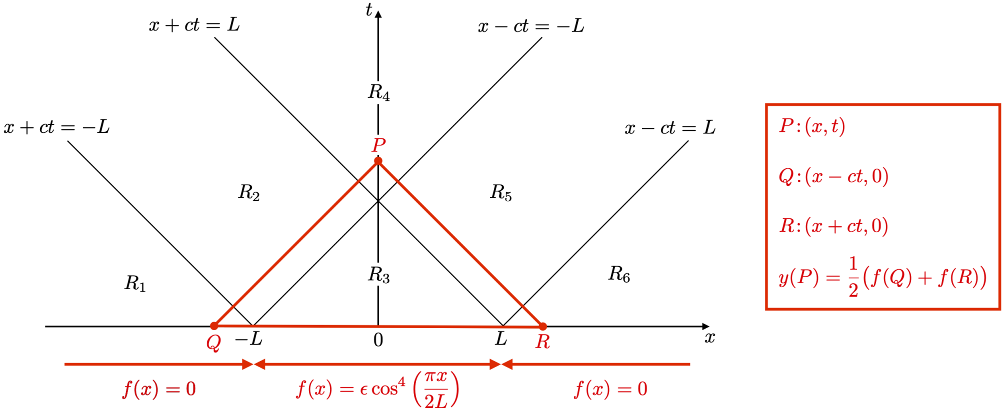


- If $P \in R_4$, then Q lies to the left of $(-L, 0)$ and R lies to the right of $(L, 0)$, so

$$f(Q) = f(R) = 0,$$

giving

$$y(x,t) = 0 \quad \text{for } (x,t) \in R_4.$$

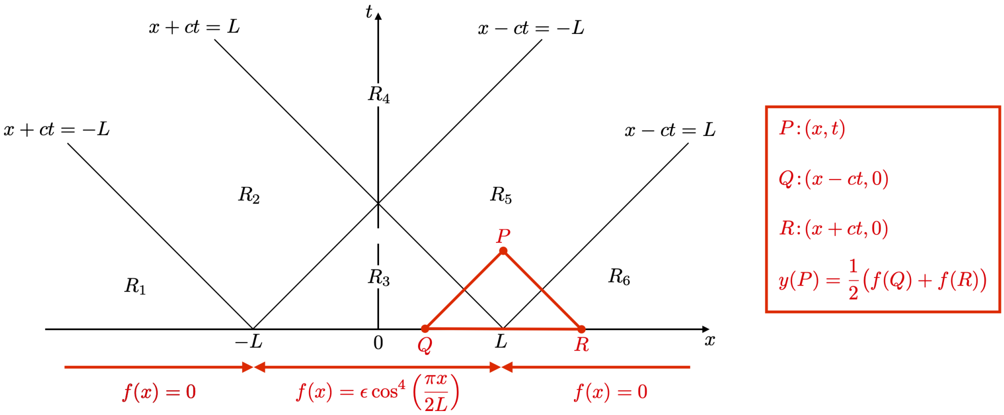


- If $P \in R_5$, then Q lies at or between $(-L, 0)$ and $(L, 0)$, while R lies at or to the right of $(L, 0)$, so

$$f(Q) = f(x - ct) = \epsilon \cos^4\left(\frac{\pi}{2L}(x - ct)\right), \quad f(R) = 0,$$

giving

$$y(x, t) = \frac{\epsilon}{2} \cos^4\left(\frac{\pi}{2L}(x - ct)\right) \quad \text{for } (x, t) \in R_5.$$

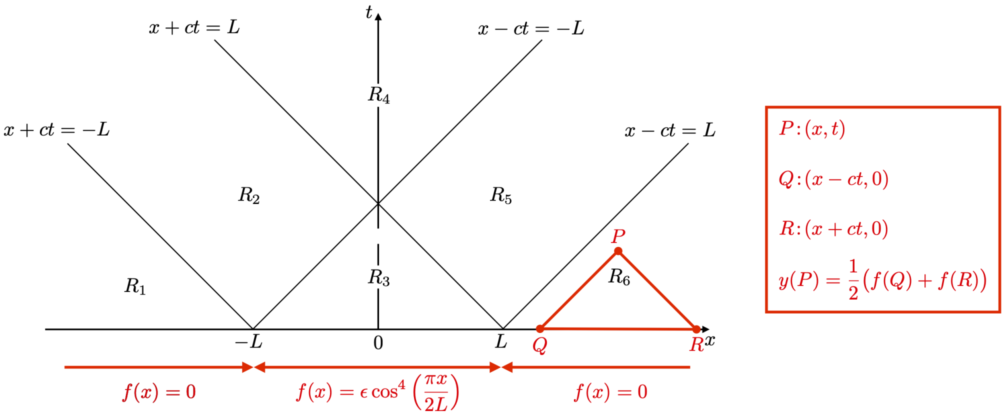


- If $P \in R_6$, then Q and R lie to the right of $(L, 0)$, so

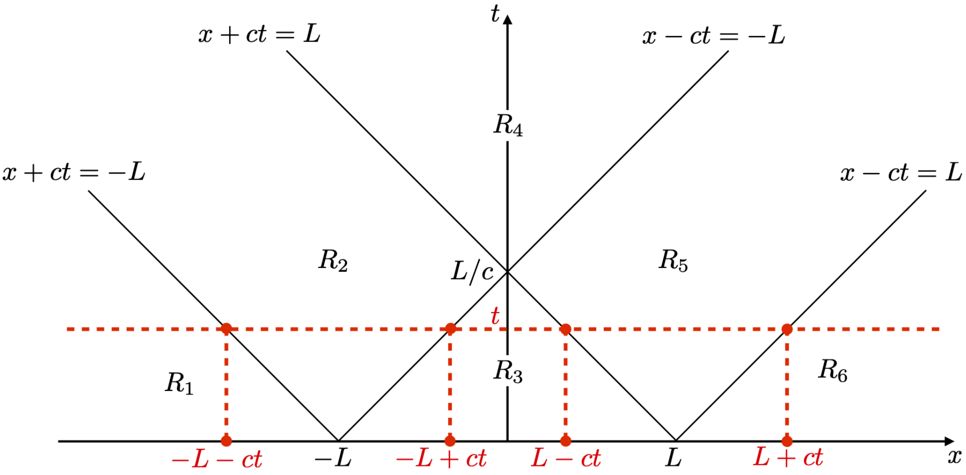
$$f(Q) = f(R) = 0,$$

giving

$$y(x, t) = 0 \quad \text{for } (x, t) \in R_6.$$



- In order to plot snapshots of the solution at some fixed time $t > 0$, we draw the corresponding horizontal line on the characteristic diagram and then write down the solution in the various different regions it crosses, *e.g.* for $0 \leq t \leq L/c$, the horizontal line crosses all but region R_4, as shown.



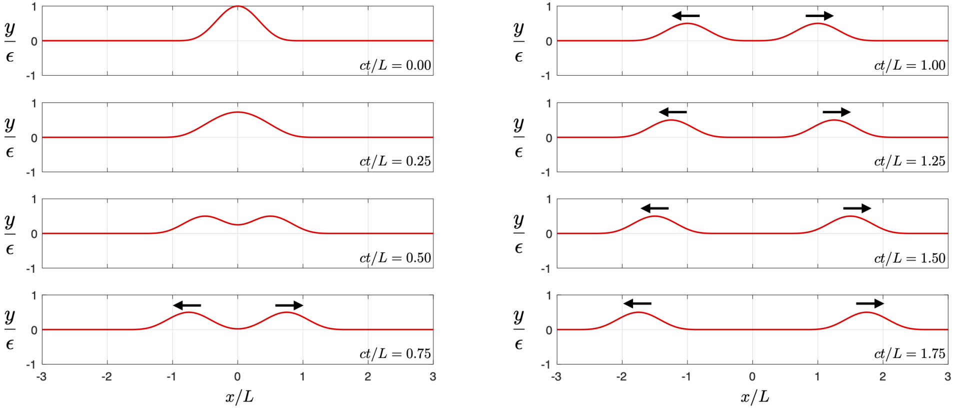
- We deduce that, for $0 < t \leq L/c$,

$$
y(x,t) = \begin{cases}
0 & \text{for } x < -L - ct, & (R_1) \\
\dfrac{\epsilon}{2}\cos^4\left(\dfrac{\pi}{2L}(x+ct)\right) & \text{for } -L - ct \leq x \leq -L + ct, & (R_2) \\
\dfrac{\epsilon}{2}\cos^4\left(\dfrac{\pi}{2L}(x-ct)\right) + \dfrac{\epsilon}{2}\cos^4\left(\dfrac{\pi}{2L}(x+ct)\right) & \text{for } -L + ct < x < L - ct, & (R_3) \\
\dfrac{\epsilon}{2}\cos^4\left(\dfrac{\pi}{2L}(x-ct)\right) & \text{for } L - ct \leq x \leq L + ct, & (R_5) \\
0 & \text{for } x > L + ct. & (R_6)
\end{cases}
$$

- Similarly, for $t > L/c$,

$$y(x,t) = \begin{cases} 0 & \text{for } x < -L - ct, & (R_1) \\ \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x+ct)\right) & \text{for } -L - ct \le x \le L - ct, & (R_2) \\ 0 & \text{for } L - ct < x < -L + ct, & (R_4) \\ \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x-ct)\right) & \text{for } -L + ct \le x \le L + ct, & (R_5) \\ 0 & \text{for } x > L + ct. & (R_6) \end{cases}$$

- We plot below snapshots of the solution with $\epsilon = vL/16c$ to illustrate the formation of two distinct compactly supported waves, one moving to the right with speed c and one to the left with speed c, each of them being the same shape as the initial profile, but half the amplitude. The arrows indicate the direction of travel of the waves.



$\frac{y}{\epsilon}$
1
0
-1
$ct/L = 0.00$
$ct/L = 0.25$
$ct/L = 0.50$
$ct/L = 0.75$
$ct/L = 1.00$
$ct/L = 1.25$
$ct/L = 1.50$
$ct/L = 1.75$
-3 -2 -1 0 1 2 3
x/L

Example 2

- Suppose that f and g are given by

$$f(x) = 0, \qquad g(x) = \begin{cases} vx/L & \text{for } |x| \le L, \\ 0 & \text{otherwise,} \end{cases}$$

  where L and v are positive constants.

- By D'Alembert's formula, we now have

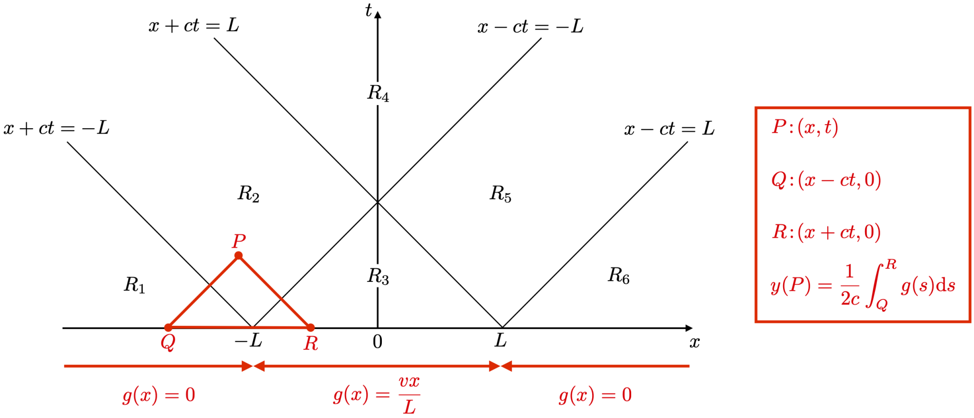
$$y(P) = \frac{1}{2c} \int_Q^R g(s)\, \mathrm{d}s,$$

  where again Q and R are the left- and right-hand intersections with the x-axis of the characteristic lines though P.

- Since g is compactly supported with support $(-L, L)$, it changes its analytic behaviour at the points $(-L, 0)$ and $(L, 0)$ on the x-axis in the (x, t)-plane.

- The characteristic diagram is therefore identical to that in Example 1, with characteristics $x \pm ct =$ constant through the points $(\pm L, 0)$, which divides the upper-half of the (x, t)-plane into six regions R_1, R_2, ..., R_6 that we take to be the same as in Example 1.

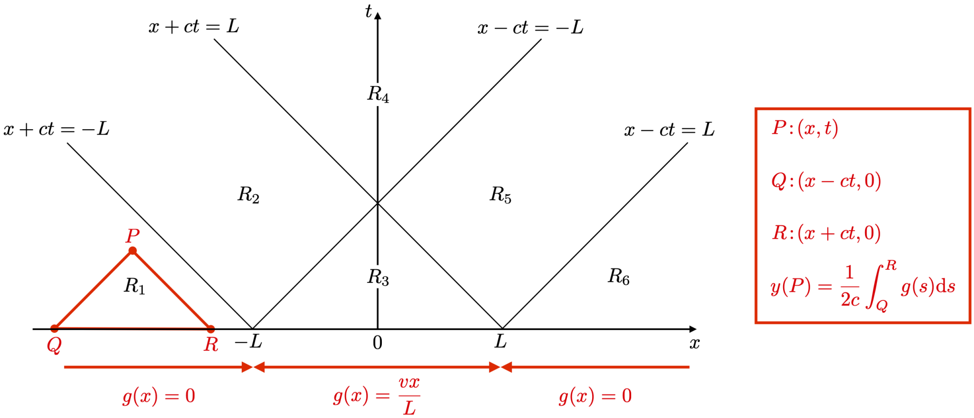
- Since PQ is parallel to the characteristics $x - ct = \pm L$, while PR is parallel to the characteristics $x + ct = \pm L$, we may construct the solution with the aid of the characteristic diagram by drawing on it the triangle PQR for P in each of the different regions.



- Thus, the locations of Q and R on the x-axis dictate their contributions, as follows.

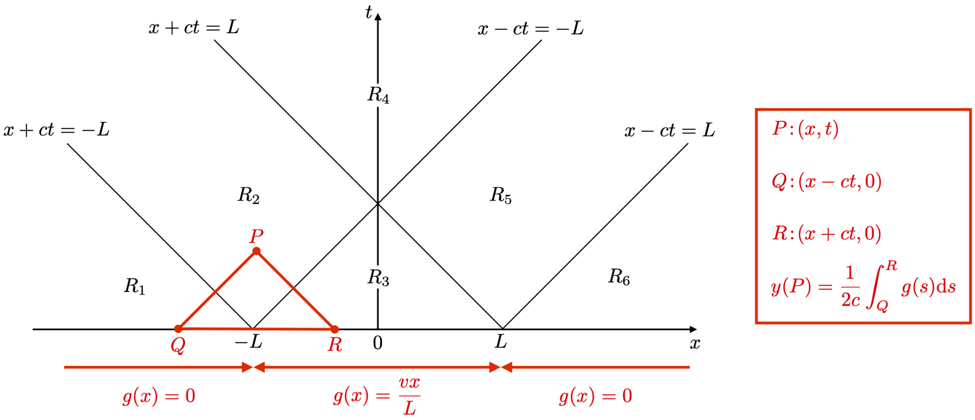
- if $P \in R_1$, then Q and R lie to the left of $(-L, 0)$, giving

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s = 0 \quad \text{for } (x,t) \in R_1.$$



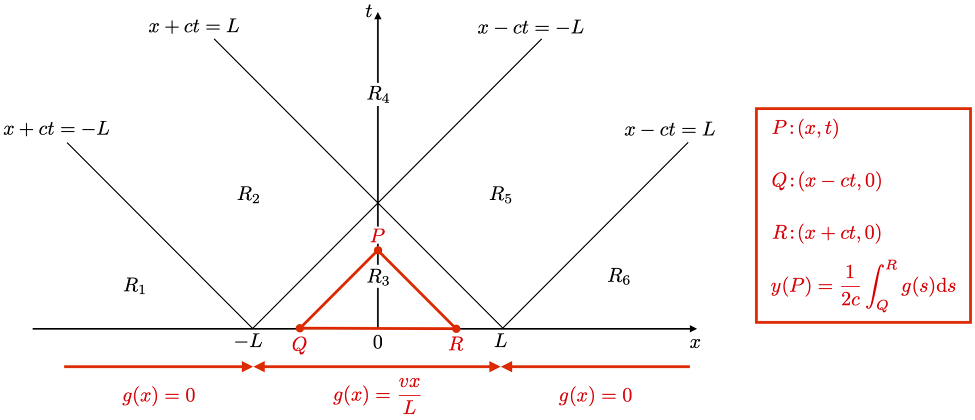
- if $P \in R_2$, then Q lies at or to the left of $(-L, 0)$, while R lies at or between $(-L, 0)$ and $(L, 0)$, giving

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{-L} 0\,\mathrm{d}s + \frac{1}{2c}\int_{-L}^{x+ct} \frac{vs}{L}\,\mathrm{d}s = \frac{v}{4Lc}\left((x+ct)^2 - L^2\right) \quad \text{for } (x,t) \in R_2.$$



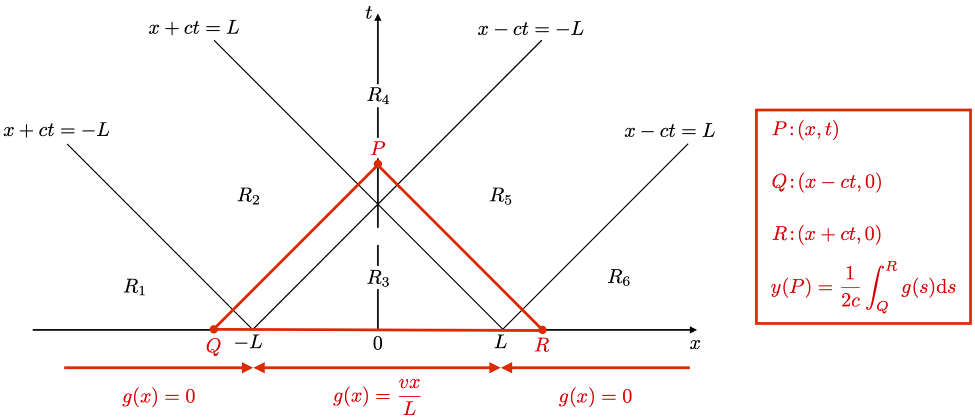
- if $P \in R_3$, then Q and R lie between $(-L, 0)$ and $(L, 0)$, giving

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{vs}{L}\,\mathrm{d}s = \frac{v}{4Lc}\left((x+ct)^2 - (x-ct)^2\right) = \frac{vxt}{L} \quad \text{for } (x,t) \in R_3.$$



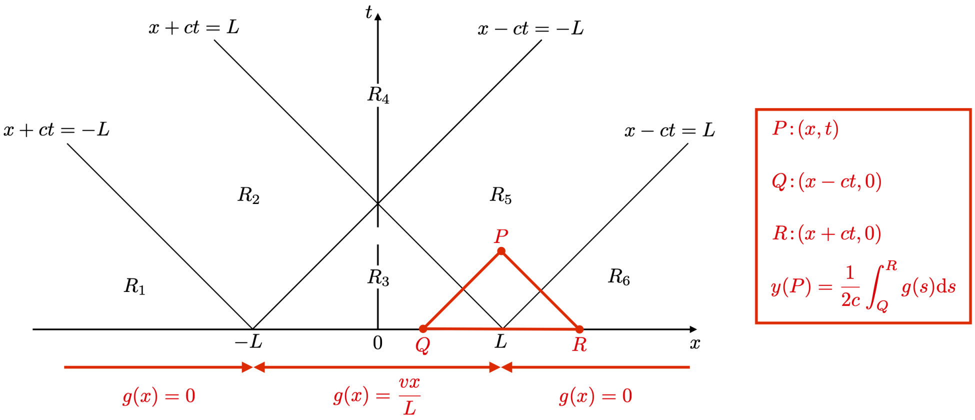
- if $P \in R_4$, then Q lies to the left of $(-L, 0)$ and R lies to the right of $(L, 0)$, giving

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{-L} 0\,\mathrm{d}s + \frac{1}{2c}\int_{-L}^{L} \frac{vs}{L}\,\mathrm{d}s + \frac{1}{2c}\int_{L}^{x+ct} 0\,\mathrm{d}s = 0 \quad \text{for } (x,t) \in R_4;.$$



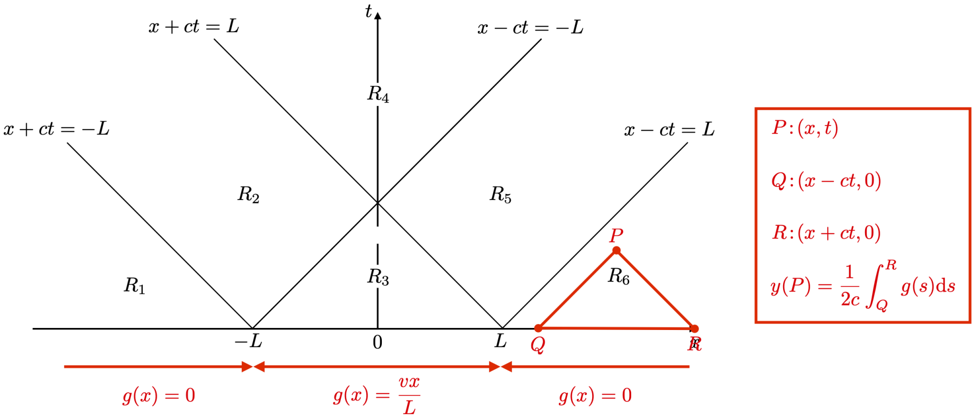
- if $P \in R_5$, then Q lies at or between $(-L, 0)$ and $(L, 0)$, while R lies at or to the right of $(L, 0)$, giving

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{L} \frac{vs}{L}\,\mathrm{d}s + \frac{1}{2c}\int_{L}^{x+ct} 0\,\mathrm{d}s = \frac{v}{4Lc}\left(L^2 - (x-ct)^2\right) \quad \text{for } (x,t) \in R_5.$$



- if $P \in R_6$, then Q and R lie to the right of $(L, 0)$, giving

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s = 0 \quad \text{for } (x,t) \in R_6.$$



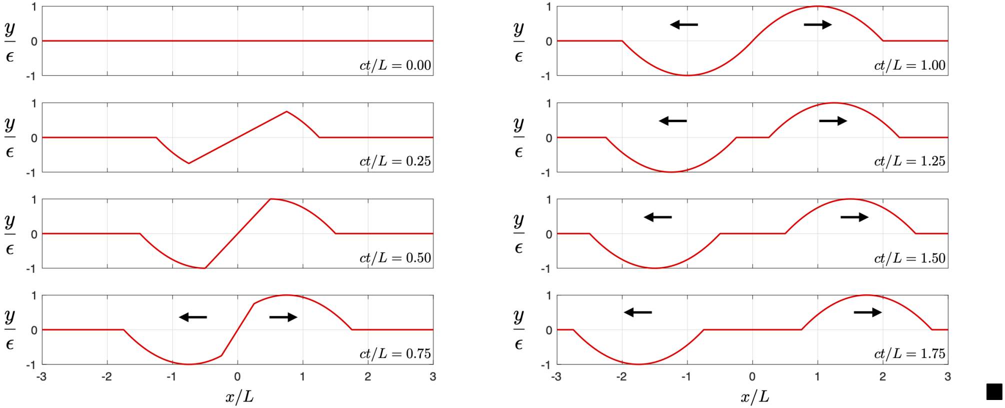
- We deduce that for $0 < t \leq L/c$,

$$y(x,t) = \begin{cases} 0 & \text{for } x < -L - ct, & (R_1) \\ \dfrac{v}{4Lc}\left((x+ct)^2 - L^2\right) & \text{for } -L - ct \leq x \leq -L + ct, & (R_2) \\ \dfrac{vxt}{L} & \text{for } -L + ct < x < L - ct, & (R_3) \\ \dfrac{v}{4Lc}\left(L^2 - (x-ct)^2\right) & \text{for } L - ct \leq x \leq L + ct, & (R_5) \\ 0 & \text{for } x > L + ct. & (R_6) \end{cases}$$

- While for $t > L/c$,

$$
y(x,t) = \begin{cases} 0 & \text{for } x < -L - ct, & (R_1) \\ \dfrac{v}{4Lc}\left((x+ct)^2 - L^2\right) & \text{for } -L - ct \le x \le L - ct, & (R_2) \\ 0 & \text{for } L - ct < x < -L + ct, & (R_4) \\ \dfrac{v}{4Lc}\left(L^2 - (x-ct)^2\right) & \text{for } -L + ct \le x \le L + ct, & (R_5) \\ 0 & \text{for } x > L + ct. & (R_6) \end{cases}
$$

- We plot below snapshots of the solution with $\epsilon = vL/16c$ to illustrate the formation of two distinct compactly supported waves, one moving to the right with speed c and one with the opposite sign to the left with speed c. The arrows indicate the direction of travel of the waves.



$\frac{y}{\epsilon}$
$ct/L = 0.00$
$ct/L = 0.25$
$ct/L = 0.50$
$ct/L = 0.75$
$ct/L = 1.00$
$ct/L = 1.25$
$ct/L = 1.50$
$ct/L = 1.75$
x/L

Notes:

(1) Since f is even in Example 1 and g is odd in Example 2, $y(x, t)$ is an even function of x in Example 1 and an odd function of x in Example 2. This provides a useful check of the solutions.

(2) While the solution that we constructed in Example 1 is twice continuously differentiable with respect to x and t and hence a classical solution, the solution in example 2 contains corners (moving with speed c) and hence is not a classical solution. As mentioned at the end of §4.4, while we do not discount such solutions, we must wait for a more sophisticated theory of PDEs in order to make sense of them.