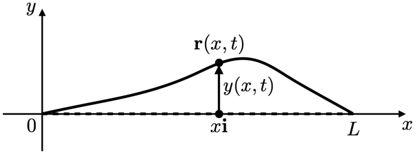
4 The wave equation

4.1 Derivation of the one-dimensional wave equation

- Consider the small transverse vibrations of a homogeneous extensible elastic string stretched initially along the x-axis at time $t = 0$ to a length L.
- A point at $x\boldsymbol{i}$ at time $t = 0$ is displaced to $\boldsymbol{r}(x, t) = x\boldsymbol{i} + y(x, t)\boldsymbol{j}$ at time $t > 0$, where the transverse displacement $y(x, t)$ is to be determined, as illustrated.



- Consider the section of the string in the fixed region $a \leq x \leq a + h$, where a and h are arbitrary constants (with $0 < a < a + h < L$).
- The linear momentum of the section of the string in $a \leq x \leq a + h$ is

$$\int_a^{a+h} \rho \frac{\partial \boldsymbol{r}}{\partial t} \, \mathrm{d}x,$$

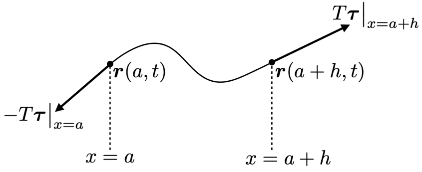
where ρ is the constant line density of the string (with $[\rho] = \mathrm{kg\,m^{-1}}$).

- The string offers no resistance to bending (*cf.* a ruler) in the sense that the string to the right of the point $\boldsymbol{r}(x,t)$ exerts at that point a tangential force $T(x,t)\boldsymbol{\tau}(x,t)$ on the string to the left, where $T(x,t)$ is the tension ($[T] = \mathrm{N} = \mathrm{kg\,m\,s^{-2}}$) and $\boldsymbol{\tau} = \boldsymbol{r}_x / |\boldsymbol{r}_x|$ is the unit tangent vector pointing in the positive x-direction.

- Note that Newton's third law implies that the string to the left of the point $\boldsymbol{r}(x,t)$ exerts at that point a tangential force $-T(x,t)\boldsymbol{\tau}(x,t)$ on the string to the right.

- Assuming the tension is so large that the effects of gravity and air resistance may be neglected, the forces acting on the ends of the section of string in $a \le x \le a+h$ are
  (i) the force $T(a+h,t)\boldsymbol{\tau}(a+h,t)$ exerted at RH end at $\boldsymbol{r}(a+h,t)$ by the string to right of section;
  (ii) the force $-T(a,t)\boldsymbol{\tau}(a,t)$ exerted at LH end at $\boldsymbol{r}(a,t)$ by string to the left of section.

  We illustrate the forces and where they act on the section in the schematic below.



- We are now in a position to apply Newton's Second Law, which states that the rate of change of the linear momentum of the section of string in $a \le x \le a+h$ is equal to the net force acting on it, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^{a+h} \rho \frac{\partial \boldsymbol{r}}{\partial t}\,\mathrm{d}x\right) = T(a+h,t)\boldsymbol{\tau}(a+h,t) - T(a,t)\boldsymbol{\tau}(a,t).$$

- Assuming $\boldsymbol{r}_{tt}$ is continuous, Leibniz's Integral Rule with a and $a+h$ constant gives

$$\frac{1}{h}\int_a^{a+h} \rho \frac{\partial^2 \boldsymbol{r}}{\partial t^2}\,\mathrm{d}x = \frac{T(a+h,t)\boldsymbol{\tau}(a+h,t) - T(a,t)\boldsymbol{\tau}(a,t)}{h},$$

  where we divided by h in anticipation of taking the limit $h \to 0$.

- To take the limit $h \to 0$,
  - apply the Fundamental Theorem of Calculus assuming $\boldsymbol{r}_{tt}$ is continuous in a neighbourhood of a;
  - use the definition of $\left(T\boldsymbol{\tau}\right)_x$ assuming it to exist at a.

- We obtain thereby the partial differential equation

$$\rho \frac{\partial^2 \boldsymbol{r}}{\partial t^2} = \frac{\partial}{\partial x}\left(T\boldsymbol{\tau}\right).$$

- Recalling the definitions of $\boldsymbol{r}$ and $\boldsymbol{\tau}$, it follows that

$$\rho \frac{\partial^2 y}{\partial t^2} \boldsymbol{j} = \frac{\partial}{\partial x} \left(\frac{T \boldsymbol{i} + T y_x \boldsymbol{j}}{(1 + y_x^2)^{1/2}} \right).$$

- But we are also assuming that the transverse displacement is small in the sense that the slope of the string is small, *i.e.* $|y_x| \ll 1$.

- Since a Taylor expansion gives

$$\left(1 + y_x^2\right)^{1/2} = 1 + \frac{1}{2}(y_x)^2 + \cdots \quad \text{for} \quad |y_x| \ll 1,$$

to a first approximation, *i.e.* neglecting quadratic and higher order terms,

$$\rho \frac{\partial^2 y}{\partial t^2} \boldsymbol{j} = \frac{\partial}{\partial x} \big(T \boldsymbol{i} + T y_x \boldsymbol{j} \big).$$

- **Remark:** We call this PDE the *linearized* version of the nonlinear PDE above.

- The x- and y-components of the linearized PDE are given by

$$\frac{\partial T}{\partial x} = 0, \qquad \rho \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\big(T y_x\big).$$

- The x-component implies that the tension T is spatially uniform, but could vary with time t, *e.g.* as when tuning a guitar string.

- We shall take the tension T to be constant, which is the case in many practical applications.

- The y-component then implies that

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2},$$

  giving the *wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

  where the *wave speed* (for reasons that will become apparent) is given by

$$c = \sqrt{\frac{T}{\rho}}.$$

- The wave equation is a second-order linear PDE.

4.2 Units and nondimensionalisation

- Consider the units of the variables (x, t and y) and parameter (c) in the wave equation.
- Since

  $$[y_{tt}] = \mathrm{m\,s}^{-2}, \quad [y_{xx}] = \mathrm{m\,m}^{-2},$$

  it follows that

  $$\left[c^2\right] = \frac{[y_{tt}]}{[y_{xx}]} = \mathrm{m}^2\,\mathrm{s}^{-2},$$

  so that $[c] = \mathrm{m\,s}^{-1}$, *i.e.* c has the units of speed.
- **Question:** On what timescale does a displacement travel a distance L?
- **Answer:** If we nondimensionalize by scaling $x = L\hat{x}$, $t = t_0\hat{t}$, $y = H\hat{y}(\hat{x}, \hat{t}\,)$, then the wave equation becomes

  $$\frac{H}{t_0^2}\frac{\partial^2\hat{y}}{\partial\hat{t}^2} = \frac{Hc^2}{L^2}\frac{\partial^2\hat{y}}{\partial\hat{x}^2};$$

  the terms balance giving

  $$\frac{\partial^2\hat{y}}{\partial\hat{t}^2} = \frac{\partial^2\hat{y}}{\partial\hat{x}^2}$$

  provided $t_0 = L/c$, which is therefore the timescale for a displacement to travel a distance L.

4.3 Normal modes of vibration for a finite string

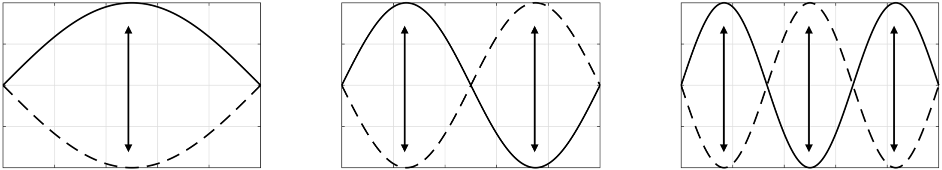
- Suppose an elastic string is stretched between $x = 0$ and $x = L$ and the ends held fixed, so that the small transverse displacement $y(x, t)$ of the string is governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L,$$

with the boundary conditions

$$y(0, t) = 0 \quad y(L, t) = 0.$$

- An experiment with a slinky suggests there exist discrete modes of vibration, as illustrated in the schematic below.



Suppose $y(x,t)$: ① $y_{tt} = c^2 y_{xx}$ for $0 < x < L$,

② $y(0,t) = 0$, $y(L,t) = 0$.

Let $T = F(x)G(t) \neq 0 \Rightarrow F(x)G''(t) = c^2 F''(x)G(t) \Rightarrow \dfrac{F''(x)}{F(x)} = \dfrac{G''(t)}{c^2 G(t)}$

LHS ind. t & RHS ind. $x \Rightarrow$ LHS = RHS ind. x & t, i.e. $= -\lambda \in \mathbb{R}$ say.

Hence, $\boxed{F'' + \lambda F = 0 \text{ for } 0 < x < L,}$ $G'' + \lambda c^2 G = 0$ for all t.

BCs ② $\Rightarrow$ $\boxed{F(0) = 0, \ F(L) = 0}$ because G is nontrivial.

But we solved before this eigenvalue problem for F and λ!

The nontrivial solutions are given for positive integers n by

$$F(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2,$$

where $B \in \mathbb{R}$.

Then $G'' + \left(\frac{n\pi c}{L}\right)^2 G = 0 \Rightarrow G(t) = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right)$, where $C, D \in \mathbb{R}$

Combo $\Rightarrow$ normal modes are given for $n \in \mathbb{N}\setminus\{0\}$ by

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right),$$

where $a_n = BC$ and $b_n = BD$ are real constants.

- To analyse mathematically the possible modes of vibration, we seek nontrivial separable solutions of the form $y = F(x)G(t)$ for which the wave equation $y_{tt} = c^2 y_{xx}$ gives

$$F(x)G''(t) = c^2 F''(x)G(t).$$

- Separating the variables for $FG \neq 0$, we obtain

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)}.$$

- The LHS of this expression is independent of t, while the RHS is independent of x. Since the LHS is equal to the RHS, they must both be independent of x and t, and therefore equal to a constant, $-\lambda \in \mathbb{R}$ say.

- Hence,

$$F'' + \lambda F = 0 \quad \text{for } 0 < x < L \qquad \text{and} \qquad G'' + \lambda c^2 G = 0 \quad \text{for all } t.$$

- Since $G(t) \neq 0$ for some t for y nontrivial, the boundary conditions imply $F(0) = 0$ and $F(L) = 0$.

- In summary, we have deduced that $F(x)$ and λ satisfy the ODE BVP given by

$$-F''(x) = \lambda F(x) \quad \text{for} \quad 0 < x < L,$$

with $F(0) = 0$ and $F(L) = 0$.

- We solved this problem in §3.4: the nontrivial solutions are given for positive integers n by

$$F(x) = B \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2,$$

where B is an arbitrary constant

- Since $G'' + \lambda c^2 G = 0$, the corresponding solution for $G(t)$ is given by

$$G(t) = C \cos\left(\frac{n\pi ct}{L}\right) + D \sin\left(\frac{n\pi ct}{L}\right),$$

where C and D are arbitrary constants.

- Since $T(x,t) = F(x)G(t)$, we conclude that the nontrivial separable solutions or the *normal modes* are given for positive integers n by

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right)$$

where a_n and b_n are arbitrary constants (with $a_n = BC$ and $b_n = BD$) and we have introduced the subscript n to enumerate the countably infinite set of such solutions.

Notes

(1) The normal mode $y_n(x,t)$ is periodic in t with prime period

$$p = \frac{2\pi}{n\pi c/L} = \frac{2L}{nc}$$

and *frequency* or *pitch*

$$\frac{1}{p} = \frac{nc}{2L}.$$

(2) The first normal mode y_1 is called the *fundamental mode*, with associated *fundamental frequency* $c/(2L)$. All of the other modes have a frequency that is an integer multiple of the fundamental frequency.

(3) The predictions are consistent with the slinky experiment.

(4) The normal modes are an example of a standing wave because y_n is equal to a function of x multiplied by an oscillatory function of time.

4.4 Initial boundary value problem for a finite string

- Consider the initial boundary value problem for the small transverse displacement $y(x, t)$ of an elastic string given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L, \; t > 0,$$

  with the Dirichlet boundary conditions

$$y(0, t) = 0, \quad y(L, t) = 0 \quad \text{for} \quad t > 0,$$

  and the two initial conditions

$$y(x, 0) = f(x), \quad \frac{\partial y}{\partial t}(x, 0) = g(x) \quad \text{for} \quad 0 < x < L,$$

  where the initial transverse displacement $f(x)$ and the initial transverse velocity $g(x)$ are given.

- **Remark:** The total number of boundary (initial) conditions is equal to the number of spatial (temporal) partial derivatives in the wave equation.

- We will use Fourier's method to find a series solution.

Step (I): Find all nontrivial separable solutions of the PDE and BCs

- We found above that these are the normal modes given for positive integers n by

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right),$$

  where a_n and b_n are arbitrary real constants.

Step (II): Apply the principle of superposition

- Since the wave equation and boundary conditions are linear and homogeneous, we can superimpose the normal modes (assuming convergence) to obtain the general series solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right).$$

Step (III): Use the theory of Fourier series to satisfy the ICs

- The initial conditions can only be satisfied if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L,$$

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L.$$

- Hence, a_n is the nth Fourier coefficient of the Fourier sine series for f, while $n\pi c b_n/L$ is the nth Fourier coefficient of the Fourier sine series for g, *i.e.*, for positive integers n,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad \frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Example: plucking a guitar string

- If the midpoint of the string is drawn aside a distance h and released from rest, then

$$f(x) = \begin{cases} 2hx/L & \text{for } 0 \le x \le L/2, \\ 2h(L-x)/L & \text{for } L/2 \le x \le L, \end{cases} \qquad g(x) = 0.$$

- Since $g(x) = 0$ we have $b_n = 0$, and integration by parts gives

$$a_n = \frac{2}{L}\int_0^{L/2} \frac{2hx}{L} \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x + \frac{2}{L}\int_{L/2}^{L} \frac{2h(L-x)}{L} \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right).$$

- Since

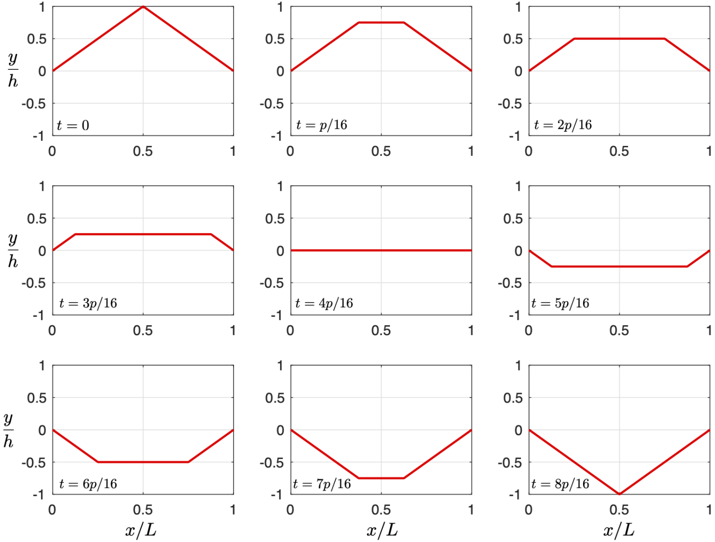
$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N} \setminus \{0\}, \\ (-1)^m & \text{for } n = 2m+1,\ m \in \mathbb{N}, \end{cases}$$

we deduce that a series solution is given by

$$y(x,t) = \frac{8h}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin\left(\frac{(2m+1)\pi x}{L}\right) \cos\left(\frac{(2m+1)\pi ct}{L}\right),$$

so that $p = 2L/c$ is the prime period of the oscillation.

- We plot below snapshots of the series solution truncated to 128 terms over the first half-period, which illustrates the persistence of corners moving with speed c.



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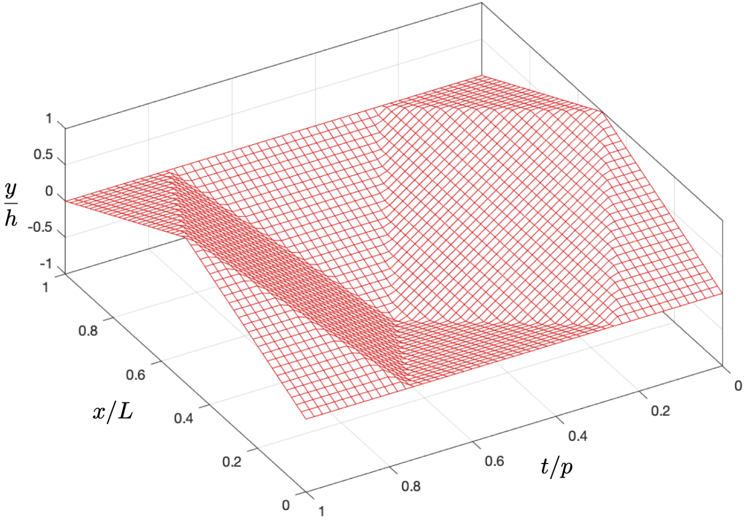
- The mesh plot below shows the series solution again truncated to 128 terms, but this time over the first period, with the orientation chosen for a good view.



Example: hammering a piano string

- Suppose we hit the string with a hammer so that

$$f(x) = 0, \quad g(x) = \begin{cases} v & \text{for } L_1 \le x \le L_2, \\ 0 & \text{otherwise,} \end{cases}$$

  where v, L_1 and L_2 are constants.

- We have $a_n = 0$ and

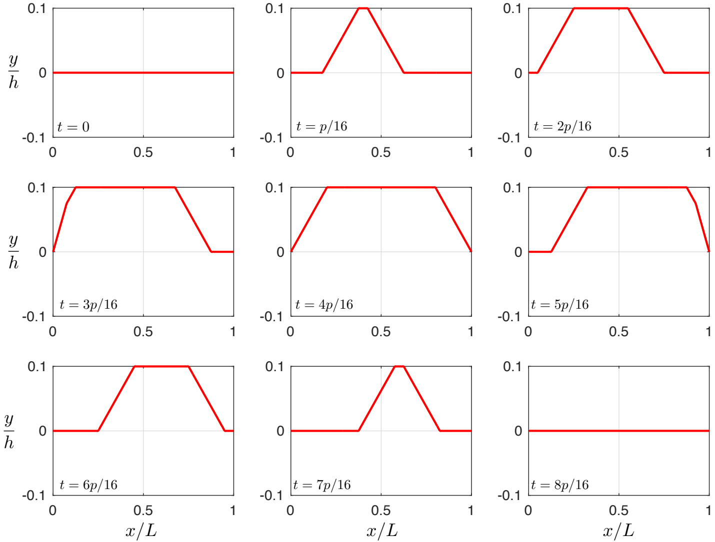
$$\frac{n\pi c}{L} b_n = \frac{2}{L} \int_{L_1}^{L_2} v \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2v}{n\pi} \left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right].$$

- It follows that a series solution is given by

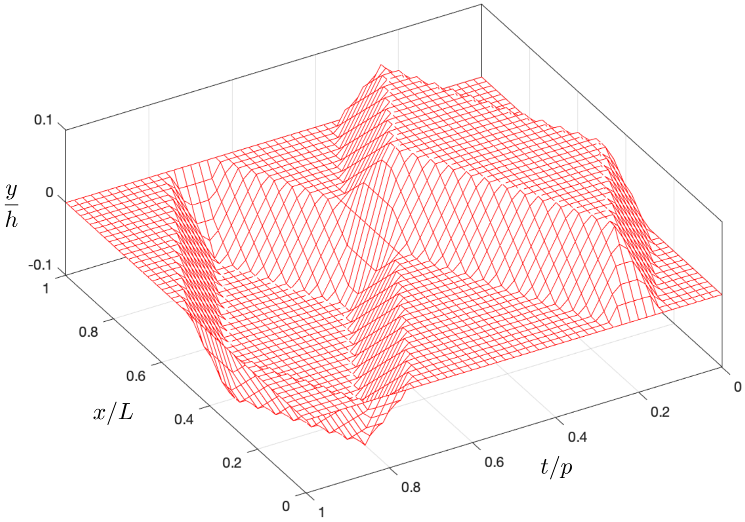
$$y(x,t) = \frac{2h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right),$$

  where $h = vL/c$ and we see that the prime period of the oscillation is again $p = 2L/c$.

- We plot below show snapshots of the evolution of the series solution truncated to 128 terms for $L_1/L = 0.3$, $L_2/L = 0.5$ over the first half-period, which again illustrates the persistence of corners moving with speed c.



- The mesh plot below shows the series solution again truncated to 128 terms, but this time over the first period, with the orientation chosen for a good view.



Notes

- Both the guitar solution and piano solution contain persistent corners travelling with speed c.
- This means that neither solution can be twice continuously differentiable with respect to x or t, and hence a so-called *classical solution* of the wave equation.
- However, if we were to modify the initial data by smoothing off the corners and jump discontinuities in small neighbourhoods of these irregularities in such a way that the new initial data is infinitely differentiable, then the new solutions would also be infinitely differentiable, and hence classical solutions, and they would be "close" in some sense to the original solutions.
- Hence, we do not want to discount the series solutions we have found, but to view them instead as motivation to *weaken* the sense in which a function can be a solution of a PDE — the resulting notion of a *weak solution* forms the basis for the modern theory of PDEs that can be studied further on in the course in *e.g.* B4.3 and B5.2.
- The differences in the makeup of the normal modes for the guitar and piano solutions contribute to the different timbres of the musical instruments

4.5 Conservation of energy

- An elastic string is stretched to a length L, line density ρ and tension T along the x-axis and its ends held fixed at $x = 0$ and $x = L$.

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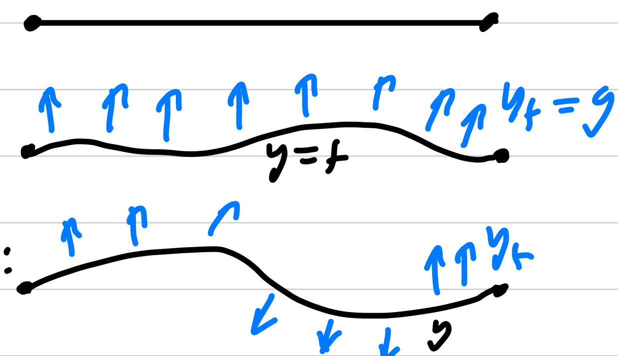
- This is the so-called <u>reference configuration</u> before the string is deformed to have its initial transverse displacement $f(x)$ and imparted with its initial transverse velocity $g(x)$ at time $t = 0$.

Reference configuration:

Initial configuration ($t=0$):

Current configuration ($t>0$):



- Measure elastic potential energy of string relative to reference configuration.

- IBVP for $y(x,t)$:
  ① $\rho y_{tt} = T y_{xx}$ for $0 < x < L,\ t > 0$,
  ② $y(0,t) = 0,\ y(L,t) = 0$ for $t > 0$,
  ③ $y(x,0) = f(x),\ y_t(x,0) = g(x)$ for $0 < x < L$.

- KE string $= \int_0^L \frac{1}{2}\rho |\underline{r}_t|^2 \, dx = \int_0^L \frac{1}{2}\rho y_t^2 \, dx$ (since $\underline{r} = x\underline{i} + y\underline{j}$)

- Elastic PE string $= \underbrace{T}_{\text{tension}} \Big(\underbrace{\int_0^L (1+y_x^2)^{1/2} \, dx - L}_{\text{extension}} \Big) = T \int_0^L (1+y_x^2)^{1/2} - 1 \, dx$.

- Transverse displacement small $\Rightarrow (1+y_x^2)^{1/2} = 1 + \frac{1}{2} y_x^2 + \text{h.o.t. as } |y_x| \ll 1$

$\Rightarrow$ Elastic PE $= \int_0^L \frac{1}{2} T y_x^2 \, dx$ to a first approx.

- Definition: Energy of string $E(t) = \int_0^L \underbrace{\frac{1}{2}\rho y_t^2}_{KE} + \underbrace{\frac{1}{2} T y_x^2}_{PE} \, dx$

Proposition: If $y(x,t)$ satisfies ① & ②, then the energy $E(t)$ is constant.

Proof:

$$\frac{dE}{dt} = \int_0^L \frac{\partial}{\partial t}\left(\frac{1}{2}\rho y_t^2 + \frac{1}{2}T y_x^2\right) dx \qquad \text{(by LIR with } L \text{ const.)}$$

$$= \int_0^L \rho y_t y_{tt} + T y_x y_{xt} \, dx$$

$$= \int_0^L T y_t y_{xx} + T y_x y_{xt} \, dx \qquad \text{(by ①)}$$

$$= \int_0^L (T y_t y_x)_x \, dx$$

$$= \Big[T y_t y_x \Big]_{x=0}^{x=L}$$

$$= 0 \qquad \text{(by ②)}$$

$\square$

- Recall Fourier's method $\Rightarrow$

$$y = \sum_{n=1}^{\infty} y_n \,, \qquad y_n = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)\right)$$

where

$$\left.\begin{aligned} \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) &= f(x) \\ \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) &= g(x) \end{aligned}\right\} \text{ for } 0 < x < L$$

so that

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$\frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

- Energy $E(t) = \int_0^L \frac{1}{2}\rho y_t^2 + \frac{1}{2}T y_x^2 \, dx$ conserved, and therefore set by the ICs: $E(t) = E(0) = \int_0^L \frac{1}{2}\rho g(x)^2 + \frac{1}{2}T f'(x)^2 \, dx$ (a)

- Since y_n satisfies (1) & (2), its energy $E_n(t) := \int_0^L \frac{1}{2}\rho y_{n,t}^2 + \frac{1}{2}T y_{n,x}^2 \, dx$ is conserved, and hence

$$E_n(t) = E_n(0) = \int_0^L \frac{1}{2}\rho y_{n,t}(x,0)^2 + \frac{1}{2}T y_{n,x}(x,0)^2 \, dx = \frac{\rho L}{4}\left(\frac{n\pi c b_n}{L}\right)^2 + \frac{TL}{4}\left(\frac{n\pi c a_n}{L}\right)^2 \quad \text{(b)}$$

- Qn: How is the energy in the string related to the energies in the normal modes?

Ans: Relate $E(t)$ and $E_n(t)$ via Parseval's identity as follows.

$$\int_0^L \tfrac{1}{2}\rho\, g(x)^2\,dx = \tfrac{1}{2}\rho \int_0^L \left(\sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right)\right) g(x)\,dx$$

$$= \tfrac{1}{2}\rho \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \left(\text{assuming } \int\Sigma = \Sigma\int\right)$$

$$= \tfrac{1}{2}\rho \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L} b_n\right)\left(\frac{n\pi c}{L} b_n \cdot \frac{L}{2}\right) \quad (\text{by defn of } b_n)$$

$$= \frac{\rho L}{4} \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L} b_n\right)^2. \quad \text{c}$$

Similarly, $\int_0^L \tfrac{1}{2} T f'(x)^2\,dx = \dfrac{TL}{4} \sum_{n=1}^{\infty} \left(\dfrac{n\pi c}{L} a_n\right)^2$ d

a – d $\Rightarrow$ $E(t) = \sum_{n=1}^{\infty} E_n(t)$, i.e. the energy in the string is equal to the sum of the energies in the normal modes!

NB: These are set at $t = 0$!

NB: Please watch "Waves & Resonance" by Jon Chapman – MI YouTube.

- An elastic string is stretched between $x = 0$ and $x = L$ along the x-axis to a line density ρ and a tension T, so that its small transverse displacement $y(x, t)$ is governed by the wave equation

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L, \; t > 0,$$

  with the Dirichlet boundary conditions

$$y(0, t) = 0, \quad y(L, t) = 0 \quad \text{for} \quad t > 0,$$

  and the initial conditions

$$y(x, 0) = f(x), \quad \frac{\partial y}{\partial t}(x, 0) = g(x) \quad \text{for} \quad 0 < x < L,$$

  where the initial transverse displacement is $f(x)$ and the initial transverse velocity is $g(x)$.

- **Remark:** Recall that the point of the string that lies at $x\mathbf{i}$ in its so-called *reference configuration* is displaced transversely to the point with position vector $\mathbf{r}(x, t) = x\mathbf{i} + y(x, t)\mathbf{j}$. When we impose the initial conditions, we must deform the string from its reference configuration along the x-axis to have transverse displacement $y(x, 0) = f(x)$ and we must impart on the string the transverse velocity given by $y_t(x, 0) = g(x)$.

- The kinetic energy of the string is given by

$$\int_0^L \frac{1}{2}\rho\left|\boldsymbol{r}_t\right|^2\,\mathrm{d}x = \int_0^L \frac{1}{2}\rho y_t^2\,\mathrm{d}x.$$

- The elastic potential energy of the string is the product of tension and extension, and therefore given by

$$T\left(\int_0^L \left|\boldsymbol{r}_x\right|\,\mathrm{d}x - L\right) = T\int_0^L \left(1+y_x^2\right)^{\frac{1}{2}} - 1\,\mathrm{d}x.$$

- Since the transverse displacement is small in the sense that $|y_x| \ll 1$, a Taylor expansion gives

$$\left(1+y_x^2\right)^{\frac{1}{2}} - 1 = \frac{1}{2}y_x^2 + \cdots .$$

- Hence, to a first approximation (*i.e.* neglecting cubic and higher order terms), the elastic potential energy is given by

$$\int_0^L \frac{1}{2}\,Ty_x^2\,\mathrm{d}x.$$

- **Definition:** The energy of the string is defined to be the sum of its kinetic and elastic potential energies, and given by
$$E(t) = \int_0^L \frac{1}{2}\rho y_t^2 + \frac{1}{2} T y_x^2 \,\mathrm{d}x.$$
- **Proposition:** If $y(x,t)$ satisfies the wave equation and the boundary conditions, then the energy $E(t)$ is constant for $t > 0$.

Proof:

- The idea is to show that the derivative of $E(t)$ is equal to zero.
- By Leibniz's Integral Rule,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^L \frac{\partial}{\partial t}\left(\frac{1}{2}\rho y_t^2 + \frac{1}{2}Ty_x^2\right)\,\mathrm{d}x = \int_0^L \rho y_t y_{tt} + Ty_x y_{xt}\,\mathrm{d}x.$$

- Substituting for ρy_{tt} from the wave equation, we deduce that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^L Ty_t y_{xx} + Ty_x y_{xt}\,\mathrm{d}x = \int_0^L (Ty_t y_x)_x\,\mathrm{d}x = [Ty_t y_x]_{x=0}^{x=L}.$$

- Since each of the boundary conditions may be differentiated with respect to t to give $y_t(0,t) = 0$ and $y_t(L,t) = 0$ for $t > 0$, we deduce that $\mathrm{d}E/\mathrm{d}t = 0$. ■

Notes

(1) We have shown that the energy of the elastic string is conserved during its motion, with the kinetic and elastic potential energy being transferred back and forth as the string oscillates.

(2) The energy of the string is set by the initial conditions to be given by

$$E(t) = E(0) = \int_0^L \frac{1}{2}\rho(g(x))^2 + \frac{1}{2}T(f'(x))^2 \,\mathrm{d}x.$$

(3) The energy of the nth normal mode $y_n(x,t)$ is given by

$$E_n(t) = \int_0^L \frac{1}{2}\rho\left(\frac{\partial y_n}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y_n}{\partial x}\right)^2 \,\mathrm{d}x.$$

Since $y_n(x,t)$ satisfies the wave equation and the boundary conditions by construction, it follows that its energy is conserved during its motion and given by

$$E_n(t) = E_n(0) = \frac{n^2\pi^2\rho c^2 b_n^2}{4L} + \frac{n^2\pi^2 T a_n^2}{4L},$$

where in the last equality we substituted for $y_n(x,0)$ and integrated.

(4) Recalling that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L,$$

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L$$

and assuming convergence, Parseval's Identity for g and f' imply that

$$\int_0^L \frac{1}{2}\rho g(x)^2 + \frac{1}{2} T f'(x)^2 \,\mathrm{d}x = \sum_{n=1}^{\infty} \left(\frac{n^2\pi^2\rho c^2 b_n^2}{4L} + \frac{n^2\pi^2 T a_n^2}{4L} \right);$$

hence,

$$E(t) = E(0) = \sum_{n=1}^{\infty} E_n(0) = \sum_{n=1}^{\infty} E_n(t),$$

i.e. the energy of the elastic string is made up of that in its normal modes.

4.6 Uniqueness Theorem

Uniqueness Theorem

There is at most one solution to the IBVP for $y(x,t)$ given by

① $y_{tt} = c^2 y_{xx}$ for $0 < x < L$, $t > 0$,

② $y(0,t) = 0$, $y(L,t) = 0$ for $t > 0$,

③ $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $0 < x < L$.

Proof: Let $w = y - \tilde{y}$ be the difference between two solutions y and $\tilde{y}$.

By linearity, w satisfies the IBVP

①' $w_{tt} = c^2 w_{xx}$ for $0 < x < L$, $t > 0$,

②' $w(0,t) = 0$, $w(L,t) = 0$ for $t > 0$,

③' $w(x,0) = 0$, $w_t(x,0) = 0$ for $0 < x < L$.

NB: Expect $w = 0$ for $0 \le x \le L$, $t \geq 0$ on physical grounds.

Let $E(t) = \int_0^L \frac{1}{2}\rho w_t^2 + \frac{1}{2}T w_x^2 \, dx$ be energy associated with w.

(1') & (2') $\Rightarrow$ $E(t) = E(0)$ for $t \geq 0$ by the Proposition of §4.5

(3') $\Rightarrow$ $E(0) = 0$

Hence, $\int_0^L \frac{1}{2}\rho w_t^2 + \frac{1}{2}T w_x^2 \, dx = E(t) = E(0) = 0$ for $t \geq 0$.

Hence, $w_x = w_t = 0$ on $R = \{(x,t) : 0 \le x \le L, t \geq 0\}$ (assuming w_t and w_x are cts there)

Since $w = 0$ on boundary of R by (2') and (3'), $w = 0$ on R (assuming w cts there). $\square$

Uniqueness Theorem:

- There is at most one solution to the IBVP for $y(x,t)$ given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with

$$y(0,t) = 0, \quad y(L,t) = 0 \quad \text{for} \quad t > 0,$$

and

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad 0 < x < L,$$

where $f(x)$ and $g(x)$ are given.

Proof:

- Our strategy is to show that the difference between any two solutions much vanish.
- We suppose that $y(x,t)$ and $\widetilde{y}(x,t)$ are solutions and let

$$w(x,t) = y(x,t) - \widetilde{y}(x,t).$$

- By linearity, $w(x,t)$ satisfies the wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

  with the boundary conditions

$$w(0,t) = 0, \quad w(L,t) = 0 \quad \text{for} \quad t > 0,$$

  and the initial conditions

$$w(x,0) = 0, \quad \frac{\partial w}{\partial t}(x,0) = 0 \quad \text{for} \quad 0 < x < L.$$

- **Remark:** Since w is the small transverse displacement of an elastic string whose initial transverse displacement and velocity are everywhere zero and whose ends are fixed thereafter, on physical grounds we expect the string to remain stationary along the x-axis, *i.e.* $w = 0$ for $0 \leq x \leq L$ and $t \geq 0$, which is what we need to show to prove uniqueness.

- The trick is to analyse the energy $E(t)$ associated with $w(x,t)$, which is given by

$$E(t) = \int_0^L \frac{1}{2}\rho w_t^2 + \frac{1}{2} T w_x^2 \,\mathrm{d}x.$$

- Since w satisfies the wave equation and homogeneous Dirichlet boundary conditions, the energy $E(t)$ is conserved.
- But $E(0) = 0$ by the initial conditions, so

$$\int_0^L \frac{1}{2}\rho w_t^2 + \frac{1}{2} T w_x^2 \,\mathrm{d}x = 0 \quad \text{for} \quad t \geq 0.$$

- We deduce that $w_t = w_x = 0$ on $R = \{(x, y) : 0 \leq x \leq L,\ t \geq 0\}$ (assuming w_t and w_x are continuous there).
- Since the boundary and initial conditions imply that $w = 0$ on the boundary of R, we deduce that $w = 0$ or $y = \widetilde{y}$ on R. ■