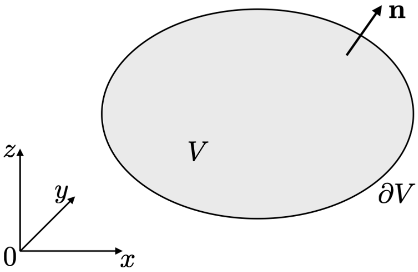
5 Laplace's equation

5.1 Preliminaries

- **Divergence Theorem:** Let V be a region of $\mathbb{R}^3$ with a piecewise smooth boundary ∂V. Let $\mathbf{F}(x, y, z)$ be a vector field with continuous first-order partial derivatives on $V \cup \partial V$. Then

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V = \iint_{\partial V} \mathbf{F} \cdot \boldsymbol{n} \, \mathrm{d}S,$$

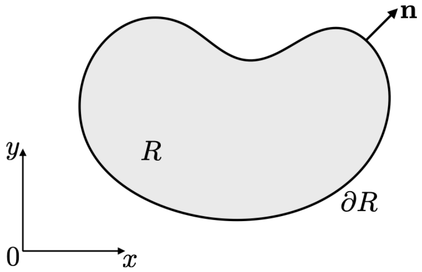
where $\boldsymbol{n}$ is the outward pointing unit normal to ∂V.



- **Green's Theorem in the plane:** Let R be a region in the (x, y)-plane, whose boundary ∂R is a piecewise smooth simple closed curve. Let $\mathbf{G}(x, y)$ be a vector field with continuous first-order partial derivatives on $R \cup \partial R$. Then

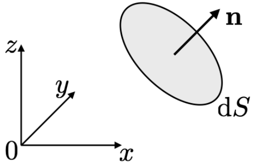
$$\iint_R \nabla \cdot \mathbf{G}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} \mathbf{G} \cdot \boldsymbol{n}\,\mathrm{d}s,$$

where $\boldsymbol{n}$ is the outward pointing unit normal to ∂R in the (x, y)-plane.



5.2 Derivation of the three-dimensional heat equation

- We begin by recalling from Multivariable Calculus the derivation of the three-dimensional heat equation because it introduces all of the quantities that we shall need.
- Let $T(\mathbf{x}, t)$ be the absolute temperature in a rigid isotropic conducting material (*e.g.* metal), with constant density ρ and specific heat c_v.
- Let $\boldsymbol{q}(\mathbf{x}, t)$ be the heat flux vector, so that $\boldsymbol{q} \cdot \boldsymbol{n}\,\mathrm{d}S$ is the rate at which thermal energy is transported through a surface element $\mathrm{d}S$ in the direction of the unit normal $\boldsymbol{n}$ that orients it.



- Let V be a fixed region in the conducting material whose boundary ∂V has outward pointing unit normal $\boldsymbol{n}$, as in the statement of the Divergence Theorem.

- We suppose that the material is heated volumetrically at a prescribed rate $Q(\mathbf{x}, t)$ per unit volume, so that conservation of thermal energy in V is given by

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho c_v T \,\mathrm{d}V}_{(1)} = \underbrace{\iint_{\partial V} \boldsymbol{q}\cdot(-\boldsymbol{n})\,\mathrm{d}S}_{(2)} + \underbrace{\iiint_V Q\,\mathrm{d}V}_{(3)},$$

  where (1) is the time rate of change of the thermal energy in V, (2) is the net rate at which thermal energy enters V through ∂V and (3) is the net rate of volumetric heating of V.

- Assuming T_t to be continuous on $V \cup \partial V$, so that we can differentiate under the integral sign in term (1), and applying the Divergence Theorem with $\boldsymbol{F} = \boldsymbol{q}$ to term (2), we obtain

$$\iiint_V \rho c_v \frac{\partial T}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - Q\,\mathrm{d}V = 0.$$

- Since V is arbitrary, the integrand must vanish if it is continuous, so we obtain

$$\rho c_v \frac{\partial T}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{q} = Q.$$

- A closed model for heat conduction is obtained by prescribing a constitutive law relating $\boldsymbol{q}$ and T.
- *Fourier's Law* states that thermal energy is transported down the temperature gradient, with

$$\boldsymbol{q} = -k \boldsymbol{\nabla} T,$$

  where k is the constant thermal conductivity.
- Recall that $-\boldsymbol{\nabla} T$ points in the direction in which T decreases most rapidly.
- Substituting Fourier's law into the PDE representing energy conservation, we deduce that T satisfies the three-dimensional inhomogeneous or forced heat equation given by

$$\rho c_v \frac{\partial T}{\partial t} = k \boldsymbol{\nabla}^2 T + Q.$$

5.3 Steady three-dimensional heat conduction

- In steady state T and Q are independent of t, so the heat equation reduces to *Poisson's equation*

$$-k\nabla^2 T = Q,$$

while conservation of energy becomes

$$\iint_{\partial V} \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}S = \iiint_V Q \, \mathrm{d}V,$$

i.e. the net rate at which thermal energy is supplied to a region by volumetric heating is equal to the net rate at which thermal energy is conducted out though its boundary.

- This result holds locally for any region V, as well as globally for the whole material.

- If in addition $Q = 0$, Poisson's equation becomes *Laplace's equation*

$$\nabla^2 T = 0,$$

while conservation of energy becomes

$$\iint_{\partial V} \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}S = 0,$$

i.e. the net rate at which thermal energy is conducted though the boundary of any region must vanish.

5.4 Steady two-dimensional heat conduction

- In this course we consider two-dimensional steady-state solutions of the heat equation.
- Setting $T = T(x, y)$ and $Q = Q(x, y)$, where (x, y) are Cartesian coordinates, we obtain Poisson's equation in the plane,

$$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = Q.$$

- If $Q = 0$, we recover Laplace's equation in the plane,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

- In terms of plane polar coordinates (r, θ) defined by $(x, y) = (r\cos\theta, r\sin\theta)$, Laplace's equation is given by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{for} \quad r > 0.$$

- We will use Fourier's method to construct solutions to boundary value problems for Laplace's equation in the plane in terms of both Cartesian and plane polar coordinates.

- If Laplace's equation holds in some region R, as in the statement of Green's Theorem in the plane, then we will need to prescribe a boundary condition on the boundary ∂R of R.

- **Definition:** The *outward normal derivative* of T on the boundary ∂R is the directional derivative of T in the direction of the outward pointing unit normal $\boldsymbol{n}$ to ∂R, and hence given by

$$\frac{\partial T}{\partial n} = \boldsymbol{n} \cdot \boldsymbol{\nabla} T \quad \text{on } \partial R.$$

- Common boundary conditions for Laplace's equation and Poisson's equation are:

  - a *Dirichlet boundary condition* in which the temperature is prescribed on the boundary,

  $$T = f \quad \text{on } \partial R;$$

  - a *Neumann boundary condition* in which the outward normal derivative of the temperature (or equivalently the heat flux $\boldsymbol{q} \cdot \boldsymbol{n} = -k\partial T/\partial n$) is prescribed on the boundary,

  $$\frac{\partial T}{\partial n} = -\frac{q}{k} \quad \text{on } \partial R;$$

  - a *Robin boundary condition* in which a linear relationship between the temperature and its outward normal derivative is prescribed on the boundary,

  $$\frac{\partial T}{\partial n} + \alpha T = \beta \quad \text{on } \partial R;$$

  here the functions f, q, α and β are prescribed on ∂R.

- **Remark:** Since $-k(T_{xx} + T_{yy}) = Q$ is equivalent to $\boldsymbol{\nabla} \cdot \boldsymbol{q} = Q$ by Fourier's law, Green's Theorem in the plane with $\boldsymbol{F} = \boldsymbol{q}$ implies that

$$\int_{\partial R} \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}s = \iint_R \boldsymbol{\nabla} \cdot \boldsymbol{q} \, \mathrm{d}x \, \mathrm{d}y = \iint_R Q \, \mathrm{d}x \, \mathrm{d}y,$$

  which has two important consequences.

- Firstly, if $Q = 0$, then the net heat flux through the boundary (per unit distance in the z-direction) must vanish, *i.e.*

$$\int_{\partial R} \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}s = 0.$$

- Secondly, if we impose the Neumann boundary condition $\boldsymbol{q} \cdot \boldsymbol{n} = q$ on ∂R, then there can only be a steady-state solution if the net heat flux though the boundary equals the net rate of volumetric heating (per unit distance in the z-direction), *i.e.*

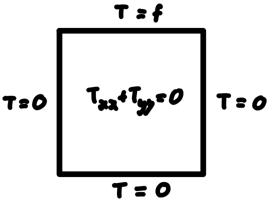
$$\int_{\partial R} q \, \mathrm{d}s = \iint_R Q \, \mathrm{d}x \, \mathrm{d}y,$$

  since otherwise the temperature must change.

5.5 Boundary value problems in Cartesian coordinates

- An infinite straight metal rod has a rectangular cross-section whose sides are of length a and b.
- The temperature $T(x,y)$ in each cross-section satisfies the boundary value problem given by Laplace's equation
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < a,\ 0 < y < b,$$
with the Dirichlet boundary conditions

$$\begin{aligned} T(0,y) &= 0 && \text{for} \quad 0 < y < b, \\ T(a,y) &= 0 && \text{for} \quad 0 < y < b, \\ T(x,0) &= 0 && \text{for} \quad 0 < x < a, \\ T(x,b) &= f(x) && \text{for} \quad 0 < x < a, \end{aligned}$$



where $f(x)$ is the prescribed temperature at which the top face of the rod is held.
- We construct a solution to the boundary value problem using Fourier's method, as follows.

Step (I) Find all nontrivial separable solutions of the PDE and homogeneous BCs

- We begin by finding all nontrivial separable solutions of Laplace's equation subject to the three homogeneous boundary conditions.
- Substituting $T(x, y) = F(x)G(y)$ into Laplace's equation and dividing through by $F(x)G(y) \neq 0$ gives
$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}.$$
- The LHS is independent of y, while the RHS is independent of x. Since the LHS is equal to the RHS, they must both be independent of x and y, and therefore equal to a constant, $-\lambda \in \mathbb{R}$ say.
- Hence,
$$-F'' = \lambda F \quad \text{for} \quad 0 < x < a,$$
with the homogeneous boundary conditions on T at $x = 0$ and $x = a$ giving the boundary conditions $F(0) = 0$ and $F(a) = 0$ for nontrivial G.
- We solved this problem for F in §3.4: the nontrivial solutions are given for positive integers n by
$$F(x) = B \sin\left(\frac{n\pi x}{a}\right) \quad \lambda = \left(\frac{n\pi}{a}\right)^2,$$
where B is an arbitrary constant.

- Since $G'' - \lambda G = 0$, the corresponding solution for $G(y)$ that satisfies the homogeneous boundary condition on $y = 0$ is given by

$$G = C \sinh\left(\frac{n\pi y}{a}\right),$$

where C is an arbitrary constant.

- Hence, the nontrivial separable solutions of Laplace's equation subject to the three homogeneous boundary conditions are given for positive integers n by

$$T_n(x, y) = b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right),$$

where $b_n = BC$ are real constants and we have introduced the subscript n on T_n and b_n to enumerate the countably infinite set of such solutions.

- **Remark:** In contrast to the wave equation for which the nontrivial separable solutions are the product of trigonometric functions in x and trigonometric functions in t, the nontrivial separable solutions of Laplace's equation are products of trigonometric functions in x with hyperbolic functions in y or *vice versa*.

Step (II) Apply the principle of superposition

- By linearity, we can superimpose the separable solutions to obtain the general series solution satisfying Laplace's equation and the three homogeneous boundary conditions (assuming convergence):

$$T(x,y)=\sum_{n=1}^{\infty}T_n(x,y)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right).$$

Step (III) Use the theory of Fourier series to satisfy the inhomogeneous BC

- The boundary condition at $y=b$ on the top side can only be satisfied if

$$f(x)=\sum_{n=1}^{\infty}b_n\sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\quad\text{for}\quad 0<x<a,$$

  so that the theory of Fourier series gives

$$b_n\sinh\left(\frac{n\pi b}{a}\right)=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x\quad\text{for}\quad n\in\mathbb{N}\backslash\{0\}.$$

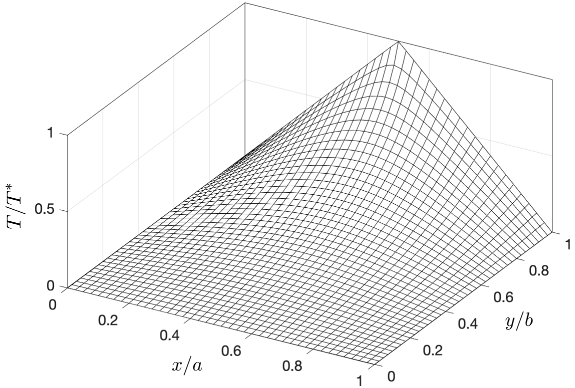
- **Remark:** If f satisfies the conditions of the Fourier Convergence Theorem, then it may be shown that the infinite series solution is termwise infinitely differentiable with respect to x and y inside the rectangular domain $0<x<a$, $0<y<b$, so that it satisfies Laplace's equation there.

Example

- If $f(x) = T^*(1 - |2x/a - 1|)$, where T^* is a constant temperature, then

$$T(x,y) = \frac{8T^*}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m \sin\big((2m+1)\pi x/a\big)\sinh\big((2m+1)\pi y/a\big)}{(2m+1)^2 \sinh\big((2m+1)\pi b/a\big)}.$$

- Series solution truncated to 100 terms, illustrating the "smoothing out" of the corner in boundary data.



5.6 Boundary value problems in plane polar coordinates

- Recall that in plane polar coordinates (r, θ), Laplace's equation for $T(r, \theta)$ becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \text{ for } r > 0.$$

- We start by finding all nontrivial separable solutions of the form $T(r, \theta) = F(r)G(\theta)$.

- Since T is a single-valued function of position on $r > 0$, we require $G(\theta)$ to be 2π-periodic.

- Substituting $T(r, \theta) = F(r)G(\theta)$ into Laplace's equation we obtain

$$F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' = 0.$$

- Separating the variables by dividing through by $F(r)G(\theta)/r^2 \neq 0$ gives

$$\frac{r^2 F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}.$$

- The LHS is independent of θ, while the RHS is independent of r. Since the LHS is equal to the RHS, they must both be independent of r and θ, and therefore equal to a constant, $\lambda \in \mathbb{R}$ say.

- Hence, we need to find all $\lambda \in \mathbb{R}$ for which $G''(\theta) + \lambda G(\theta) = 0$ has a nontrivial, 2π-periodic, solution $G(\theta)$. There are three cases to consider.

Case (i) $\lambda = -\omega^2$ ($\omega > 0$ wlog)

- If $G'' - \omega^2 G = 0$, then $G(\theta) = A \cosh \omega\theta + B \sinh \omega\theta$, where $A,\ B \in \mathbb{R}$.
- If G is 2π-periodic, then $G(0) = G(\pm 2\pi)$, which implies $A = A \cosh 2\pi\omega \pm B \sinh 2\pi\omega$, so that $A(\cosh 2\pi\omega - 1) = 0$ and $B \sinh 2\pi\omega = 0$, giving $A = B = 0$ and $G = 0$.

Case (ii) $\lambda = 0$

- If $G'' = 0$, then $G(\theta) = A + B\theta$, where $A,\ B \in \mathbb{R}$.
- If G is 2π-periodic, then $B = 0$, but arbitrary A is admissible.
- For $\lambda = 0$, $r^2 F'' + rF' = 0$, so $(rF')' = 0$, giving $F = c + d \log r$ for $r > 0$, where $c,\ d \in \mathbb{R}$.
- Hence for $\lambda = 0$ there is a nontrivial, 2π-periodic, separable solution in $r > 0$ of the form

$$T_0 = A_0 + B_0 \log r,$$

  where $A_0 = cA$ and $B_0 = dA$ are real constants.
- Since this solution is independent of θ it is called a *cylindrically-symmetric solution*.

Case (iii) $\lambda = \omega^2$ ($\omega > 0$ wlog)

- If $G'' + \omega^2 G = 0$, then $G(\theta) = R\cos(\omega\theta + \Phi)$, where R, $\Phi \in \mathbb{R}$.
- If G is nontrivial, then $R \neq 0$ and G has prime period $p = 2\pi/\omega$. Hence, G can only be nontrivial and 2π-periodic if there exists a positive integer n such that $np = 2\pi$, *i.e.* $\omega = n$ for some positive integer n, which the graph of G would reveal to be a geometrically obvious result.
- In anticipation of the need to write the solution in the form of a Fourier series, we write the resulting solution for $\omega = n$ in the form $G(\theta) = A\cos n\theta + B\sin n\theta$, where $A = R\cos\Phi$, $B = -R\sin\Phi$ are arbitrary real constants.
- If $\lambda = \omega^2 = n^2$, then we obtain for $F(r)$ Euler's ODE in the form

$$r^2F'' + rF' - n^2F = 0 \quad \text{for} \quad r > 0.$$

- As in Introductory Calculus, we derive the general solution of this ODE by making the change of variable $r = e^t$, $F(r) = W(t)$.

- By the chain rule,
$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = r\frac{\mathrm{d}F}{\mathrm{d}r},$$
so that
$$\frac{\mathrm{d}^2 W}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right)\frac{\mathrm{d}r}{\mathrm{d}t} = r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right) = r^2F'' + rF' = n^2F = n^2W.$$
Hence, $W = Ce^{nt} + De^{-nt}$, where C, $D \in \mathbb{R}$, and we conclude that the general solution for $F(r)$ is given by
$$F(r) = Cr^n + Dr^{-n}.$$

- **Remark:** An alternative method is to seek a solution of the form $F(r) = r^\mu$ for which $\mu(\mu-1) + \mu - \mu^2 = 0$, so that $\mu = \pm n$, from which follows the general solution.

- We conclude that for $\lambda = \omega^2$ there are a countably infinite set of nontrivial, 2π-periodic, separable solution in $r > 0$ given for positive integers n by
$$T_n = (A_n r^b + B_n r^{-n})\cos n\theta + (C_n r^n + D_n r^{-n})\sin n\theta,$$
where $A_n = AC$, $B_n = AD$, $C_n = BC$, $D_n = BD$ are real constants and we have introduced the subscript n on T_n and these constants to enumerate the countably infinite set of such solutions.

Summary

- Superimposing the nontrivial separable solutions in $r > 0$, we obtain the general series solution

$$T(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty} \Big((A_n r^n + B_n r^{-n}) \cos n\theta + (C_n r^n + D_n r^{-n}) \sin n\theta \Big).$$

Notes

(1) The solutions $\log r$, $r^{-n} \cos n\theta$ and $r^{-n} \sin n\theta$ are unbounded as $r \to 0+$, and hence not defined at $r = 0$. This means that these solutions are not admissible if the origin belongs to the domain in which T is defined.

(2) Similarly, if the domain in which T is defined extends to infinity and T is bounded there, then the solutions $\log r$, $r^n \cos n\theta$ and $r^n \sin n\theta$ are not admissible. We illustrate these results below with some concrete examples.

5.6 Boundary value problems in plane polar coordinates

Example 1

- Consider the boundary value problem for T given by

$$\nabla^2 T = 0 \quad \text{in} \quad a < r < b,$$

with

$$T = T_0^\star \quad \text{on} \quad r = a, \qquad T = T_1^\star \quad \text{on} \quad r = b,$$

where a and b are constant radii, while $T_0^\star$ and $T_1^\star$ are constant temperatures.

- Since T satisfies Laplace's equation in $a < r < b$, it has the general series solution

$$T(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty} \Big((A_n r^n + B_n r^{-n}) \cos n\theta + (C_n r^n + D_n r^{-n}) \sin n\theta \Big).$$

- Hence, the boundary conditions can only be satisfied if

$$T_0^\star = A_0 + B_0 \log a + \sum_{n=1}^{\infty} \left((A_n a^n + B_n a^{-n}) \cos n\theta + (C_n a^n + D_n a^{-n}) \sin n\theta \right),$$

$$T_1^\star = A_0 + B_0 \log b + \sum_{n=1}^{\infty} \left((A_n b^n + B_n b^{-n}) \cos n\theta + (C_n b^n + D_n b^{-n}) \sin n\theta \right),$$

for $-\pi < \theta \leq \pi$, say.

- Since the Fourier coefficients of a Fourier series are unique, we can equate them on the left- and right-hand sides of these equalities to obtain, for positive integers n,

$$\begin{bmatrix} 1 & \log a \\ 1 & \log b \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} T_0^\star \\ T_1^\star \end{bmatrix}, \qquad \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- Since $0 < a < b$, the matrices have nonzero determinant, so we can invert each of them to obtain

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \frac{1}{\log\left(\frac{b}{a}\right)} \begin{bmatrix} \log b & -\log a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_0^\star \\ T_1^\star \end{bmatrix}, \qquad \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- Hence, the solution is cylindrically symmetric and given by

$$T = \frac{T_0^\star \log b - T_1^\star \log a}{\log(b/a)} + \frac{T_1^\star - T_0^\star}{\log\left(\frac{b}{a}\right)} \log r.$$

Notes

(1) The solution may be written in the form

$$\frac{T}{T_0^\star} = \frac{\log(r/b)}{\log(a/b)} + \frac{T_1^\star}{T_0^\star}\frac{\log(r/a)}{\log(b/a)}.$$

Since all of the fractions in this expression are dimensionless, it is dimensionally correct.

(2) We could have sought a circularly-symmetric solution $T = T(r)$ from the outset because the boundary data is independent of θ. However, the method above generalises to $T_0^\star$ and $T_1^\star$ being functions of θ.

Example 2

- Consider the boundary value problem for T given by

$$\nabla^2 T = 0 \quad \text{in} \quad r < a,$$

with

$$T(a,\theta) = T^\star \sin^3\theta \quad \text{for} \quad -\pi < \theta \le \pi,$$

where a is a constant radius and $T^\star$ is a constant temperature.

- Recall that in $r > 0$ Laplace's equation has the general series solution

$$T(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty} \Big((A_n r^n + B_n r^{-n}) \cos n\theta + (C_n r^n + D_n r^{-n}) \sin n\theta \Big).$$

- Since in this example T satisfies Laplace's equation in the disk $r < a$, it must be twice differentiable with respect to x and y in a neighbourhood of the origin, and therefore continuous and bounded at the origin.

- Hence, the general series solution pertains, but with $B_0 = 0$ and $B_n = D_n = 0$ for positive integers n, giving

$$T(r,\theta) = A_0 + \sum_{n=1}^{\infty} \Big(A_n r^n \cos n\theta + C_n r^n \sin n\theta \Big).$$

- The boundary condition can only be satisfied if

$$T^\star \sin^3\theta = A_0 + \sum_{n=1}^{\infty} \left(A_n a^n \cos n\theta + C_n a^n \sin n\theta\right) \quad \text{for} \quad -\pi < \theta \le \pi.$$

- Since the Fourier series for the left-hand side of this expression is given by the identity

$$T^\star \sin^3\theta = \frac{3T^\star}{4}\sin\theta - \frac{T^\star}{4}\sin 3\theta,$$

  we can equate Fourier coefficients to deduce that

$$C_1 a = \frac{3T^\star}{4}, \; C_3 a^3 = -\frac{T^\star}{4}$$

  while the remainder must vanish.

- Hence, a solution is given by

$$T = \frac{3T^\star}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{T^\star}{4}\left(\frac{r}{a}\right)^3 \sin 3\theta.$$

- **Question:** What is the heat flux out of the disc through $r = a$?

- **Answer:** The heat flux vector $\boldsymbol{q} = -k\boldsymbol{\nabla} T$ according to Fourier's Law and we need the component in the direction of the outward pointing unit normal $\boldsymbol{n} = \boldsymbol{e}_r$ to the boundary $r = a$, namely

$$\boldsymbol{q} \cdot \boldsymbol{n}|_{r=a} = (-k\boldsymbol{\nabla} T) \cdot \boldsymbol{e}_r|_{r=a} = -k\frac{\partial T}{\partial r}(a, \theta) = -k\left(\frac{3T^\star}{4a}\sin\theta - \frac{3T^\star}{4a}\sin 3\theta\right),$$

where in the last equality we substituted the solution.

Since there is no volumetric heating, the net heat flux though $r = a$ must vanish, *i.e.*

$$\int_{r=a} \boldsymbol{q} \cdot \boldsymbol{n}\,\mathrm{d}s = 0,$$

which may be verified by substituting for $\boldsymbol{q} \cdot \boldsymbol{n}$ and integrating. ■

5.7 Poisson's Integral Formula

- Consider the boundary value problem for T given by

$$\nabla^2 T = 0 \quad \text{in} \quad r < a,$$

with

$$T(a,\theta) = f(\theta) \quad \text{for} \quad -\pi < \theta \le \pi,$$

where a is a constant radius and the temperature profile f is given.

- As in Example 2, the general series solution that satisfies Laplace's equation in $r < a$ is given by

$$T(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \Big(A_n r^n \cos n\theta + C_n r^n \sin n\theta \Big),$$

where we replaced A_0 with $A_0/2$ for algebraic convenience.

- Hence, the boundary condition can only be satisfied if

$$f(\phi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n a^n \cos(n\phi) + C_n a^n \sin(n\phi) \right) \text{ for } -\pi < \phi \le \pi,$$

where we replaced the dummy variable θ with ϕ in anticipation of the following analysis.

- The theory of Fourier series then gives the Fourier coefficients

$$a^n A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \cos(n\phi)\,\mathrm{d}\phi \quad \text{for } n \in \mathbb{N},$$

$$a^n C_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \sin(n\phi)\,\mathrm{d}\phi \quad \text{for } n \in \mathbb{N}\backslash\{0\}.$$

- While these integral expressions can evaluated in simple cases (such as in Example 2), it is a remarkable fact that the series solution may be summed for a wide class of functions f (namely those that are sufficiently regular that the following analysis is valid).

- We being by substituting the integral expressions for the Fourier coefficients into the series solution and assuming that the orders of summation and integration may be interchanged, *viz.*

$$T(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\,\mathrm{d}\phi + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^n [\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi)] f(\phi)\,\mathrm{d}\phi \right)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right) f(\phi)\,\mathrm{d}\phi.$$

- Now, if we let $\alpha = \theta - \phi$ and $z = \frac{r}{a}\mathrm{e}^{\mathrm{i}\alpha}$, then

$$
\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n\alpha &= \mathrm{Re}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \mathrm{e}^{\mathrm{i}n\alpha}\right) \\
&= \mathrm{Re}\left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right) \\
&= \mathrm{Re}\left(\frac{1}{2}\frac{1+z}{1-z}\right) \\
&= \mathrm{Re}\left(\frac{1}{2}\frac{a + r\mathrm{e}^{\mathrm{i}\alpha}}{a - r\mathrm{e}^{\mathrm{i}\alpha}}\right) \\
&= \mathrm{Re}\left(\frac{1}{2}\frac{(a + r\cos\alpha + \mathrm{i}r\sin\alpha)(a - r\cos\alpha + \mathrm{i}r\sin\alpha)}{(a - r\cos\alpha - \mathrm{i}r\sin\alpha)(a - r\cos\alpha + \mathrm{i}r\sin\alpha)}\right) \\
&= \frac{1}{2}\frac{(a + r\cos\alpha)(a - r\cos\alpha) + (\mathrm{i}r\sin\alpha)^2}{(a - r\cos\alpha)^2 + (r\sin\alpha)^2} \\
&= \frac{a^2 - r^2}{2(a^2 - 2ar\cos\alpha + r^2)},
\end{aligned}
$$

where the summation of the geometric series in the third equality is valid for $|z| < 1$.

- Hence,

$$T(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right) f(\phi)\,\mathrm{d}\phi,$$

where

$$\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta-\phi) = \frac{a^2-r^2}{2(a^2-2ar\cos(\theta-\phi)+r^2)}$$

for $0 \le r < a$.

- Substituting the latter into the former, we obtain *Poisson's Integral Formula* in the form

$$T(r,\theta) = \frac{(a^2-r^2)}{2\pi}\int_{-\pi}^{\pi} \frac{f(\phi)\,\mathrm{d}\phi}{a^2-2ar\cos(\theta-\phi)+r^2},$$

which is valid for $0 \le r < a$.

Notes

(1) The value of T at the centre of the disc is given by

$$T(0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\,\mathrm{d}\phi,$$

which is the mean value of T over the boundary.

(2) More generally, we can now see that if T satisfies Laplace's equation in some region R and if $D(x, y, a)$ is a disk inside R with centre (x, y) and radius a, then

$$T(x, y) = \frac{1}{2\pi a} \int_{\partial D(x,y,a)} T\,\mathrm{d}s,$$

where $\partial D(x, y, a)$ is the boundary of $D(x, y, a)$ and $\mathrm{d}s$ an element of arclength. That the mean value over a circle is equal to its value at the centre is called the *mean-value property of Laplace's equation* and has important consequences. For example, it explains why solutions of Laplace's equation are infinitely differentiable, since local averages over a circle vary smoothly as the circle moves.

5.8 Uniqueness Theorems

- We will state and prove uniqueness theorems for the two-dimensional inhomogeneous Dirichlet and Neumann problems and illustrate with examples their implications for the application of Fourier's method.

- **Uniqueness Theorem (Dirichlet problem):** Consider the Dirichlet problem for $T(x, y)$ given by

$$-k\nabla^2 T = Q \quad \text{in} \quad R,$$

with

$$T = f \quad \text{on} \quad \partial R,$$

where R is a path-connected region as in the statement of Green's theorem in the plane, Q is a given function on R and f is a given function on ∂R. Then the boundary value problem has at most one solution.

Proof:

- Let W be the difference between two solutions, then

$$\nabla^2 W = 0 \quad \text{in } R \quad \text{with} \quad W = 0 \quad \text{on } \partial R.$$

- The trick is to apply Green's theorem in the plane with $\boldsymbol{F} = W\nabla W$ to obtain the integral identify

$$\iint_R \nabla \cdot (W\nabla W)\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} (W\nabla W) \cdot \boldsymbol{n}\,\mathrm{d}s.$$

- Since $\nabla^2 W = 0$ in R, we have $\nabla \cdot (W\nabla W) = W\nabla^2 W + \nabla W \cdot \nabla W = |\nabla W|^2$ in R.

- Since $W = 0$ on ∂R, we have $W\nabla W \cdot \boldsymbol{n} = 0$ on ∂R.

- Hence, the integral identity becomes

$$\iint_R |\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y = 0.$$

- Assuming ∇W is continuous on $R \cup \partial R$, we deduce that $\nabla W = \mathbf{0}$ on R, so that W is constant on R because R is path connected.

- But $W = 0$ on ∂R, so assuming W is continuous on $R \cup \partial R$, the constant must vanish, and we deduce that $W = 0$ on $R \cup \partial R$. ■

Example 1

- Find T such that
$$\nabla^2 T = 0 \quad \text{in } r < a$$
with
$$T = \frac{T^\star x}{a} \quad \text{on } r = a,$$
where a and $T^\star$ are constants.

- If we can find any solution, then the uniqueness theorem guarantees it is the only solution.

- We could proceed via Fourier's method or Poisson's Integral Formula, but it is quicker to spot that the solution is simply
$$T = \frac{T^\star x}{a}.$$

- **Uniqueness Theorem (Neumann problem):** Consider the Neumann problem for $T(x, y)$ given by

$$-k\nabla^2 T = Q \quad \text{in} \quad R,$$

with

$$-k\frac{\partial T}{\partial n} = q \quad \text{on} \quad \partial R,$$

where R is a bounded and path-connected region as in the statement of Green's theorem in the plane, Q is a given function on R and q is a given function on ∂R. Then the boundary value problem has no solution unless Q and q satisfy the solvability condition

$$\iint_R Q \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} q \,\mathrm{d}s.$$

When a solution exists, it is not unique: any two solutions differ by a constant.

- **Remark:** The solvability condition is precisely the global energy balance derived earlier on.

Proof:

- Suppose there is a solution T, then

$$
\begin{aligned}
\iint\limits_R Q \, \mathrm{d}x \, \mathrm{d}y &= -k \iint\limits_R \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} T \, \mathrm{d}x \, \mathrm{d}y \\
&= -k \int\limits_{\partial R} \boldsymbol{n} \cdot \boldsymbol{\nabla} T \, \mathrm{d}s \\
&= -k \int\limits_{\partial R} \frac{\partial T}{\partial n} \, \mathrm{d}s \\
&= \int\limits_{\partial R} q \, \mathrm{d}s,
\end{aligned}
$$

  where we used Poisson's equation in the first equality, Green's theorem in the plane with $\boldsymbol{F} = \boldsymbol{\nabla} T$ in the second equality and the boundary conditions in the final equality.

- Now let W be the difference between two solutions, so that linearity gives

$$\boldsymbol{\nabla}^2 W = 0 \quad \text{in} \quad R \quad \text{with} \quad \frac{\partial W}{\partial n} = 0 \quad \text{on} \quad \partial R.$$

- Then, as in the uniqueness proof for the Dirichlet problem,

$$\begin{aligned}
\iint_R |\nabla W|^2 \,\mathrm{d}x\,\mathrm{d}y &= \iint_R W\nabla^2 W + \nabla W \cdot \nabla W \,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_R \nabla \cdot (W\nabla W)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\partial R} W\nabla W \cdot \boldsymbol{n}\,\mathrm{d}s \\
&= \int_{\partial R} W\frac{\partial W}{\partial \boldsymbol{n}}\,\mathrm{d}s \\
&= 0,
\end{aligned}$$

  where we used Laplace's equation for W in the first equality, Green's theorem in the plane with $\boldsymbol{F} = W\nabla W$ in the second equality and the boundary conditions for W in the final equality.

- Assuming ∇W is continuous on $R \cup \partial R$, we deduce that $\nabla W = \mathbf{0}$ on R, so that W is constant on R because R is path connected.

- Hence, W is constant on $R \cup \partial R$, assuming W is continuous there. ■

Example 2

- Find T such that

$$\nabla^2 T = 0 \quad \text{in } r < a,$$

with

$$-k\frac{\partial T}{\partial r}(a,\theta) = q(\theta) \quad \text{for } -\pi < \theta \leq \pi,$$

where the heat flux $q(\theta)$ is given.

- As in §5.5 the general series solution of Laplace's equation in $r < a$ is given by

$$T = A_0 + \sum_{n=1}^{\infty} \left(A_n r^n \cos n\theta + C_n r^n \sin n\theta\right).$$

so the boundary condition on $r = a$ can be satisfied only if

$$q(\theta) = \sum_{n=1}^{\infty} \big(- knA_n a^{n-1} \cos n\theta - knC_n a^{n-1} \sin n\theta \big) \quad \text{for} \quad -\pi < \theta \leq \pi.$$

- The theory of Fourier series then requires

$$0 = \frac{1}{\pi} \int_{-\pi}^{\pi} q(\theta)\, \mathrm{d}\theta, \tag{†}$$

while for positive integers n,

$$-knA_n a^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} q(\theta) \cos n\theta\, \mathrm{d}\theta,$$

$$-knC_n a^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} q(\theta) \sin n\theta\, \mathrm{d}\theta.$$

- Hence, there are two cases:
  (i) if q is such that (†) is not satisfied, then there is no solution;
  (ii) if q is such that (†) is satisfied, then there is a solution but it is not unique because A_0 is arbitrary (while the other Fourier coefficients are uniquely determined). ■

Notes

(1) This conclusion is in agreement with the Uniqueness Theorem, which also guarantees that in case (ii) we've found all possible solutions.

(2) In case (i) there is no solution because the temperature cannot be in steady state if the net heat flux through $r = a$ is non-zero.

(3) In case (ii) there can be a steady state solution because the net heat flux through $r = a$ vanishes, but we cannot pin down the temperature without additional information — in practice this would usually be provided by the evolution toward the steady state.

The end — thank you for listening

Please email comments & corrections to *oliver@maths.ox.ac.uk*