Fourier Analysis **HT22**

Problem Sheet 4

Problem 1. [*A theorem of Paley and Wiener.*]
(a) Let $f \in \mathrm{L}^2(\mathbb{R})$ and assume that $f(x) = 0$ for a. e. $x > 0$. Prove that

$$F(\zeta) := \int_{-\infty}^{0} f(x) \mathrm{e}^{-\mathrm{i}\zeta x} \, \mathrm{d}x$$

is a well-defined and holomorphic function for all $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\zeta) > 0$.
Show that F satisfies the bound

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\xi + \mathrm{i}\eta)|^2 \, \mathrm{d}\xi = \int_{-\infty}^{0} |f(x)|^2 \mathrm{e}^{2\eta x} \, \mathrm{d}x \le \|f\|_2^2$$

for all $\eta > 0$. Next, show that

$$F(\cdot + \mathrm{i}\eta) \to \widehat{f} \text{ in } \mathrm{L}^2(\mathbb{R}) \text{ as } \eta \searrow 0.$$

(b) (*Optional.*) Assume that $F \colon H \to \mathbb{C}$ is a holomorphic function defined on the upper half-plane $H = \big\{ \zeta \in \mathbb{C} : \mathrm{Im}(\zeta) > 0 \big\}$ that satisfies the bound

$$\sup_{\eta > 0} \int_{-\infty}^{\infty} \big| F(\xi + \mathrm{i}\eta) \big|^2 \, \mathrm{d}\xi < \infty.$$

Prove that $F(\cdot + \mathrm{i}\eta)$ converges in $\mathrm{L}^2(\mathbb{R})$ as $\eta \searrow 0$ to the Fourier transform $\widehat{f}$ of a function $f \in \mathrm{L}^2(\mathbb{R})$ vanishing on $(0, \infty)$.

Problem 2.
(a) Show that the Fourier transform of the function

$$f(x) = \big(1 - |x|\big)^+ = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

is $\mathrm{sinc}^2(\xi/2)$. (*See Example 1.4 in the lecture notes for the definition of* sinus cardinalis. *However, the calculation is probably easiest via the differentiation rule and double-angle formula.*)
Explain why the Poisson summation formula applies to f and use it to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^2} = \frac{\pi^2}{\sin^2(\pi x)} \tag{1}$$

holds for all $x \in \mathbb{R} \setminus \mathbb{Z}$.

(b) For instance by use of (a) show that

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n+x}=\pi\cot(\pi x) \tag{2}$$

holds for all $x\in\mathbb{R}\setminus\mathbb{Z}$. (*Hint: Consider the identity first for* $x\in(0,1)$. *The difference of the two sides is a distribution in* $\mathscr{D}'(0,1)$*–what is its derivative?*)
(c) Explain why the identity (2) can be extended to hold in $\mathbb{C}\setminus\mathbb{Z}$:

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n+z}=\pi\cot(\pi z)$$

for all $z\in\mathbb{C}\setminus\mathbb{Z}$. Using this formula deduce *Lipschitz's formula:*

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^k}=\frac{\left(-2\pi\mathrm{i}\right)^k}{(k-1)!}\sum_{n=1}^{\infty}n^{k-1}\mathrm{e}^{2\pi\mathrm{i}nz}$$

holds for all $z\in\mathbb{C}$ with $\mathrm{Im}(z)>0$ and $k\in\mathbb{N}$, $k\geq 2$.

Problem 3. (*Optional.*) Consider the 2π-periodic function $g\colon\mathbb{R}\to\mathbb{C}$ given by

$$g(x)=\frac{\pi}{\sin\pi\alpha}\mathrm{e}^{\mathrm{i}(\pi-x)\alpha}$$

for $x\in(0,2\pi]$, where $\alpha\in\mathbb{R}\setminus\mathbb{Z}$. Calculate its Fourier series and use Parseval's identity for Fourier series to recover (1). In what sense does the Fourier series for g converge and could we use it to recover (2)?

Problem 4. For each $\varphi\in\mathscr{S}(\mathbb{R})$ we define its *periodisation* as

$$P\varphi(x)=\sum_{k\in\mathbb{Z}}\varphi(x-2\pi k),\quad x\in\mathbb{R}.$$

(a) Check that $P\varphi$ is a 2π periodic C^∞ function, and show that

$$P\varphi(x)=\sum_{k\in\mathbb{Z}}\frac{1}{2\pi}\widehat{\varphi}(k)\mathrm{e}^{ikx}$$

holds for all $x\in\mathbb{R}$. Deduce Poisson's summation formula:

$$2\pi\sum_{k\in\mathbb{Z}}\varphi(2\pi k)=\sum_{k\in\mathbb{Z}}\widehat{\varphi}(k).$$

(c) Show that

$$\sum_{k\in\mathbb{Z}}\mathrm{e}^{-4\pi^2tk^2}=\frac{1}{\sqrt{4\pi t}}\sum_{k\in\mathbb{Z}}\mathrm{e}^{-\frac{k^2}{4t}}$$

holds for all $t > 0$.

Problem 5. [*Fast convergence of Riemann sums for test functions.*]
Let $\varphi \in \mathscr{D}(\mathbb{R})$ and $m \in \mathbb{N}$. Show that

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = \frac{1}{N} \sum_{n \in \mathbb{Z}} \varphi\left(\frac{n}{N}\right) + O\left(\frac{1}{N^m}\right) \text{ as } N \to \infty.$$

Here the implied constant may of course depend on φ and m.

Problem 6.
(a) Let $f \colon \mathbb{R} \to \mathbb{C}$ be a 2π-periodic $\mathrm{L}^1_{\mathrm{loc}}$ function. Show that for $n \in \mathbb{Z} \setminus \{0\}$ its n-th Fourier coefficient c_n satisfies

$$c_n = \frac{1}{4\pi} \int_{-\pi}^{\pi} \big(f(x) - f(x + \frac{\pi}{n})\big) \mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x.$$

Deduce that $c_n \to 0$ as $|n| \to \infty$. This result is also called the *Riemann–Lebesgue lemma*.
(b) Show that the Fourier coefficients of a continuous 2π-periodic function can tend to 0 arbitrarily slowly by proving that for every sequence (t_n) of positive numbers converging to 0, there exists a continuous function f whose Fourier coefficients satisfy $|c_n| \geq t_n$ for infinitely many values of n. (*Hint: Choose a subsequence* (t_{n_k}) *so* $\sum_{k=1}^{\infty} t_{n_k} < \infty$.)
(c) Assume that $f \colon \mathbb{R} \to \mathbb{C}$ is 2π-periodic and α-Hölder continuous for some $\alpha \in (0, 1]$:

$$\big|f(x) - f(y)\big| \leq c|x - y|^{\alpha}$$

holds for all x, $y \in \mathbb{R}$, where $c \geq 0$ is a constant. Using (a) prove that the Fourier coefficients c_n satisfy

$$c_n = O(|n|^{-\alpha}) \text{ as } |n| \to \infty.$$

(d) Assume that $f \colon \mathbb{R} \to \mathbb{C}$ is a 2π-periodic $\mathrm{H}^1_{\mathrm{loc}}$ function (so both f and f' belong to $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$). Prove that the Fourier series for f converges absolutely and uniformly.