General relativity II

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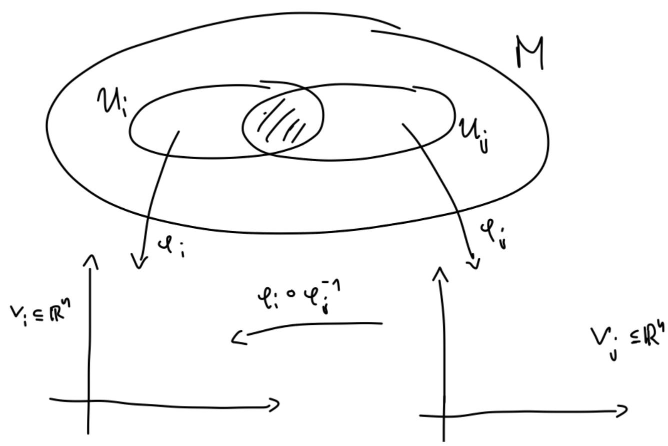
1 Mathematical preliminaries

1.1 Review of differential geometry from GR I & notation

A **smooth manifold** M **of dimension** n is a topological space M (second countable, Hausdorff) together with a collection (U_i, φ_i) of homeomorphisms[1] $\varphi_i : M \supseteq U_i \to V_i \subseteq \mathbb{R}^n$, U_i, V_i open sets, such that

1. every point $p \in M$ is contained in some U_i
2. if $U_i \cap U_j \neq \emptyset$, then $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a smooth diffeomorphism[2]

Each (U_i, φ_i) is called a **chart** for the manifold M.



A **function** $f : M \to \mathbb{R}$ is called **smooth** iff for all charts (U_i, φ_i) we have that $f \circ \varphi_i^{-1} : U_i \to \mathbb{R}$ is smooth. We denote the space of all smooth functions on M with $C^\infty(M)$.

A **derivation** X at $p \in M$ is a linear map $X : C^\infty(M) \to \mathbb{R}$ which satisfies Leibniz's rule

$$X(fg) = X(f) \cdot g + f \cdot X(g) \qquad \text{for all } f, g \in C^\infty(M)\ .$$

The **tangent space** T_pM at the point $p \in M$ is the linear space of all derivations X at p.

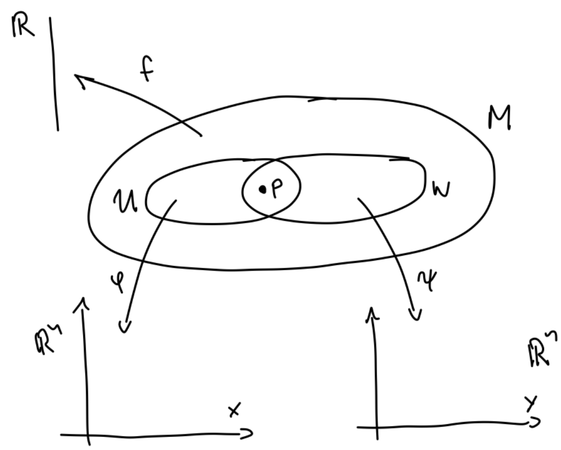
Given a chart (U, φ) for M with x^j coordinates and $p \in U$, the coordinate derivations

$$\left.\frac{\partial}{\partial x^j}\right|_p (f) := \frac{\partial}{\partial x^j}(f \circ \varphi^{-1})\big(\varphi(p)\big)$$

form a basis for T_pM. Given $X \in T_pM$ we can thus write $X = X^i \frac{\partial}{\partial x^i}$, where the $X^i \in \mathbb{R}$ are the **components of the vector** X **in the chart** φ.

Change of coordinates: Let (W, ψ) be another chart with $p \in M$ with y^i coordinates.

[1] I.e. a continuous map with a continuous inverse.
[2] I.e. a smooth map with a smooth inverse.



We compute using the chain rule[3]

$$\frac{\partial}{\partial x^j}\Big|_p f = \frac{\partial}{\partial x^j}\big(f \circ \psi^{-1} \circ (\psi \circ \varphi^{-1})\big)\big(\varphi(p)\big) = \frac{\partial}{\partial y^i}(f \circ \psi^{-1})\big(\psi(p)\big) \cdot \frac{\partial (\psi \circ \varphi^{-1})^i}{\partial x^j}\big(\varphi(p)\big) = \frac{\partial}{\partial y^i}\Big|_p f \cdot \frac{\partial y^i}{\partial x^j} \ .$$

This gives the transformation rule

$$\frac{\partial}{\partial x^j} = \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i} \ . \tag{1.1}$$

The **cotangent space** T_p^*M is the dual space of T_pM with coordinate basis dx^i, i.e., $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$. Given $\alpha \in T_p^*M$ we can write $\alpha = \alpha_i dx^i$, where the $\alpha_i \in \mathbb{R}$ are the components of the covector α in the chart φ. Using (1.1) we compute

$$dx^j(\frac{\partial}{\partial y^i}) = dx^j(\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^k}) = \frac{\partial x^j}{\partial y^i} \ .$$

This gives the transformation rule

$$dx^j = \frac{\partial x^j}{\partial y^i} dy^i \ . \tag{1.2}$$

The (k,l)**-tensor space** $T_p^{(k,l)}M$ consists of all multilinear maps

$$T : \underbrace{T_p^*M \times \ldots \times T_p^*M}_{k-\text{times}} \times \underbrace{T_pM \times \ldots \times T_pM}_{l-\text{times}} \to \mathbb{R} \ .$$

Note that a $(0,1)$-tensor is just a covector and a $(1,0)$-tensor is a vector (use here that $(T_p^*M)^* = T_pM$)). Define the elements $\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_l}$ with $1 \le i_1, \ldots, i_k, j_1, \ldots, j_l \le n$ by

$$\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_l}(dx^{a_1}, \ldots, dx^{a_k}, \frac{\partial}{\partial x^{b_1}}, \ldots, \frac{\partial}{\partial x^{b_l}}) = \delta^{a_1}_{i_1} \cdots \delta^{a_k}_{i_k} \delta^{j_1}_{b_1} \cdots \delta^{j_l}_{b_l} \ .$$

They form a basis of $T_p^{(k,l)}M$. Given $T \in T_p^{(k,l)}M$ we can thus write

$$\begin{aligned} T &= T^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_l} \\ &= \tilde{T}^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} \frac{\partial}{\partial y^{a_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{a_k}} \otimes dy^{b_1} \otimes \ldots \otimes dy^{b_l} \ , \end{aligned}$$

[3]We are using the Einstein summation convention of summing over repeated indices.

with

$$\tilde{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} = T(dy^{a_1}, \ldots, dy^{a_k}, \frac{\partial}{\partial y^{b_1}}, \ldots, \frac{\partial}{\partial y^{b_l}}) \,.$$

Using (1.1) and (1.2) in the right hand side we obtain the transformation rule for the components of a general (k,l)-tensor

$$\tilde{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} = T^{i_1\ldots i_k}{}_{j_1\ldots j_l} \frac{\partial y^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{a_k}}{\partial x^{i_k}} \frac{\partial x^{j_1}}{\partial y^{b_1}} \cdots \frac{\partial x^{j_l}}{\partial y^{b_l}} \,. \tag{1.3}$$

A **smooth** (k,l)**-tensor field** T is a map $M \ni p \mapsto T(p) \in T_p^{(k,l)}M$ for all $p \in M$ such that in local coordinates $\varphi : M \supseteq U \to V \subseteq \mathbb{R}^n$ the components $T^{i_1\ldots i_k}{}_{j_1\ldots j_l} : V \to \mathbb{R}$ are smooth functions.

We denote by $\mathfrak{X}^\infty(M)$ the **space of smooth vector fields** on M and by $\Omega^1(M)$ the **space of smooth** 1**-covector fields** (1-forms) on M.

Given $f \in C^\infty(M)$ we define the **derivative of** f, $df \in \Omega^1(M)$, by $df(X) = X(f)$ for $X \in \mathfrak{X}^\infty(M)$. In coordinates we have $df = \partial_i f dx^i$ (easy exercise).[4]

A **Lorentzian metric** g on M is a smooth $(0,2)$-tensor field such that at every point $p \in M$ $g(p) : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate[5] symmetric bilinear form of signature $(-,+,\ldots,+)$. In local coordinates we have $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ and by convention we write for the inverse metric, a smooth $(2,0)$-tensor field, $g^{-1} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$.

Tensor operations:

- Contraction of contravariant (upper) and covariant (lower) indices (**trace**). For example let $T = T^i_{jk}\partial_i \otimes dx^j \otimes dx^k$ be a $(1,2)$-tensor. Contract i and j to get $\mathrm{tr}T = T^i_{ik}dx^k$, a $(0,1)$-tensor.

- **Raising/lowering index with the metric**. For example let $X = X^\mu \partial_\mu$ be a vector, then $X^\flat := g(X,\cdot)$ is a covector with $(X^\flat)_\nu = X^\mu g_{\mu\nu}$. We also often write just X_ν for $(X^\flat)_\nu$.

  Similarly, let α be a covector. Then $\alpha^\sharp := g^{-1}(\alpha,\cdot)$ is a vector with $(\alpha^\sharp)^\mu = g^{\mu\nu}\alpha_\nu$. Again, we often write just α^μ for $(\alpha^\sharp)^\mu$.

- **Tensor product**. For example let $\alpha = \alpha_i dx^i$, $\beta = \beta_j dx^j$ be covectors, then $\alpha \otimes \beta = \underbrace{\alpha_i\beta_j}_{=(\alpha\otimes\beta)_{ij}} dx^i \otimes dx^j$ is a $(0,2)$-tensor.

Given two vector fields X, Y we define their **Lie bracket** $[X,Y]f := X(Yf) - Y(Xf)$ for $f \in C^\infty(M)$. Clearly $[X,Y] : C^\infty(M) \to C^\infty(M)$ is linear. We show that $[X,Y]$ also satisfies the Leibniz

[4]Note that we also write ∂_i for $\frac{\partial}{\partial x^i}$ if we use generic coordinates or if no confusion can arise which coordinates we are using.

[5]Recall that non-degenerate means that $g_p(X,Y) = 0$ for all $Y \in T_pM$ implies $X = 0$.

rule, from which it then follows that it is a vector field:

$$\begin{aligned}
[X,Y](fg) &= XY(fg) - YX(fg) \\
&= X\big((Yf)\cdot g + f\cdot(Yg)\big) - Y\big((Xf)\cdot g + f\cdot(Xg)\big) \\
&= (XYf)\cdot g + \underbrace{(Yf)\cdot(Xg)} + \underbrace{(Xf)\cdot(Yg)} + f\cdot(XYg) - (YXf)\cdot g - \underbrace{(Xf)\cdot(Yg)} \\
&\qquad - \underbrace{(Yf)\cdot(Xg)} - f\cdot(YXg) \\
&= \big([X,Y]f\big)\cdot g + f\cdot\big([X,Y]g\big)\ .
\end{aligned}$$

In coordinates $X = X^\mu \partial_\mu$, $Y = Y^\nu \partial_\nu$, and thus[6]

$$[X,Y]f = X^\mu \partial_\mu (Y^\nu \partial_\nu f) - Y^\nu \partial_\nu (X^\mu \partial_\mu f) = X^\mu (\partial_\mu Y^\nu)\cdot \partial_\nu f - Y^\nu (\partial_\nu X^\mu)\cdot \partial_\mu f\ .$$

Thus we obtain $[X,Y] = [X,Y]^\nu \partial_\nu$ with

$$[X,Y]^\nu = X(Y^\nu) - Y(X^\nu)\ . \tag{1.4}$$

The Lie bracket satisfies the following properties (exercise):

- Antisymmetric: $[X,Y] = -[Y,X]$
- Bilinear over $\mathbb{R}$: $[X, aY + bZ] = a[X,Y] + b[X,Z]$ for $a,b \in \mathbb{R}$
- Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$
- For $f \in C^\infty(M)$ we have $[X, fY] = f[X,Y] + (Xf)\cdot Y$.

An **affine connection** (covariant derivative) is a map $\nabla : \mathfrak{X}^\infty(M)\times\mathfrak{X}^\infty(M) \to \mathfrak{X}^\infty(M)$ such that for $X, Y, Z \in \mathfrak{X}^\infty(M)$

- $C^\infty(M)$-linear in first entry: $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ for $f,g \in C^\infty(M)$.
- $\mathbb{R}$-linear in second entry: $\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z$ for $a,b \in \mathbb{R}$
- Leibniz rule: $\nabla_X(fY) = f\nabla_X Y + X(f)\cdot Y$ for $f \in C^\infty(M)$.

Given local coordinates we call the quantities $dx^\kappa(\nabla_{\partial_\mu}\partial_\nu) =: \Gamma^\kappa_{\mu\nu}$ the **Christoffel symbols** of the connection. Using the above defining properties of an affine connection we then obtain

$$\nabla_X Y = (X^\mu \partial_\mu Y^\nu + \Gamma^\nu_{\mu\kappa} X^\mu Y^\kappa)\partial_\nu\ . \tag{1.5}$$

It follows from the $C^\infty(M)$-linearity in the first argument that ∇Y is a $(1,1)$-tensor field (problem sheet 1). For the components in local coordinates we thus obtain from (1.5) $\nabla_\mu Y^\nu = dx^\nu(\nabla_{\partial_\mu} Y) = \partial_\mu Y^\nu + \Gamma^\nu_{\mu\kappa} Y^\kappa$.

We can extend the affine connection in a unique way to all tensor fields by requiring

1) $\nabla_X f = X(f)$ for $f \in C^\infty(M)$.

[6] It also follows directly from the coordinate expression that $[X,Y]$ is a derivation.

2) $\nabla_X(\alpha \otimes \beta) = (\nabla_X \alpha) \otimes \beta + \alpha \otimes (\nabla_X \beta)$, the Leibniz rule, where α, β are arbitrary tensor fields.

3) ∇_X commutes with all contractions: $\mathrm{tr}(\nabla_X \alpha) = \nabla_X(\mathrm{tr}\alpha)$.

Example 1.6. *We compute the covariant derivative of a 1-form: Let $\alpha \in \Omega^1(M)$, $X, Y \in \mathfrak{X}^\infty(M)$. Then*

$$\begin{aligned}(\nabla_X \alpha)(Y) &= \mathrm{tr}(\nabla_X \alpha \otimes Y) \\ &= \mathrm{tr}(\nabla_X(\alpha \otimes Y) - \alpha \otimes \nabla_X Y) \qquad \text{using 2)} \\ &= X(\alpha(Y)) - \alpha(\nabla_X Y) \qquad \text{using 3) and 1).}\end{aligned}$$

This gives in coordinates $\nabla_\mu \alpha_\nu = (\nabla_{\partial_\mu}\alpha)(\partial_\nu) = \partial_\mu(\alpha_\nu) - \alpha_\kappa \Gamma^\kappa_{\mu\nu}$.

For a general (n, m)-tensor field T we obtain in coordinates (see GR I)

$$\begin{aligned}\nabla_a T^{b_1\dots b_n}{}_{c_1\dots c_m} = \partial_a(T^{b_1\dots b_n}{}_{c_1\dots c_m}) + \Gamma^{b_1}_{ad} \cdot T^{db_2\dots b_n}{}_{c_1\dots c_m} + \dots + \Gamma^{b_n}_{ad} \cdot T^{b_1\dots b_{n-1}d}{}_{c_1\dots c_m} \\ - \Gamma^d_{ac_1} \cdot T^{b_1\dots b_n}{}_{dc_2\dots c_m} - \dots - \Gamma^d_{ac_m} \cdot T^{b_1\dots b_n}{}_{c_1\dots c_{m-1}d} \,.\end{aligned}$$

The **torsion tensor** T of an affine connection ∇ is defined by $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$, where $X, Y \in \mathfrak{X}^\infty(M)$. (Exercise: Check that T is indeed a $(1, 2)$-*tensor field*.)

The affine connection ∇ is called **symmetric** $:\iff T = 0 \overset{[\partial_\mu,\partial_\nu]=0}{\iff} \Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}$.

Theorem 1.7. *Let (M, g) be a Lorentzian manifold. There exists exactly one affine connection ∇ which is*

1. ***metric**: $\nabla g = 0$*

2. *symmetric: $T = 0$.*

*∇ is called the **Levi-Civita connection**. The Christoffel symbols are given in coordinates by*

$$\Gamma^\mu_{\nu\kappa} = \frac{1}{2} g^{\mu\sigma}(\partial_\kappa g_{\nu\sigma} + \partial_\nu g_{\sigma\kappa} - \partial_\sigma g_{\nu\kappa}) \,.$$

From now on we will always use the Levi-Civita connection.

A vector field X is **parallel** along a curve $\gamma : I \to M$ iff $\nabla_{\dot\gamma} X = 0$. A curve $\gamma : I \to M$ is an affinely parametrised **geodesic** iff $\nabla_{\dot\gamma}\dot\gamma = 0$. For more details on parallel transport see GR I.

We define the Riemann **curvature tensor** as the map $R(\cdot,\cdot)\cdot : \mathfrak{X}^\infty(M) \times \mathfrak{X}^\infty(M) \times \mathfrak{X}^\infty(M) \to \mathfrak{X}^\infty(M)$,

$$R(X, Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \,, \qquad X, Y, Z \in \mathfrak{X}^\infty(M) \,.$$

One can show (GR I) that R is indeed a $(1, 3)$-*tensor field*. The coordinate components are given by $R(\partial_\mu, \partial_\nu)\partial_\kappa = R^\sigma{}_{\kappa\mu\nu}\partial_\sigma$ with

$$R^\sigma{}_{\kappa\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\kappa} - \partial_\nu \Gamma^\sigma_{\mu\kappa} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\nu\kappa} - \Gamma^\sigma_{\nu\rho}\Gamma^\rho_{\mu\kappa} \,.$$

We can lower the first index with the metric: $R_{\sigma\kappa\mu\nu} = g_{\sigma\rho} R^\rho{}_{\kappa\mu\nu}$. The curvature tensor satisfies the following symmetries:

- $R_{\sigma\kappa\mu\nu} = -R_{\kappa\sigma\mu\nu}$
- $R_{\sigma\kappa\mu\nu} = -R_{\sigma\kappa\nu\mu}$
- $R_{\sigma\kappa\mu\nu} = R_{\mu\nu\sigma\kappa}$
- $R_{\sigma\kappa\mu\nu} + R_{\sigma\mu\nu\kappa} + R_{\sigma\nu\kappa\mu} = 0$ (first Bianchi identity)
- $\nabla_\rho R_{\sigma\kappa\mu\nu} + \nabla_\sigma R_{\kappa\rho\mu\nu} + \nabla_\kappa R_{\rho\sigma\mu\nu} = 0$ (second Bianchi identity) .

Recall the interpretation of curvature via geodesic deviation from GR I. On problem sheet 1 you find another interpretation of curvature as determining parallel transport around an infinitesimal loop.

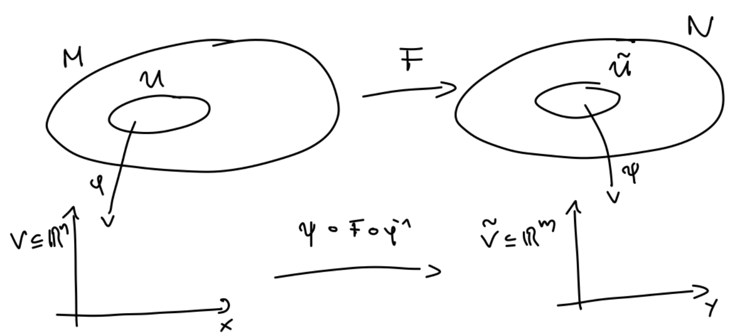
We define the **Ricci tensor** $R_{\kappa\nu} = R^\sigma{}_{\kappa\sigma\nu}$, the **scalar curvature** $R = g^{\kappa\nu}R_{\kappa\nu}$, and the **Einstein tensor** $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. The **Einstein equations** in geometrised units ($G = c = 1$) are

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \ ,$$

where $T_{\mu\nu}$ is the stress-energy tensor of matter.

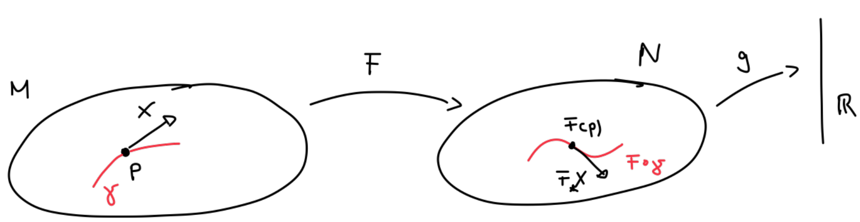
1.2 Smooth maps between manifolds

Let M, N be smooth manifolds. A **map** $F : M \to N$ **is smooth** iff it is smooth in coordinates, i.e., if $\varphi : M \supseteq U \to V \subseteq \mathbb{R}^n$ is a chart for M and $\psi : N \supseteq \tilde{U} \to \tilde{V} \subseteq \mathbb{R}^m$ a chart for N with $F(U) \subseteq \tilde{U}$, then $\psi \circ F \circ \varphi^{-1} : V \to \tilde{V}$ is smooth.



Let $X \in T_pM$. Define the **pushforward** $F_*X \in T_{F(p)}N$ **of** X **via** F by

$$(F_*X)(g) := X(\underbrace{g \circ F}_{\in C^\infty(M)}) \ , \qquad \text{for } g \in C^\infty(N) \ .$$



Thus a smooth map $F : M \to N$ induces a map $F_* : T_pM \to T_{F(p)}N$ for all $p \in M$.

Exercise: Check that F_*X is indeed a derivation, i.e., that $F_*X \in T_{F(p)}N$, and also that $F_* : T_pM \to T_{F(p)}N$ is a linear map.

We compute the coordinate expression. Let x^μ be local coordinates around $p \in M$ and y^ν be local coordinates around $F(p) \in N$. Then

$$(F_* \frac{\partial}{\partial x^\mu})^\nu|_{F(p)} = (F_* \frac{\partial}{\partial x^\mu})|_{F(p)}(y^\nu) = \frac{\partial}{\partial x^\mu}(y^\nu \circ F)|_p = \frac{\partial F^\nu}{\partial x^\mu}|_p \; .$$

Thus for $X = X^\mu \frac{\partial}{\partial x^\mu}$ we obtain by linearity

$$F_*X = X^\mu \frac{\partial F^\nu}{\partial x^\mu} \cdot \frac{\partial}{\partial y^\nu} \; . \tag{1.8}$$

F_* is also often denoted by DF (or df in case f is scalar valued), the **derivative** of F. It generalises the derivative of a map $F : \mathbb{R}^n \to \mathbb{R}^m$.

Example 1.9. *Let $\gamma : \mathbb{R} \to M$ be a smooth curve. Then $\frac{\partial}{\partial s}$ is a tangent vector in $\mathbb{R}$ and we have $\gamma_*(\frac{\partial}{\partial s}) = \frac{\partial \gamma^\mu}{\partial s}\frac{\partial}{\partial x^\mu}$, which is the tangent vector of the curve γ in M.*

Consider a covector $\alpha \in T^*_{F(p)}N$. Then define $F^*\alpha \in T^*_pM$, the **pullback of** α **via** F, by

$$(F^*\alpha)|_p(X) := \alpha|_{F(p)}(F_*X) \qquad \text{for } X \in T_pM \; .$$

We compute in local coordinates

$$(F^*\alpha)_\mu = (F^*\alpha)(\frac{\partial}{\partial x^\mu}) = \alpha(F_* \frac{\partial}{\partial x^\mu}) = \alpha(\frac{\partial F^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}) = \frac{\partial F^\nu}{\partial x^\mu}\alpha_\nu \; . \tag{1.10}$$

1.3 Diffeomorphisms & Einstein's hole argument

A smooth map $F : M \to N$ is a **diffeomorphism** iff F is bijective and $F^{-1} : N \to M$ is also smooth. (Exercise: Show that $\dim M = \dim N$.)

In general, if $F : M \to N$ is a smooth map, then we can only push forward vectors and pull back covectors. If F is a diffeomorphism, however, we can pull back a vector $Y \in T_{F(p)}N$ by pushing it forward with F^{-1}. Similarly, we can push forward covectors by pulling them back with F^{-1}.

Let now (M, g) be a Lorentzian manifold and let $F : M \to M$ be a diffeomorphism. We can define another Lorentzian metric F^*g on M by $(F^*g)|_p(X, Y) = g|_{F(p)}(F_*X, F_*Y)$, where $X, Y \in T_pM$. In coordinates we have

$$(F^*g)_{\mu\nu} = g_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^\mu}\frac{\partial F^\beta}{\partial x^\nu} \; . \tag{1.11}$$

Note that since $\frac{\partial F^\alpha}{\partial x^\mu}$ is an invertible matrix, it is clear that F^*g is indeed a Lorentzian metric. Also note that (1.11) looks formally like a coordinate transformation, cf. (1.3). Indeed, there are two different, but closely related viewpoints on diffeomorphisms:

i) **Active viewpoint:** F^{-1} maps point $F(p)$ to p, F^* maps tensors from $F(p)$ to tensors at p. In particular, starting with metric g we obtain new metric F^*g on M.

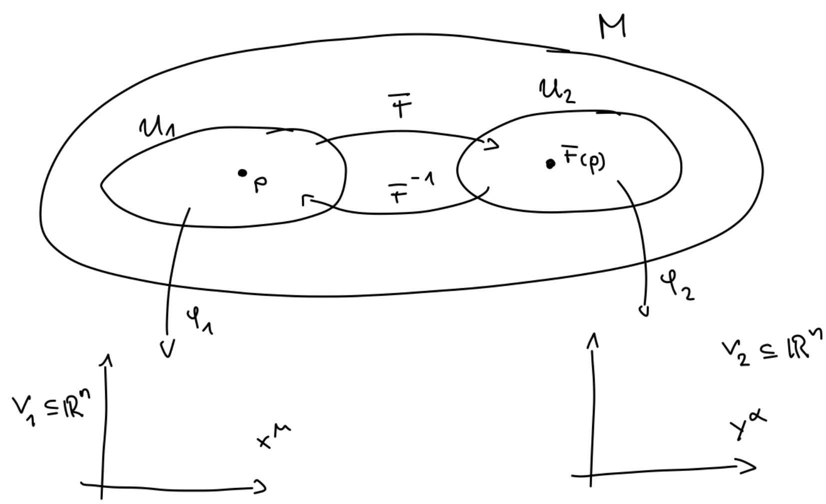
   This is the viewpoint we took above and which we will also usually take.

ii) **Passive viewpoint:** Diffeomorphism induces a change of coordinates. If x^μ are coordinates on $U_1 \subseteq M$, y^α are coordinates on $U_2 \subseteq M$ (see below diagram), and $F(U_1) = U_2$, then we can introduce a new coordinate chart $\psi = \varphi_1 \circ F^{-1} : U_2 \to V_1$ on U_2. The transition function is

$$\varphi_2 \circ \psi^{-1} = \varphi_2 \circ F \circ \varphi_1^{-1} : V_1 \to V_2 \ ,$$

which is a smooth diffeomorphism. Moreover, we have $\frac{\partial(\varphi_2 \circ F \circ \varphi_1^{-1})^\alpha}{\partial x^\mu} = \frac{\partial F^\alpha}{\partial x^\mu}$. Thus, in this new coordinate system ψ the components of g are $g_{\mu\nu} = g_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^\mu} \frac{\partial F^\beta}{\partial x^\nu}$, which is the same as the right hand side of (1.11)!

We see that although the two viewpoints are philosophically very different, computationally they are equivalent.



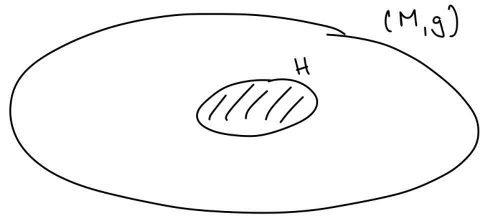
Assume now that we have a solution (M, g) to the vacuum Einstein equations $R_{\mu\nu}(g) = 0$. Let $F : M \to M$ be a diffeomorphism. Consider the new metric F^*g on M. By the above the coordinate components of F^*g are the same as those of g in a different chart. We note that since the Ricci curvature $R_{\mu\nu}(g)$ is a tensor, it does not depend on in which coordinate system one computes it. We thus infer that we also have $R_{\mu\nu}(F^*g) = 0$.

Theorem 1.12. *Let (M, g) be a solution of the vacuum Einstein equations $Ric(g) = 0$ and $F : M \to M$ a diffeomorphism. Then (M, F^*g) is also a solution of the vacuum Einstein equations.*

Exercise: Show more generally that if (M, g, T) satisfies the Einstein equations $G = 8\pi T$, and if $F : M \to M$ is a diffeomorphism, then (M, F^*g, F^*T) is also a solution.

This shows that Einstein's equations are **diffeomorphism invariant**. This has the following implication:

Einstein's hole argument: ***If*** one gives reality to spacetime points $p \in M$, then the following problem occurs: Let (M, g) be a solution of the vacuum Einstein equations, and let $H \subseteq M$ be a compact set (the "hole").



Choose a diffemorphism $F : M \to M$ such that $F|_{M\setminus H} = \mathrm{id}|_{M\setminus H}$, but which scrambles the points inside H. Then we get a new solution F^*g, which agrees with g on $M \setminus H$, but is different in H.

It follows that the physical configuration of the gravitational field in $M \setminus H$ does *not* determine the gravitational field inside H. Thus, if one gives reality to spacetime points $p \in M$, Einstein's equations are not deterministic, so that their content is truly vacuous.

Einstein's resolution: **Spacetime points p in the manifold M do not have physical reality that is independent of the metric g.** Only in conjunction with the metric do spacetime points acquire physical reality.

$\implies$ (M, g) **and** (M, F^*g) **are physically the same. The group of diffeomorphisms forms the gauge group in general relativity.**

1.4 One-parameter groups of diffeomorphisms

A **one-parameter group of diffeomorphisms** on a smooth manifold M is a smooth map

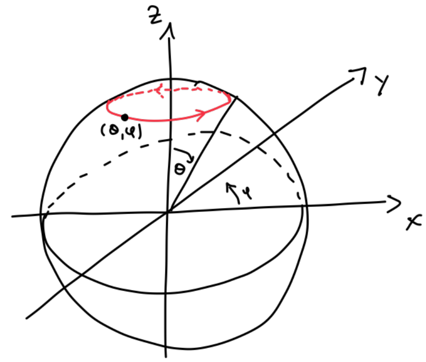
$$\begin{aligned} F :&\mathbb{R} \times M \to M \\ &(t, x) \mapsto F_t(x) \end{aligned}$$

such that

1. $\forall t \in \mathbb{R} \quad F_t : M \to M$ is a diffeomorphism
2. $F_0 = \mathrm{id}_M$
3. $F_s \circ F_t = F_{s+t} \qquad \forall s, t \in \mathbb{R} \qquad$ (group action)

A one-parameter group of diffeomorphisms is also called a **global flow**.

Example 1.13. *Consider $\mathbb{S}^2 \subseteq \mathbb{R}^3$ with standard coordinates (θ, φ), i.e., $x = \cos\varphi \sin\theta$, $y = \sin\varphi\sin\theta$, $z = \cos\theta$.*



Then $F : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{S}^2$, $F_t(\theta, \varphi) = (\theta, \varphi + t)$ is a one-parameter group of diffeomorphisms, where for fixed $t \in \mathbb{R}$, $F_t : \mathbb{S}^2 \to \mathbb{S}^2$ is the rotation around the z-axis by an angle $\Delta\varphi = t$.

Also note that for fixed $(\theta, \varphi) \in \mathbb{S}^2$, $t \mapsto F_t(\theta, \varphi) = (\theta, \varphi + t)$ is a curve with tangent $\frac{d}{dt} F_t(\theta, \varphi) = \frac{\partial}{\partial \varphi}$.

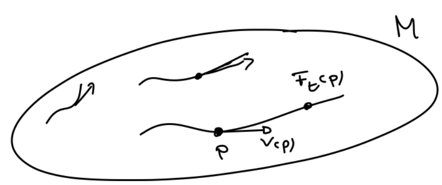
Let V be a smooth vector field on M. An **integral curve** of V is a curve $\gamma : \mathbb{R} \to M$ such that $\dot{\gamma}(s) = V(\gamma(s))$, i.e., such that V is tangent to the curve.

Example 1.14. *a) In the above example if $V = \frac{\partial}{\partial \varphi}$, then $t \mapsto (\theta, \varphi + t)$ are integral curves of V.*

b) $M = (-1, 1)^2$, $V = \frac{\partial}{\partial x}$. Integral curve $s \mapsto (x(s), y(s))$ has to satisfy $\dot{x}(s) = 1$, $\dot{y}(s) = 0$. Thus $s \mapsto (x + s, y)$ are the integral curves.[7]

Given a one-parameter group of diffeomorphisms $F : \mathbb{R} \times M \to M$, we obtain smooth curves $\gamma(t) = F_t(p)$ for each $p \in M$. Define a smooth vector field V on M by

$$V(p) := \frac{d}{dt}|_{t=0} F_t(p) = \dot{\gamma}(0) .$$



The vector field V is called the **infinitesimal generator of** F for the following reason: $t \mapsto F_t(p)$ are integral curves of V.

Proof.

$$\frac{d}{dt}\Big|_{t=t_0} F_t(p) = \frac{d}{dt}\Big|_{t=t_0} F_{t-t_0}\big(F_{t_0}(p)\big) = \frac{d}{ds}\Big|_{s=0} F_s\big(F_{t_0}(p)\big) = V\big(F_{t_0}(p)\big) .$$

□

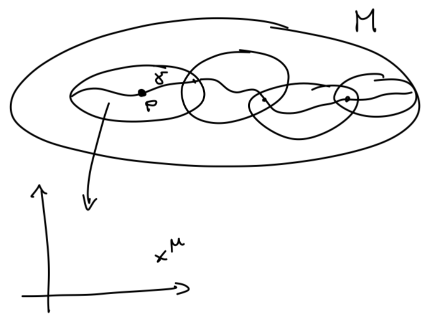
Theorem 1.15 (Relation between one-parameter groups of diffeomorphisms and vector fields)**.**

[7] Note that they are not defined for all $t \in \mathbb{R}$. For the sake of simplicity of the following presentation we will always assume that integral curves *are* defined on all of $\mathbb{R}$. The general case is, however, not much more complicated.

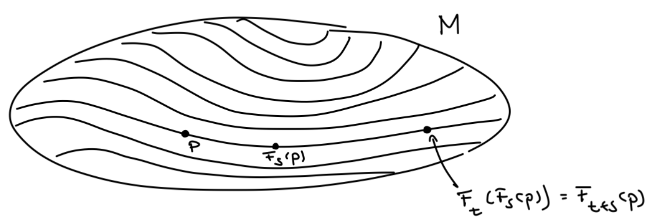
i) Given a one-parameter group of diffeomorphisms $F : \mathbb{R} \times M \to M$, we associate the smooth vector field $V(x) := \frac{d}{dt}\Big|_{t=0} F_t(x)$ and its integral curves are given by $t \mapsto F_t(x)$.

ii) Given a smooth vector field $V \in \mathfrak{X}^\infty(M)$ (whose integral curves are defined on all of $\mathbb{R}$), there exists a unique one-parameter group of diffeomorphisms $F : \mathbb{R} \times M \to M$ with $V(x) := \frac{d}{dt}\Big|_{t=0} F_t(x)$.

Sketch of proof: We have already proven i). For ii) choose local coordinates x^μ around $p \in M$. We then consider the ODE $\dot\gamma^\mu(s) = V^\mu(\gamma(s))$ with initial condition $\gamma(0) = p$. By the fundamental theorem on ODEs, there exists a unique solution in this chart which depends smoothly on the initial data. Now cover M with charts and repeat.

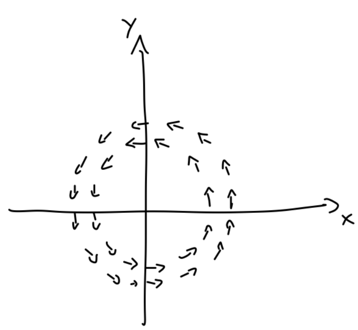


In this way we obtain a **foliation** of M by integral curves of V, i.e., a family of integral curves of V such that through every $p \in M$ there passes exactly one such integral curve.



For $t \in \mathbb{R}$ define $F_t : M \to M$ by flowing points for time t along the integral curves. Clearly $F_{t+s}(p) = F_t(F_s(p))$. Inverse of F_t is given by F_{-t}. By smooth dependence of integral curves on initial data the map $F : \mathbb{R} \times M \to M$ is smooth and thus is the wanted one-parameter group of diffeomorphisms. □

Example 1.16. $M = \mathbb{R}^2 \setminus \{0\}$, $V = x\partial_y - y\partial_x$.



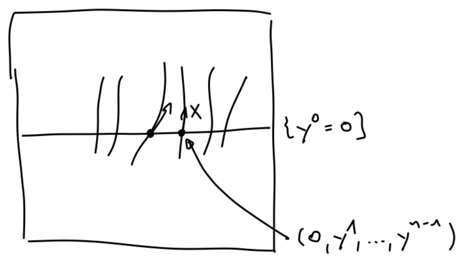
Use polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$. Then $\partial_\varphi = x\partial_y - y\partial_x = V$.

The integral curves solve $\dot\varphi(s) = 1$ and $\dot r(s) = 0$, which gives $s \mapsto (\varphi + s, r)$. Thus the associated one-parameter group of diffeomorphisms $F : \mathbb{R} \times M \to M$ is given in polar coordinates by $(s, (\varphi, r)) \mapsto (\varphi + s, r)$. V is the infinitesimal generator of rotations around the origin.

Proposition 1.17 (Coordinates adapted to a non-vanishing vector field)**.**

Let M be a smooth manifold, $X \in \mathfrak{X}^\infty(M)$, and $X(p) \neq 0$. Then there exists smooth coordinates x^μ on a neighbourhood of p such that $X = \frac{\partial}{\partial x^0}$.

Proof. Take a coordinate chart $\varphi : U \to \mathbb{R}^n$ such that the coordinates y^ν are centred at p (i.e. $y^\nu(p) = 0$), and without loss of generality such that the hypersurface $\{y^0 = 0\}$ is not tangent to X at p (thus $X^0(p) \neq 0$).



Define a map

$$(x^0, x^1, \dots, x^{n-1}) \overset{\Psi}{\mapsto} F_{x^0}(0, x^1, \dots, x^{n-1}),$$

i.e., we flow the point $(0, x^1, \dots, x^{n-1})$ on $\{y^0 = 0\}$ for time x^0 along the integral curve of X. Then by Theorem 1.15 $\frac{\partial}{\partial x^0} = X$. It remains to show that ψ is a local diffeomorphism, then we can choose $\Psi^{-1} \circ \varphi$ as a new coordinate chart in a neighbourhood of p. We compute

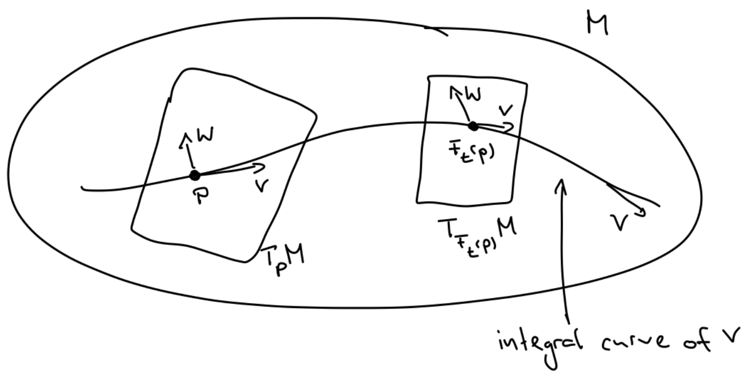
$$D\Psi(0) = \begin{pmatrix} X^0(p) & 0 & \cdots & 0 \\ X^1(p) & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ X^n(p) & 0 & \cdots & 1 \end{pmatrix}$$

and since $X^0(p) \neq 0$ this matrix is invertible. By the inverse function theorem there exists a small neighbourhood of 0 on which Ψ^{-1} exists and is smooth. □

1.5 Lie derivative

Let V be a smooth vector field on M, $F : \mathbb{R} \times M \to M$ the associated one-parameter group of diffeomorphisms, and W another smooth vector field on M. We want to take the derivative of W along V in a way which does not resort to a metric but only depends on the smooth structure of the manifold.

Problem: $W(p)$ and $W(F_t(p))$ lie in different tangent spaces which we cannot compare.



<u>Solution:</u> Recall that F_t is diffeomorphism with inverse $(F_t)^{-1} = F_{-t}$. We have $F_{-t}(F_t(p)) = p$, thus $(F_{-t})_* : T_{F_t(p)}M \to T_pM$ gives identification of tangent spaces depending on the flow of V. Define the **Lie derivative $\mathcal{L}_V W$ of W with respect to V** by

$$\begin{aligned} \mathcal{L}_V W(p) :&= \frac{d}{dt}\Big|_{t=0} (F_{-t})_*(W_{F_t(p)}) \qquad\qquad \Big(\overset{Def}{=} \frac{d}{dt}\Big|_{t=0} (F_t)^*(W_{F_t(p)}) \Big) \\ &= \lim_{t\to 0} \frac{(F_{-t})_* W_{F_t(p)} - W_p}{t} \in T_pM \ . \end{aligned} \tag{1.18}$$

Recall from (1.8) that in local coordinates we have

$$(F_{-t})_* W_{F_t(p)} = \underbrace{\frac{\partial F^\mu_{-t}}{\partial x^\nu}(F_t(p)) \cdot W^\nu(F_t(p))}_{\text{smooth expression in } t,x} \cdot \frac{\partial}{\partial x^\mu} \ .$$

Thus we see that

$$(\mathcal{L}_V W)^\mu = \frac{d}{dt}\Big|_{t=0} \Big(\frac{\partial F^\mu_{-t}}{\partial x^\nu}(F_t(p)) \cdot W^\nu(F_t(p)) \Big) \tag{1.19}$$

depends smoothly on x and thus $\mathcal{L}_V W$ is a smooth vector field.

Let now $R \subseteq M$ be the open set of all $p \in M$ such that $V(p) \neq 0$. By Proposition 1.17 we can find local coordinates x^μ such that $V = \frac{\partial}{\partial x^0}$. Then $F_t(x^0, \ldots, x^{n-1}) = (x^0 + t, x^1, \ldots, x^{n-1})$ and thus $\frac{\partial F^\mu_{-t}}{\partial x^\nu} = \delta^\mu_\nu$. Equation (1.19) simplifies to

$$(\mathcal{L}_V W)^\mu(x^0, \ldots, x^{n-1}) = \frac{d}{dt}\Big|_{t=0} \Big(W^\mu(x^0 + t, x^1, \ldots, x^{n-1}) = (\partial_{x^0} W^\mu)(x^0, \ldots, x^{n-1}) \ .$$

Moreover, note that in these coordinates we have $[V, W]^\mu = [\partial_0, W]^\mu = (\partial_0 W^\mu)$, see (1.4), and thus we have shown that $\mathcal{L}_V W = [V, W]$ on R. By continuity this also holds on the closure of R in M, and on $M \backslash R$, where $V = 0$, it is easy to see that both expressions vanish. <u>We thus conclude that $\mathcal{L}_V W = [V, W]$.</u> This gives us another interpretation of the Lie bracket. The Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ directly gives

$$\mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z = \mathcal{L}_{[X,Y]} Z \ .$$

The definition (1.18) of the Lie derivative can easily be extended to general (k, l)-tensor fields T by

$$(\mathcal{L}_V T)(p) := \frac{d}{dt}\Big|_{t=0} (F_t^* T)(p) \ ,$$

where the pullback of T is defined as follows: for $\alpha_1, \dots, \alpha_k \in T_p^*M$, $X_1, \dots, X_l \in T_pM$ we set

$$\begin{aligned}(F_t^*T)(p)\big(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l\big) &:= T(F_t(p))\big((F_t)_*\alpha_1, \dots, (F_t)_*\alpha_k, (F_t)_*X_1, \dots, (F_t)_*X_l\big) \\ &= T(F_t(p))\big((F_{-t})^*\alpha_1, \dots, (F_{-t})^*\alpha_k, (F_t)_*X_1, \dots, (F_t)_*X_l\big)\end{aligned}$$

If $f \in C^\infty(M)$, then we define $\mathcal{L}_V f(p) := \frac{d}{dt}\Big|_{t=0}(F_t^* f)(p) = \frac{d}{dt}\Big|_{t=0}(f \circ F_t)(p) = V|_p(f)$.

We summarise the properties of the Lie derivative in the following

Proposition 1.20 (Properties of the Lie derivative)**.**

i) $\mathcal{L}_V f = Vf$ *for* $f \in C^\infty(M)$

ii) $\mathcal{L}_V(aT + bS) = a\mathcal{L}_V T + b\mathcal{L}_V S$ *for* $a, b \in \mathbb{R}$ *and* T, S *tensor fields* *(linearity over* $\mathbb{R}$*)*

iii) $\mathcal{L}_V(T \otimes S) = (\mathcal{L}_V T) \otimes S + T \otimes (\mathcal{L}_V S)$ *(Leibniz rule)*

iv) $\mathcal{L}_V(\mathrm{tr}T) = \mathrm{tr}(\mathcal{L}_V T)$ *(commutes with contractions)*

v) $\mathcal{L}_V W = [V, W]$ *for* W *a vector field*

vi) $\mathcal{L}_V(df) = d(\mathcal{L}_V f) = d(Vf)$ *for* $f \in C^\infty(M)$

vii) $\mathcal{L}_V\mathcal{L}_W T - \mathcal{L}_W\mathcal{L}_V T = \mathcal{L}_{[V,W]}T$ *for* T *a tensor field*

viii) For a (k,l)*-tensor field* T *in local coordinates we have*

$$\begin{aligned}(\mathcal{L}_V T)^{a_1\dots a_k}{}_{b_1\dots b_l} = V^c\partial_c T^{a_1\dots a_k}{}_{b_1\dots b_l} - T^{ca_2\dots a_k}{}_{b_1\dots b_l}\cdot\partial_c V^{a_1} - \dots - T^{a_1\dots a_{k-1}c}{}_{b_1\dots b_l}\cdot\partial_c V^{a_k} \\ + T^{a_1\dots a_k}{}_{cb_2\dots b_l}\cdot\partial_{b_1}V^c + \dots + T^{a_1\dots a_k}{}_{b_1\dots b_{l-1}c}\cdot\partial_{b_l}V^c\ .\end{aligned}$$

ix) In adapted coordinates such that $V = \partial_0$ *we have*

$$(\mathcal{L}_V T)^{a_1\dots a_k}{}_{b_1\dots b_l} = \partial_0\big(T^{a_1\dots a_k}{}_{b_1\dots b_l}\big)\ .$$

Proof. i) is by definition, ii) is an easy exercise. For iii) we compute

$$\begin{aligned}\mathcal{L}_V(T\otimes S)|_p &= \lim_{t\to 0}\frac{F_t^*(T\otimes S)|_p - T\otimes S|_p}{t} \\ &= \lim_{t\to 0}\frac{(F_t^*T)\otimes(F_t^*S)|_p - T\otimes S|_p}{t} \\ &= \lim_{t\to 0}\frac{F_t^*T\otimes F_t^*S|_p - F_t^*T\otimes S|_p}{t} + \lim_{t\to 0}\frac{(F_t^*T)\otimes S|_p - T\otimes S|_p}{t} \\ &= T\otimes\mathcal{L}_V S|_p + \mathcal{L}_V T\otimes S|_p\ .\end{aligned}$$

For iv) consider first $\alpha \in \Omega^1(M)$ and $X \in \mathfrak{X}^\infty(M)$. Then

$$\begin{aligned}
\mathrm{tr}(\mathcal{L}_V(\alpha \otimes X))|_p &= \mathrm{tr}\Big(\frac{d}{dt}\Big|_{t=0} F_t^*(\alpha \otimes X)\Big)\Big|_p \\
&= \mathrm{tr}\Big(\frac{d}{dt}\Big|_{t=0} (F_t^*\alpha)|_p \otimes (F_{-t})_* X|_p\Big) \\
&= \delta^\mu_\kappa \frac{d}{dt}|_{t=0}\Big(\frac{\partial F_t^\nu}{\partial x^\mu}(p)\cdot \alpha_\nu(F_t(p)) \cdot \frac{\partial F_{-t}^\kappa}{\partial x^\sigma}(F_t(p)) \cdot X^\sigma(F_t(p))\Big) \\
&= \frac{d}{dt}\Big|_{t=0}\big(\alpha_\nu(F_t(p)) \cdot X^\nu(F_t(p))\big) \\
&= \frac{d}{dt}\Big|_{t=0} F_t^*\big(\alpha(X)\big)|_p \\
&= \mathcal{L}_V(\mathrm{tr}(\alpha \otimes X))|_p \ .
\end{aligned}$$

Here we used the coordinate expression for the pushforward (1.8) and pullback (1.10) in the third line – and we used $F_{-t} \circ F_t = \mathrm{id}$ and thus $\frac{\partial F^\kappa_{-t}}{\partial x^\sigma}(F_t(p)) \cdot \frac{\partial F^\sigma_t}{\partial x^\mu}(p) = \delta^\kappa_\mu$ in the fourth line. The general case follows from this together with iii).

We have already shown v) and for vi) we compute with $X \in \mathfrak{X}^\infty(M)$

$$\begin{aligned}
\big(\mathcal{L}_V(df)\big)(X) &= \mathrm{tr}(\mathcal{L}_V df \otimes X) \\
&= \mathcal{L}_V\big(df(X)\big) - df(\mathcal{L}_V X) \qquad\qquad \text{using iii), iv)} \\
&= V(X(f)) - [V, X]f \qquad\qquad \text{using v)} \\
&= X(V(f)) \\
&= d(Vf)(X) \ .
\end{aligned}$$

Note that this in particular implies for the differentials of local coordinates

$$\mathcal{L}_V dx^\mu = dV^\mu = \partial_\kappa V^\mu dx^\kappa \ . \tag{1.21}$$

vii) is an exercise, and for viii) we compute

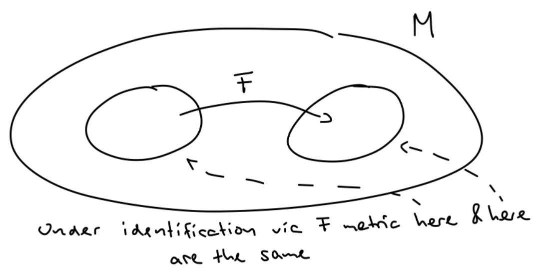
$$\begin{aligned}
(\mathcal{L}_V T)^{a_1\ldots a_k}{}_{b_1\ldots b_l} &= (\mathcal{L}_V T)(dx^{a_1}, \ldots, dx^{a_k}, \partial_{b_1}, \ldots, \partial_{b_l}) \\
&= \mathrm{tr}(\mathcal{L}_V T \otimes dx^{a_1} \otimes \ldots \otimes dx^{a_k} \otimes \partial_{b_1} \otimes \ldots \otimes \partial_{b_l}) \\
&= \mathcal{L}_V(T^{a_1\ldots a_k}{}_{b_1\ldots b_l}) - T(\mathcal{L}_V dx^{a_1}, \ldots, dx^{a_k}, \partial_{b_1}, \ldots \partial_{b_l}) - \ldots - T(dx^{a_1}, \ldots, \mathcal{L}_V dx^{a_k}, \partial_{b_1}, \ldots, \partial_{b_l}) \\
&\qquad - T(dx^{a_1}, \ldots, dx^{a_k}, \mathcal{L}_V \partial_{b_1}, \ldots, \partial_{b_l}) - \ldots - T(dx^{a_1}, \ldots, dx^{a_k}, \partial_{b_1}, \ldots, \mathcal{L}_V \partial_{b_l}) \\
&= V^c \partial_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} - T^{c a_2 \ldots a_k}{}_{b_1\ldots b_l} \cdot \partial_c V^{a_1} - \ldots - T^{a_1\ldots a_{k-1} c}{}_{b_1\ldots b_l} \cdot \partial_c V^{a_k} \\
&\qquad + T^{a_1\ldots a_k}{}_{c b_2\ldots b_l} \cdot \partial_{b_1} V^c + \ldots + T^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1} c} \cdot \partial_{b_l} V^c \ ,
\end{aligned}$$

where we used iii), iv) in the third line and i), (1.21), and (1.4) in the fourth line.

Finally ix) follows directly from viii). □

1.6 Killing vector fields & isometries

Let (M, g) be a Lorentzian (Riemannian) manifold. A diffeomorphism $F : M \to M$ is an **isometry** iff $F^* g = g$.



Now let $F : \mathbb{R} \times M \to M$ be a one-parameter group of isometries, i.e., $F_t : M \to M$ is an isometry for every $t \in \mathbb{R}$. Let $V \in \mathfrak{X}^\infty(M)$ denote the infinitesimal generator. We then have

$$\mathcal{L}_V g = \frac{d}{dt}|_{t=0} F_t^* g = \frac{d}{dt}|_{t=0} g = 0 \ .$$

A vector field V on (M, g) satisfying $\mathcal{L}_V g = 0$ is called a **Killing vector field**.

Vice versa, let V be a Killing vector field (assuming also that its integral curves are defined on all of $\mathbb{R}$) and let $F : \mathbb{R} \times M \to M$ be the associated one-parameter group of diffeomorphisms. Thus we have $\frac{d}{dt}|_{t=0} F_t^* g = \mathcal{L}_V g = 0$. Fix a point $p \in M$ and consider the curve $t \overset{\psi}{\mapsto} F_t^* g|_p$. We have

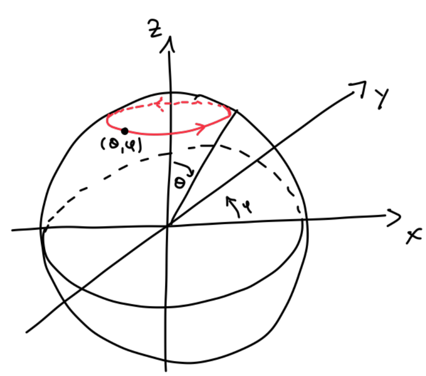
$$\begin{aligned}\psi'(t_0) &= \frac{d}{dt}|_{t=t_0} F_t^* g|_p = \frac{d}{dt}|_{t=t_0} F_{t_0}^* F_{t-t_0}^* g|_p = F_{t_0}^* \Big(\frac{d}{dt}|_{t=0} F_t^* g|_{F_{t_0}(p)} \Big) \\ &= F_{t_0}^* \Big(\underbrace{\mathcal{L}_V g|_{F_{t_0}(p)}}_{=0} \Big) = 0 \ .\end{aligned}$$

It follows that ψ is a constant curve and since $\psi(0) = g|_p$ it follows that $\psi(t) = g|_p$ for all $t \in \mathbb{R}$. Thus we have $F_t^* g = g$ for all $t \in \mathbb{R}$. Hence, we have shown the following

Proposition 1.22. *Let (M, g) be a Lorentzian (Riemannian) manifold and $F : \mathbb{R} \times M \to M$ a one-parameter group of diffeomorphisms. Then F is a one-parameter group of isometries if, and only if, the infinitesimal generator V is a Killing vector field.*

Let us also remark that by Proposition 1.20 ix) a vector field V is a Killing vector field if, and only if, in adapted coordinates x^μ such that $V = \partial_0$ we have $\partial_0 g_{\mu\nu} = 0$, i.e., iff the metric components in these adapted coordinates are independent of x^0.

Example 1.23. *Consider example 1.13, the sphere $\mathbb{S}^2$ with local coordinates (θ, φ) and metric $g = d\theta^2 + \sin^2\theta\, d\varphi^2$.*



Let $V = \partial_\varphi$ be the infinitesimal generator of the rotations $F : \mathbb{R} \times M \to M$ around the z-axis: $F(t, (\theta, \varphi)) = (\theta, \varphi + t)$. Since the components of g in the coordinates (θ, φ) are independent of φ, it follows that $V = \partial_\varphi$ is a Killing vector field and F is a one-parameter group of isometries.

Proposition 1.24 (Properties of Killing vector fields)**.**

Let (M, g) be a Lorentzian (Riemannian) manifold.

i) Killing vector fields form a Lie algebra: if V, K are Killing vector fields, then so is $[V, K]$.

ii) V is a Killing vector field if, and only if, $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$.

iii) If V is a Killing vector field, then $\nabla_a \nabla_b V_c = -R_{adbc} V^d$.

iv) Let V be a Killing vector field and $\gamma : I \to M$ an affinely parametrised geodesic ($\nabla_{\dot\gamma} \dot\gamma = 0$). Then $g(V, \dot\gamma)$ is constant along γ.

Note that iv) shows that Killing vector fields give rise to first integrals/conserved quantities for geodesics.

Proof. i) follows from $\mathcal{L}_{[V,K]} g = \mathcal{L}_V \mathcal{L}_K g - \mathcal{L}_K \mathcal{L}_V g = 0$, where we used Proposition 1.20 vii).

To see ii) we compute

$$\begin{aligned}
(\mathcal{L}_V g)(\partial_\mu, \partial_\nu) &= V(g_{\mu\nu}) - g([V, \partial_\mu], \partial_\nu) - g(\partial_\mu, [V, \partial_\nu]) && \text{using Proposition 1.20 iii), iv), v)} \\
&= V(g_{\mu\nu}) - g(\nabla_V \partial_\mu, \partial_\nu) - g(\partial_\mu, \nabla_V \partial_\nu) && \text{using } \nabla \text{ is symmetric} \\
&\qquad + g(\nabla_{\partial_\mu} V, \partial_\nu) + g(\partial_\mu, \nabla_{\partial_\nu} V) \\
&= (\underbrace{\nabla_V g}_{=0})(\partial_\mu, \partial_\nu) + g(\nabla_{\partial_\mu} V, \partial_\nu) + g(\partial_\mu, \nabla_{\partial_\nu} V) && \text{using Leibnizrule for } \nabla \\
&= \nabla_\mu V_\nu + \nabla_\nu V_\mu && \text{using } \nabla \text{ is metric .}
\end{aligned}$$

For iii) recall from first problem sheet that $\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = R_{cdab} V^d = -R_{dcab} V^d$. Also recall the first Bianchi identity $R_{dcab} + R_{dabc} + R_{dbca} = 0$. Together this gives

$$\begin{aligned}
0 &= \nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c + \nabla_b \nabla_c V_a - \nabla_c \nabla_b V_a + \nabla_c \nabla_a V_b - \nabla_a \nabla_c V_b \\
&\overset{ii)}{=} \nabla_a \nabla_b V_c + \nabla_b \nabla_c V_a + \nabla_b \nabla_c V_a - \nabla_c \nabla_b V_a - \nabla_c \nabla_b V_a + \nabla_a \nabla_b V_c \\
&= 2(\nabla_a \nabla_b V_c + \nabla_b \nabla_c V_a - \nabla_c \nabla_b V_a) ,
\end{aligned}$$

and thus $\nabla_a \nabla_b V_c = -R_{adbc} V^d$.

For iv) we compute:

$$\dot\gamma(g(V, \dot\gamma)) = \nabla_\gamma(g(V, \dot\gamma)) = \underbrace{g(\nabla_{\dot\gamma} V, \dot\gamma)}_{=0} + g(V, \underbrace{\nabla_{\dot\gamma} \dot\gamma}_{=0}) ,$$

where the first term is zero by ii), which says that $\nabla_\mu V_\nu$ is antisymmetric, and the last term is zero by virtue of the geodesic equation. □

Example 1.25. *Consider Minkowski spacetime $M = \mathbb{R}^4$, $g = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$. Then $V = \partial_t$ is a Killing vector field, since the metric components are independent of t. It generates the one-parameter group of isometries $F : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4$, $F_{t_0}(t, \underline{x}) = (t + t_0, \underline{x})$, the time translations.*

Let $U \in \mathbb{R}^4$ be a unite timelike vector. Then $s \overset{\gamma}{\mapsto} s \cdot U \in \mathbb{R}^4$ is an affinely parametrised timelike geodesic with $\dot{\gamma} = U$. Then $-g(\partial_t, U) = U^0$ is the conserved energy of the particle.

1.7 Submanifolds

Let M be an n-dimensional smooth manifold. A subset $S \subseteq M$ is called a **k-dimensional embedded submanifold of** M ($k \leq n$), iff for all $p \in S$ there exists a coordinate chart $\varphi : M \supseteq U \to V \subseteq \mathbb{R}^n$ with $p \in U$ such that

$$S \cap U = \{(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) \mid x^{k+1} = \ldots = x^n = 0\} .$$

A **hypersurface** $S \subseteq M$ is an $(n-1)$-dimensional embedded submanifold.

Proposition 1.26. *Let $f : M \to \mathbb{R}$ be a smooth function such that $df|_p \neq 0$ for all $p \in f^{-1}(0)$. Then $f^{-1}(0) =: S \subseteq M$ is a hypersurface in M.*

Proof. Let $p \in S$ and $(x^1, \ldots, x^n)$ a coordinate system centred at p and let without loss of generality $\frac{\partial f}{\partial x^n}(0) \neq 0$. Consider the map

$$(x^1, \ldots, x^n) \overset{\psi}{\mapsto} (x^1, \ldots, x^{n-1}, f(x_1, \ldots, x_n)) = (y^1, \ldots, y^n) .$$

Then

$$D\psi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & 1 & 0 \\ \frac{\partial f}{\partial x^1} & \cdots & \cdots & \frac{\partial f}{\partial x^n} \end{pmatrix}$$

is invertible at $x = 0$ and thus ψ is a diffeomorphism in a neighbourhood of 0 and thus gives rise to a new coordinate system $(y^1, \ldots, y^n)$ in which S is locally the level set $y^n = 0$. □

Example 1.27.
1. *Let $M = \mathbb{R}^n$, then $S = \{x^{k+1} = \ldots = x^n = 0\} \simeq \mathbb{R}^k$ is a k-dimensional submanifold. Indeed, all submanifolds are locally modelled on this one.*

2. *Let $M = \mathbb{R}^n$ and let $f(x) = x_1^2 + \ldots + x_n^2 - 1$. Then $f^{-1}(0) = \mathbb{S}^{n-1}$, and since $df(x) = 2(x_1 dx_1 + \ldots + x_n dx_n) \neq 0$ for $x \neq 0$, it follows from Proposition 1.26 that $\mathbb{S}^{n-1}$ is a hypersurface.*

3. *Let M be an n-dimensional manifold, V a vector field with $V(p) \neq 0$ for all $p \in M$. Let γ_I be the family of integral curves of V. Then each γ is locally[8] a 1-dimensional embedded submanifold, since one can choose locally adapted coordinates $\{x^1, \ldots, x^n\}$ such that $V = \partial_1$ and thus locally $\gamma = \{x^2 = \ldots = x^n = 0\}$.*

[8]In general each integral curve is only an *immersed* submanifold – a notion we do not discuss in this course. By 'locally' we mean here that if $\gamma : I \to M$, then for each $s_0 \in I$ there is $\varepsilon > 0$ such that $\gamma((s_0 - \varepsilon, s_0 + \varepsilon))$ is an embedded submanifold.

One way to think about the last example is that we prescribed one dimensional tangent spaces at each point of M by prescribing a non-vanishing vector field V and then we showed that one can 'integrate' them, i.e., find one dimensional submanifolds which have the prescribed tangent spaces. In the next section we generalise this to higher dimensions.

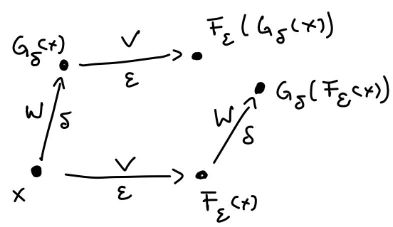
1.8 Integral manifolds

Give now two smooth vector fields V, W on M such that at every point $p \in M$ $V(p)$ and $W(p)$ are linearly independent. Then $\mathrm{span}\{V, W\}$ is called a **2-dimensional distribution** of the tangent bundle TM. It is called **integrable** if there exists locally a family of 2-dimensional submanifolds which have $\mathrm{span}\{V, W\}$ as their tangent spaces.

Assume there exists such a submanifold S. Then V, W restrict to vector fields on S and thus $[V, W]$ is also tangent to S, i.e., $[V, W] \subseteq \mathrm{span}\{V, W\}$. **This shows that** $[V, W] \subseteq \mathrm{span}\{V, W\}$ **is a necessary condition for the integrability of the distribution.** We show in the following that it is also a sufficient condition. But before we do so we look at some heuristics:

Heuristics: Geometric interpretation of $[V, W]$.

Let $V, W \in \mathfrak{X}^\infty(M)$, $F_t : \mathbb{R} \times M \to M$ the one-parameter group of diffeomorphisms generated by V and $G_t : \mathbb{R} \times M \to M$ that generated by W. Let $\{x^1, \ldots, x^n\}$ be a local coordinate chart. We compute in those coordinates $F^\mu_\varepsilon(G_\delta(x))$ and $G^\mu_\delta(F_\varepsilon(x))$ to second order in the small parameters $\varepsilon, \delta > 0$.



Taylor expanding around $\delta = 0$ and using $\frac{d}{d\delta}\Big|_{\delta=0} G^\nu_\delta(y) = W^\nu(y)$ we obtain $G^\mu_\delta(y) = y^\mu + \delta W^\mu(y) + \mathcal{O}(\delta^2)$. Using this with $y = F_\varepsilon(x)$ we get

$$\begin{aligned} G^\mu_\delta(F_\varepsilon(x)) &\simeq F^\mu_\varepsilon(x) + \delta W^\mu(F_\varepsilon(x)) \\ &\simeq x^\mu + \varepsilon V^\mu(x) + \delta W^\mu(x) + \delta\varepsilon V^\nu(x)\partial_\nu W^\mu(x) \ , \end{aligned}$$

where in the last line we have analogously Taylor expanded $F^\mu_\varepsilon(x)$ and $W^\mu(F_\varepsilon(x))$ around $\varepsilon = 0$. Analogously we obtain

$$F^\mu_\varepsilon(G_\delta(x)) \simeq x^\mu + \delta W^\mu(x) + \varepsilon V^\mu(x) + \varepsilon\delta W^\nu(x)\partial_\nu V^\mu(x) \ .$$

Subtracting we obtain

$$G^\mu_\delta(F_\varepsilon(x)) - F^\mu_\varepsilon(G_\delta(x)) \simeq \varepsilon\delta(V^\nu\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu)(x) = \varepsilon\delta[V, W]^\mu(x) \ . \tag{1.28}$$

We see that $[V, W]$ measures the lack of commutation of the associated flows F_t and G_s.[9]

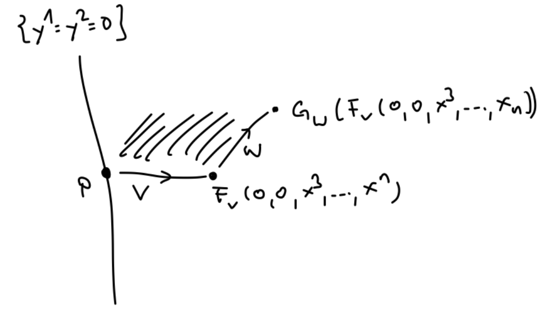
[9] One can show indeed rigorously that $F_s \circ G_t = G_t \circ F_s$ if, and only if, $[V, W] = 0$.

If $[V, W] = 0$ the idea is now to sweep out the integral manifold through a point p by flowing along the flows starting from p. The heuristics suggest that if $[V, W] = 0$ we should obtain a 2-dimensional surface.

Proposition 1.29. *Let M be an n-dimensional manifold, V, W smooth vector fields which are pointwise linearly independent and satisfy $[V, W] = 0$. Then, locally, there are coordinates $(v, w, x^3, \ldots, x^n)$ for M such that $V = \frac{\partial}{\partial v}$ and $W = \frac{\partial}{\partial w}$.*

Proof. Let $p \in M$, then we can find local coordinates $(y^1, \ldots, y^n)$ centred at p such that $\left.\frac{\partial}{\partial y^1}\right|_p = V|_p$ and $\left.\frac{\partial}{\partial y^2}\right|_p = W|_p$ (exercise). Let F denote the one-parameter group of diffeomorphisms generated by V and G that generated by W. Define

$$(v, w, x^3, \ldots, x^n) \overset{\psi}{\mapsto} G_w\big(F_v(0, 0, x^3, \ldots, x^n)\big) \ .$$



Then

$$D\psi|_0 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} ,$$

thus ψ is a local diffeomorphism around p and $(v, w, x^3, \ldots, x^n)$ form new coordinates on M. Clearly we have $W = \frac{\partial}{\partial w}$, but in general we only have $V = a(v, w, x^i)\frac{\partial}{\partial v} + b(v, w, x^i)\frac{\partial}{\partial w} + \sum_{j=3}^n c^j(v, w, x^i)\frac{\partial}{\partial x^j}$ with $a(v, 0, x^i) = 1$, $b(v, 0, x^i) = 0 = c^j(v, 0, x^i)$ for $j = 3, \ldots n$. Now we compute

$$\begin{aligned} 0 = [W, V] &= \frac{\partial}{\partial w}\Big(a\frac{\partial}{\partial v} + b\frac{\partial}{\partial w} + \sum_{j=3}^n c^j\frac{\partial}{\partial x^j}\Big) - \Big(a\frac{\partial}{\partial v} + b\frac{\partial}{\partial w} + \sum_{j=3}^n c^j\frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial w} \\ &= \partial_w a \cdot \frac{\partial}{\partial v} + \partial_w b \cdot \frac{\partial}{\partial w} + \sum_{j=3}^n \partial_w c^j \cdot \frac{\partial}{\partial x^j} \ . \end{aligned}$$

Since $\frac{\partial}{\partial v}$, $\frac{\partial}{\partial w}$, $\frac{\partial}{\partial x^j}$, $j = 3, \ldots, n$ are linearly independent we obtain $\partial_w a = \partial_w b = \partial_w c^j = 0$ for $j = 3, \ldots, n$. With the initial conditions of a, b, c^j for $w = 0$ this gives $a \equiv 1$ and $b \equiv 0 \equiv c^j$, $j = 3, \ldots, n$, from which $V = \frac{\partial}{\partial v}$ follows. □

In particular if $[V, W] = 0$, then there exist locally 2-dimensional submanifolds $S := \{x^3 = c^3, \ldots, x^n = c^n\}$ with $TS = \operatorname{span}\{V, W\}$. The following lemma reduces the general case to this one.

Lemma 1.30. *Let M be a smooth manifold and $V, W \in \mathfrak{X}^\infty(M)$ pointwise linearly independent with $[V, W] \subseteq \operatorname{span}\{V, W\}$. Then there exist $\hat{V}, \hat{W} \in \mathfrak{X}^\infty(M)$ with $\operatorname{span}\{\hat{V}, \hat{W}\} = \operatorname{span}\{V, W\}$ and $[\hat{V}, \hat{W}] = 0$.*

Proof. Let $\hat{V} = \lambda \cdot V$ and $\hat{W} = \mu \cdot W$ with $\lambda, \mu \in C^\infty(M)$. We compute with $f \in C^\infty(M)$

$$\begin{aligned}
0 \overset{!}{=} [\hat{V}, \hat{W}]f &= [\lambda V, \mu W]f \\
&= \lambda[V, \mu W]f - \mu W(\lambda) \cdot (Vf) \\
&= \lambda\mu[V, W]f + \lambda V(\mu) \cdot (Wf) - \mu W(\lambda) \cdot (Vf) \\
&= \lambda\mu a V(f) + \lambda\mu b W(f) + \lambda V(\mu) \cdot (Wf) - \mu W(\lambda) \cdot (Vf) \ ,
\end{aligned}$$

where we have used $[V, W] = a \cdot V + b \cdot W$ with $a, b \in C^\infty(M)$ in the last line. We choose μ such that

$$\mu \cdot b + V(\mu) = 0 \qquad \Longleftrightarrow \qquad V(\ln \mu) = -b$$

by integrating along the integral curves of V, and similarly we choose λ such that

$$\lambda \cdot a - W(\lambda) = 0 \qquad \Longleftrightarrow \qquad W(\ln \lambda) = a$$

by integrating along the integral curves of W. This then gives $[\hat{V}, \hat{W}] = 0$. □

Theorem 1.31 (Frobenius)**.** *Let M be an n-dimensional smooth manifold, $V_1, \ldots, V_k \in \mathfrak{X}^\infty(M)$, $k < n$, pointwise linearly independent with $[V_i, V_j] \subseteq \operatorname{span}\{V_1, \ldots, V_k\}$ for all $1 \leq i, j \leq k$. Then $\operatorname{span}\{V_1, \ldots, V_k\}$ is integrable, i.e., locally there exists coordinates $\{v_1, \ldots, v_k, x_{k+1}, \ldots, x_n\}$ such that $\operatorname{span}\{V_1, \ldots, V_k\}$ are the tangent spaces of the family of submanifolds $\{x_{k+1} = c_{k+1}, \ldots, x_n = c_n\}$, $c_i \in \mathbb{R}$.*

Proof. For $k = 1$ this is Proposition 1.17, for $k = 2$ this follows from Proposition 1.29 and Lemma 1.30. The general case is by induction (not examinable). □

1.8.1 k-forms and dual version of Frobenius

Recall that a $(0, k)$-tensor field ω is at every point $p \in M$ a multilinear map $\omega|_p : \underbrace{T_pM \times \ldots \times T_pM}_{k-\text{times}} \to \mathbb{R}$. If $\omega|_p$ is totally antisymmetric for every $p \in M$ then we say ω is a k-**form**. If α is a $(0, k)$-tensor field, then

$$\alpha_{[a_1 \ldots a_k]} := \frac{1}{k!} \sum_\sigma \operatorname{sgn}(\sigma) \alpha_{\sigma(a_1) \ldots \sigma(a_k)} \qquad \text{with } \operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{for even permutations} \\ -1 & \text{for odd permutations} \end{cases}$$

is the **total antisymmetrisation** of α, a k-form. Given a k-form α and an l-form β, we define their **wedge product** $\alpha \wedge \beta$, a $(k + l)$-form, by

$$(\alpha \wedge \beta)_{a_1 \ldots a_{k+l}} := \frac{(k + l)!}{k! l!} \alpha_{[a_1 \ldots a_k} \beta_{a_{k+1} \ldots a_{k+l}]} \ ,$$

the total antisymmetrisation of their tensor product with a normalising factor.

Example 1.32. *Let $\alpha, \beta \in \Omega^1(M)$. Then for $X, Y \in \mathfrak{X}^\infty(M)$ we have*

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \beta(X)\alpha(Y) \ .$$

Also note that $\alpha \wedge \alpha = 0$.

Given a k-form α, then $\nabla\alpha$ is a $(0, k+1)$-tensor field. We define the **exterior derivative** $d\alpha$, a $(k+1)$-form, by

$$(d\alpha)_{a_1 \ldots a_{k+1}} := (k+1)\nabla_{[a_1}\alpha_{a_2 \ldots a_{k+1}]} \ .$$

Note that

$$\begin{aligned}\nabla_{[a_1}\alpha_{a_2 \ldots a_{k+1}]} &= \partial_{[a_1}\alpha_{a_2 \ldots a_{k+1}]} - \Gamma^b_{[a_1 a_2}\alpha_{b a_3 \ldots a_{k+1}]} - \ldots - \Gamma^b_{[a_1 a_{k+1}}\alpha_{a_2 \ldots a_k b]} \\ &= \partial_{[a_1}\alpha_{a_2 \ldots a_{k+1}]} \ ,\end{aligned}$$

where we have used the symmetry of the connection. We thus obtain an operator d : k-forms $\to$ $(k+1)$-forms which is independent of the metric g (and ∇). It only depends on the smooth manifold structure.

Exercise: Show that $d \circ d = 0$.

Proposition 1.33. *Let $\omega \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}^\infty(M)$. Then*

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X, Y]\big) \ .$$

Proof. We compute

$$\begin{aligned} d\omega(X, Y) &= (d\omega)_{ab}X^a Y^b \\ &= 2\partial_{[a}\omega_{b]}X^a Y^b \\ &= X(\omega_b) \cdot Y^b - Y(\omega_b)X^b \\ &= X(\omega_b Y^b) - \omega_b \cdot X(Y^b) - Y(\omega_b X^b) + \omega_b \cdot Y(X^b) \\ &= X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) \ ,\end{aligned}$$

where we have used in the last line $[X, Y]^b = X(Y^b) - Y(X^b)$. □

Let M be a an n-dimensional smooth manifold and $\alpha \in \Omega^1(M)$ non-vanishing. For $p \in M$, $\alpha|_p : T_pM \to \mathbb{R}$ is linear, so $\ker \alpha_p$ is $(n-1)$-dimensional. It follows that there are locally $(n-1)$-smooth pointwise linearly independent vector fields $V_1, \ldots, V_{n-1}$ such that $\ker \alpha = \operatorname{span}\{V_1, \ldots, V_{n-1}\}$. Vice versa, given an $(n-1)$-dimensional distribution $\operatorname{span}\{V_1, \ldots, V_{n-1}\}$, there is locally a 1-form α with $\ker \alpha = \operatorname{span}\{V_1, \ldots, V_{n-1}\}$. We conclude that 1-forms are an easy way to specify $(n-1)$-dimensional distributions.

Proposition 1.34. *Let M be an n-dimensional smooth manifold and $\alpha \in \Omega^1(M)$ pointwise non-vanishing. Then the following are equivalent:*

i) For $V, W \in \ker \alpha$ smooth vector fields we have $[V, W] \in \ker \alpha$ ($\iff$ $\ker \alpha$ is integrable by Theorem 1.31) *ii)* $d\alpha|_{\ker\alpha} \equiv 0$

iii) $\alpha \wedge d\alpha \equiv 0$.

Proof. By Proposition 1.33 for $V, W \in \ker\alpha$ we have

$$d\alpha(V,W) = V\big(\underbrace{\alpha(W)}_{=0}\big) - W\big(\underbrace{\alpha(V)}_{=0}\big) - \alpha\big([V,W]\big) = -\alpha\big([V,W]\big) \ .$$

We thus obtain $d\alpha(V,W) = 0$ iff $[V,W] \in \ker\alpha$, which shows the equivalence of i) and ii).

To see ii) $\iff$ iii), let $p \in M$ and let $\alpha_1, \ldots, \alpha_n$ be a basis of T_p^*M with $\alpha_1 = \alpha(p)$. Then $\alpha_i \wedge \alpha_j$, $1 \le i < j \le n$ is a basis for all antisymmetric $(0,2)$-tensors at p (exercise). Thus $d\alpha|_p = \sum_{1\le i<j\le n} f_{ij}\alpha_i \wedge \alpha_j$ with $f_{ij} \in \mathbb{R}$.

Now, if $d\alpha|_{\ker\alpha} = 0$, then $d\alpha|_p = \sum_{1<j\le n f_{1j}} \alpha \wedge \alpha_j$ and thus $\alpha \wedge d\alpha|_p = \sum_{1<j\le n} f_{1j} \underbrace{\alpha \wedge \alpha}_{=0} \wedge \alpha_j = 0$.

Vice versa, if $0 \overset{!}{=} \alpha \wedge d\alpha = \alpha \wedge \sum_{1\le i<j\le n} f_{ij}\alpha_i \wedge \alpha_j$, then it follows that $d\alpha = \sum_{1<j\le n} f_{1j}\alpha \wedge \alpha_j$ and thus $d\alpha|_{\ker\alpha} = 0$. □

Corollary 1.35. *Let $\alpha \in \Omega^1(M)$ pointwise non-vanishing with $\alpha \wedge d\alpha = 0$. Then there exist local coordinates $\{x^1, \ldots, x^n\}$ and a local function f such that $\alpha = f \cdot dx^n$.*

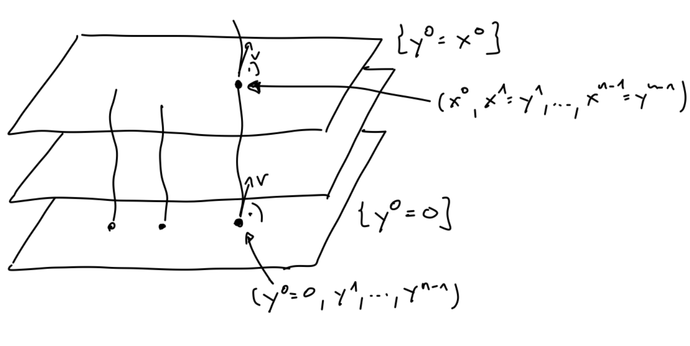
Proof. By Proposition 1.34 and Theorem 1.31 $\ker\alpha$ is integrable, thus there exist local coordinates $\{x^1, \ldots, x^n\}$ such that $\{x^n = c_n\}$, $c_n \in \mathbb{R}$, are integral manifolds of $\ker\alpha$. Thus dx^n is proportional to α. □

Let now (M, g) be a Lorentzian manifold and let $V \in \mathfrak{X}^\infty(M)$ be nowhere vanishing. We say that V is **hypersurface orthogonal** iff the distribution orthogonal to it, i.e., $V^\perp = \{X \in \mathfrak{X}^\infty(M) \mid g(V,X) = 0\}$, is integrable. Note that $V^\perp = \ker V^\flat$. Proposition 1.34 shows that V is hypersurface orthogonal iff $V^\flat \wedge dV^\flat = 0$ ($\iff$ $V_{[a}\partial_b V_{c]} = 0$).

Corollary 1.36. *Let (M, g) be a Lorentzian manifold and let $V \in \mathfrak{X}^\infty(M)$ be hypersurface orthogonal and nowhere null. Then there exist local coordinates $\{x^0, \ldots, x^{n-1}\}$ such that $\partial_0 \sim V$ and*

$$g = g_{00}dx_0^2 + \sum_{1\le i,j\le n-1} g_{ij}dx_i \otimes dx_j \ .$$

Proof. Let $p \in M$. By the Frobenius theorem we can choose local coordinates $\{y^0, \ldots, y^{n-1}\}$ such that $\{y^0 = c_0\}$ are integral manifolds of $V^\perp$, and without loss of generality assume they are centred at p.



Pick $\{y^0 = 0\}$, on which we have coordinates $\{y^1, \ldots, y^{n-1}\}$. Note that $0 \neq g(V,V) = V^\flat(V)$, and thus $V \notin \ker V^\flat$, i.e., V is transverse and orthogonal for $\{y^0 = 0\}$. Consider the family of integral curves of V and let $(x^0, x^1, \ldots, x^{n-1})$ refer to the point that is the intersection of the integral curve of V through $\{y^0 = 0, y^i = x^i\}$ with $\{y^0 = x^0\}$. In a small enough neighbourhood $\{x^0, \ldots, x^{n-1}\}$ form coordinates (exercise) and since x_i, $i = 1, \ldots, n-1$ are constant along the integral curves of V we have $V \sim \partial_0$. Moreover, we clearly have $\partial_i \subseteq V^\perp$ by construction. This shows that $g_{0i} = 0$ for $i = 1, \ldots, n-1$. $\square$

Example 1.37. *Consider the FLRW cosmologies from GR I, where $M = I \times \overline{M}$, $g = -dt^2 + a(t)^2 \overline{g}$. Here, $(\overline{M}, \overline{g})$ is a Riemannian manifold of constant curvature. It is clear from the form of the metric that ∂_t is hypersurface orthogonal with orthogonal hypersurfaces $\{t = t_0\}$.*

We call a spacetime (M, g) **static** iff there is a timelike and hypersurface orthogonal Killing vector field V. In such a spacetime one can locally introduce coordinates $\{x^0, \ldots, x^{n-1}\}$ such that $V = \partial_0$ and $g = g_{00} dx_0^2 + \sum_{1 \leq i,j \leq n-1} g_{ij} dx_i \otimes dx_j$ with $g_{\mu\nu}$ being independent of x^0 (problem sheet).

Example 1.38. *The Schwarzschild spacetime is static, which is easily seen from the form of the metric*

$$g = -(1 - \frac{2M}{r})\, dt^2 + \frac{1}{1 - \frac{2M}{r}}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\ .$$

For $r > 2M$, ∂_t is a timelike and hypersurface orthogonal Killing vector field.

We call a spacetime (M, g) **stationary** iff there exists a timlike Killing vector field (which is not necessarily hypersurface orthogonal).[10] The Kerr spacetime, which we encounter later, is stationary but not static.

2 Linearised general relativity

2.1 Einstein equations with matter

A continuum of matter has an associated stress-energy tensor T_{ab}, a symmetric $(0,2)$-tensor field. In flat spacetime $\partial^a T_{ab} = 0$ expresses the conservation laws (of energy, momentum, ...), see the second problem sheet. In curved spacetime the local conservation laws are expressed by $\nabla^a T_{ab} = 0$.

Recall that $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ and the second Bianchi equations give $\nabla^a G_{ab} = 0$. So Einstein tried $G_{ab} = \lambda \cdot T_{ab}$ with λ being a constant which will be determined by comparison with the Newtonian theory. We will find in Section 2.3 that $\lambda = 8\pi$ in geometrised units where $G = c = 1$, or $\lambda = \frac{8\pi G}{c^4}$ in non-geometrised units.

Example 2.1. *1.* ***Perfect fluid:*** *A perfect fluid is described by*

- 4-*velocity u of a fluid element, a unit timelike vector field*

[10] Usually one does not require that the Killing vector field is timelike throughout the whole spacetime, but only in an asymptotic region. This remark also applies to the notion of a static spacetime. In the Schwarzschild interior the Killing vector field ∂_t is spacelike.

- *the mass-energy density ρ in the rest frame (scalar)*
- *the pressure p in the rest frame (scalar)*
- *equation of state $p = p(\rho)$*

The stress-energy tensor is given by $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$. Choosing an orthonormal frame field $e_0, \ldots, e_3$ with $e_0 = u$ ($\iff$ rest frame), then in this frame

$$T_{ab} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} ,$$

from which we see that the interpretation of ρ and p is as given above. A direct computation shows

$$\nabla^a T_{ab} = 0 \iff \begin{cases} u^a \nabla_a \rho + (\rho + p)\nabla^a u_a = 0 \\ (p + \rho) u^a \nabla_a u_b + (g_{ab} + u_a u_b)\nabla^a p = 0 \ , \end{cases}$$

which, together with the equation of state, give the equations of motion for the fluid.

2. ***Dust:*** *This is a perfect fluid with $p = 0$. Thus $T_{ab} = \rho u_a u_b$.*

3. ***Electromagnetic field:*** *Described by the Faraday tensor F_{ab}, a 2-form. The stress-energy tensor is given by $T_{ab} = \frac{1}{4\pi}\big(F_{ac}F_b{}^c - \frac{1}{4} g_{ab} F_{de}F^{de}\big)$. A direct computation gives*[11] *(see problem sheet)*

$$\nabla^a T_{ab} = 0 \impliedby \begin{cases} dF = 0 \\ \nabla^a F_{ab} = 0 \ . \end{cases}$$

Rewriting the Einstein equations: Taking the trace of $R_{ab} - \frac{1}{2} g_{ab} R = \lambda T_{ab}$ yields $R - 2R = \lambda T$ and thus $R = -\lambda T$. We thus see that the Einstein equations are equivalent to

$$R_{ab} = \lambda (T_{ab} - \frac{1}{2} g_{ab} T) \ . \tag{2.2}$$

In vacuum this reduces to $R_{ab} = 0$.

2.2 Linearising the Einstein equations around Minkowski spacetime

We start out with Minkowski spacetime $(\mathbb{R}^4, \eta)$ in inertial (Cartesian) coordinates x^μ. **In the following the Greek indices $\mu, \nu, \kappa, \ldots$ will not be abstract indices but will always refer to this chosen coordinate system on $\mathbb{R}^4$.**

We look for an approximate solution $(\mathbb{R}^4, g = \eta + \varepsilon h, T = \varepsilon \overset{(1)}{T})$, where h is a symmetric $(0,2)$-tensor field on $\mathbb{R}^4$, $0 < \varepsilon \ll 1$ a small parameter. We require that g satisfies the Einstein equations $R_{ab} = \lambda(T_{ab} - \frac{1}{2} g_{ab} T)$ to order ε, ignoring higher orders of ε. This is a good approximation if the

[11] In general $\nabla^a T_{ab} = 0$ does not imply the Maxwell equtions.

gravitational field is weak, the mass-energy density and the material stresses are small. In this way we obtain a theory for a symmetric $(0,2)$-tensor field h on Minkowski spacetime. We now compute $R_{\mu\nu}(\eta+\varepsilon h)$ to order ε:

- **Inverse of $g_{\mu\nu}$:** Ansatz $(g^{-1})^{\nu\kappa}=\eta^{\nu\kappa}-\varepsilon s^{\nu\kappa}$, with s a symmetric $(2,0)$-tensor field. Then

$$g_{\mu\nu}(g^{-1})^{\nu\kappa}=(\eta_{\mu\nu}+\varepsilon h_{\mu\nu})(\eta^{\nu\kappa}-\varepsilon s^{\nu\kappa})=\delta^{\kappa}{}_{\mu}-\varepsilon\eta_{\mu\nu}s^{\nu\kappa}+\varepsilon h_{\mu\nu}\eta^{\nu\kappa}+\mathcal{O}(\varepsilon^2)\ ,$$

whence $\eta_{\mu\nu}s^{\nu\kappa}=h_{\mu\nu}\eta^{\nu\kappa}$ and thus $s^{\rho\kappa}=h_{\mu\nu}\eta^{\nu\kappa}\eta^{\mu\rho}$.

Note that we have the two Lorentzian metrics η and g, so we now make the convention that we **raise and lower all indices with the Minkowski metric** $\eta_{\mu\nu}$, i.e., $h^{\rho\kappa}=h_{\mu\nu}\eta^{\nu\kappa}\eta^{\mu\rho}$. We thus obtain

$$(g^{-1})^{\nu\kappa}=\eta^{\nu\kappa}-\varepsilon h^{\nu\kappa}+\mathcal{O}(\varepsilon^2)\ .$$

- **Christoffel symbols:**

$$\begin{aligned}\Gamma^{\mu}_{\nu\kappa}&=\frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\kappa\sigma}+\partial_\kappa g_{\nu\sigma}-\partial_\sigma g_{\nu\kappa})\\&=\varepsilon\frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\kappa\sigma}+\partial_\kappa h_{\nu\sigma}-\partial_\sigma h_{\nu\kappa})+\mathcal{O}(\varepsilon^2)\ ,\end{aligned}$$

where we have used in the last line that the x^μ are Cartesian coordinates.

- **Curvature:**

$$\begin{aligned}R^{\mu}{}_{\kappa\rho\nu}&=\partial_\rho\Gamma^{\mu}_{\nu\kappa}-\partial_\nu\Gamma^{\mu}_{\rho\kappa}+\underbrace{\Gamma^{\mu}_{\rho\sigma}\Gamma^{\sigma}_{\nu\kappa}-\Gamma^{\mu}_{\nu\sigma}\Gamma^{\sigma}_{\rho\kappa}}_{=\mathcal{O}(\varepsilon^2)}\\&=\varepsilon\frac{1}{2}\eta^{\mu\sigma}(\partial_\rho\partial_\nu h_{\kappa\sigma}+\partial_\rho\partial_\kappa h_{\nu\sigma}-\partial_\rho\partial_\sigma h_{\nu\kappa})\\&\qquad-\varepsilon\frac{1}{2}\eta^{\mu\sigma}(\partial_\nu\partial_\rho h_{\kappa\sigma}+\partial_\nu\partial_\kappa h_{\rho\sigma}-\partial_\nu\partial_\sigma h_{\rho\kappa})+\mathcal{O}(\varepsilon^2)\\&=\varepsilon\frac{1}{2}\eta^{\mu\sigma}(\partial_\rho\partial_\kappa h_{\nu\sigma}-\partial_\rho\partial_\sigma h_{\nu\kappa}-\partial_\nu\partial_\kappa h_{\rho\sigma}+\partial_\nu\partial_\sigma h_{\rho\kappa})+\mathcal{O}(\varepsilon^2)\end{aligned}\tag{2.3}$$

- **Ricci curvature:**

$$R_{\kappa\nu}=R^{\mu}{}_{\kappa\mu\nu}=\varepsilon\frac{1}{2}(\partial^\mu\partial_\kappa h_{\nu\mu}-\Box h_{\nu\kappa}-\partial_\nu\partial_\kappa h+\partial_\nu\partial^\mu h_{\mu\kappa})+\mathcal{O}(\varepsilon^2)\ ,$$

where $h=h_{\rho\sigma}\eta^{\rho\sigma}$ (the trace) and $\Box=\eta^{\mu\sigma}\partial_\mu\partial_\sigma$ (the **wave operator** on Minkowski spacetime).

We now introduce $\overline{h}_{\mu\nu}:=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h$, the **trace reversed** metric perturbations, and compute

$$\begin{aligned}\partial_\kappa(\partial^\mu\overline{h}_{\mu\nu})+\partial_\nu(\partial^\mu\overline{h}_{\mu\kappa})&=\partial_\kappa\partial^\mu h_{\mu\nu}-\frac{1}{2}\partial_\kappa\partial_\nu h+\partial_\nu\partial^\mu h_{\mu\kappa}-\frac{1}{2}\partial_\nu\partial_\kappa h\\&=\partial_\kappa\partial^\mu h_{\mu\nu}+\partial_\nu\partial^\mu h_{\mu\kappa}-\partial_\kappa\partial_\nu h\ .\end{aligned}$$

Thus

$$R_{\kappa\nu}(\eta+\varepsilon h)=\frac{1}{2}\varepsilon\Big(-\Box h_{\nu\kappa}+\partial_\kappa(\partial^\mu\overline{h}_{\mu\nu})+\partial_\nu(\partial^\mu\overline{h}_{\mu\kappa})\Big)+\mathcal{O}(\varepsilon^2)\ .$$

We thus obtain that the field equations (2.2) to order ε are

$$\frac{1}{2}\Big(-\Box h_{\nu\kappa}+\partial_\kappa\partial^\mu\overline{h}_{\mu\nu}+\partial_\nu\partial^\mu\overline{h}_{\mu\kappa}\Big)=\lambda(\overset{(1)}{T}{}_{\nu\kappa}-\frac{1}{2}\eta_{\nu\kappa}\overset{(1)}{T})\ .\tag{2.4}$$

We next simplify these equations by choosing a suitable gauge.

Recall from Section 1.3 that if $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ is a diffeomorphism and g is a solution of $R_{\mu\nu}(g) = \lambda(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$, then Φ^*g is a solution of

$$R_{\mu\nu}(\Phi^*g) = \big(\Phi^*R(g)\big)_{\mu\nu} = \lambda\big((\Phi^*T)_{\mu\nu} - \frac{1}{2}(\Phi^*g)_{\mu\nu} \cdot \Phi^*T\big) \ .$$

The solutions $(\mathbb{R}^4, g, T)$ and $(\mathbb{R}^4, \Phi^*g, \Phi^*T)$ are regarded as physically equivalent.

Now, let $\xi \in \mathfrak{X}^\infty(\mathbb{R}^4)$ and let Φ_t denote the associated one-parameter group of diffeomorphisms. Then

$$\begin{aligned}(\Phi_\varepsilon^*g)_{\mu\nu} &= (\Phi_\varepsilon^*\eta)_{\mu\nu} + \varepsilon(\Phi_\varepsilon^*h)_{\mu\nu} \\ &= \eta_{\mu\nu} + \varepsilon\frac{d}{dt}\Big|_{t=0}(\Phi_t^*\eta)_{\mu\nu} + \mathcal{O}(\varepsilon^2) + \varepsilon h_{\mu\nu} + \mathcal{O}(\varepsilon^2) \\ &= \eta_{\mu\nu} + \varepsilon\big(h_{\mu\nu} + (\mathcal{L}_\xi\eta)_{\mu\nu}\big) + \mathcal{O}(\varepsilon^2)\end{aligned}$$

and

$$(\Phi_\varepsilon^*T)_{\mu\nu} = (\Phi_\varepsilon^*\varepsilon \overset{(1)}{T}_{\mu\nu}) = \varepsilon \overset{(1)}{T}_{\mu\nu} + \mathcal{O}(\varepsilon^2) \ .$$

(We see that since T is already of order ε it is *gauge invariant* to order ε.)

Thus if $(\underbrace{\eta_{\mu\nu} + \varepsilon h_{\mu\nu}}_{=g_{\mu\nu}}, \varepsilon \overset{(1)}{T}_{\mu\nu})$ satisfy the Einstein equations (2.2) to order ε, then so does $(\eta_{\mu\nu} + \varepsilon(h_{\mu\nu} + (\mathcal{L}_\xi\eta)_{\mu\nu}), \varepsilon\overset{(1)}{T}_{\mu\nu})$ for any $\xi \in \mathfrak{X}^\infty(\mathbb{R}^4)$. The vector field ξ is often called an **infinitesimal diffeomorphism** in this context. We summarise our finding in the following[12]

Proposition 2.5. *If for given* $\overset{(1)}{T}_{\mu\nu}$ *we have that* $h_{\mu\nu}$ *satisfies* (2.4)*, then so does*

$$\tilde{h}_{\mu\nu} := h_{\mu\nu} + (\mathcal{L}_\xi\eta)_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

for any $\xi \in \mathfrak{X}^\infty(\mathbb{R}^4)$*.*

We now choose a gauge ξ by solving[13]

$$\Box\xi_\mu = -\partial^\rho\overline{h}_{\mu\rho} \ . \tag{2.6}$$

The gauge (2.6) is called the **wave gauge** or **harmonic gauge**. The gauge condition (2.6) determines ξ up to the addition of a solution $\chi \in \mathfrak{X}^\infty(\mathbb{R}^4)$ of the homogeneous wave equation $\Box\chi_\mu = 0$. We thus have a **residual gauge freedom**.

We now compute

$$\begin{aligned}\partial^\mu\overline{\tilde{h}}_{\mu\nu} &= \partial^\mu(\tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}) \\ &= \partial^\mu(h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu) - \frac{1}{2}\partial_\nu(h + 2\partial^\mu\xi_\mu) \\ &= \partial^\mu h_{\mu\nu} + \Box\xi_\nu + \partial_\nu\partial^\mu\xi_\mu - \frac{1}{2}\partial_\nu h - \partial_\nu\partial^\mu\xi_\mu \\ &= \partial^\mu\overline{h}_{\mu\nu} + \Box\xi_\nu \ , \\ &= 0\end{aligned}$$

[12] The proposition could have also been verified by direct computation.

[13] The linear wave equation with a right hand side can be easily solved in Minkowski spacetime. There remains the freedom to prescribe, for example, $\xi^\mu|_{t=0}$ and $\partial_t\xi^\mu|_{t=0}$.

where we used (2.6) in the last line. Thus $\tilde{h}_{\mu\nu}$ satisfies

$$
\begin{aligned}
\Box\tilde{h}_{\mu\nu} &= -2\lambda(\overset{(1)}{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overset{(1)}{T}) = -2\lambda\overline{\overset{(1)}{T}}_{\mu\nu} \\
\partial^\mu\bar{\tilde{h}}_{\mu\nu} &= 0 \ ,
\end{aligned}
\tag{2.7}
$$

the **linearised Einstein equations around Minkowski spacetime in the wave gauge**.

Remark 2.8. *It will be convenient later to rephrase the first equation in* (2.7) *also in terms of* $\bar{\tilde{h}}_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h}$ *to obtain*

$$
\begin{aligned}
\Box\bar{\tilde{h}}_{\mu\nu} &= -2\lambda\overset{(1)}{T}_{\mu\nu} \\
\partial^\mu\bar{\tilde{h}}_{\mu\nu} &= 0 \ .
\end{aligned}
\tag{2.9}
$$

Note that $\bar{\bar{\tilde{h}}}_{\mu\nu} = \tilde{h}_{\mu\nu}$*. Also note that if* $\partial^\mu\overset{(1)}{T}_{\mu\nu} = 0$*, then*

$$
\Box\partial^\mu\bar{\tilde{h}}_{\mu\nu} = -2\lambda\partial^\mu\overset{(1)}{T}_{\mu\nu} = 0 \ ,
$$

i.e., $\partial^\mu\bar{\tilde{h}}_{\mu\nu}$ *satisfies the homogeneous wave equation. If we solve the first equation in* (2.9) *using for example the retarded solution of* $\Box$ *then it follows automatically that the second equation in* (2.9) *is also satisfied.*[14] *Note that solving the first equation of* (2.9) *using the retarded solution to obtain* $\bar{\tilde{h}}_{\mu\nu}$*, and then computing* $\bar{\bar{\tilde{h}}}_{\mu\nu} = \tilde{h}_{\mu\nu}$ *from it, gives the same result as solving the first equation of* (2.7) *using the retarded solution. Thus we infer that if* $\partial^\mu\overset{(1)}{T}_{\mu\nu} = 0$ *holds, solving the first equation of* (2.7) *using the retarded solution again ensures that the second equation of* (2.7) *is satisfied.*

Remark 2.10 (Why is it called wave gauge?)**.**

Let (M,g) *be a Lorentzian manifold and let* $\Box_g\psi := g^{\rho\lambda}\nabla_\rho\nabla_\lambda\psi$ *be the* ***wave operator on*** (M,g)*, where* $\psi \in C^\infty(M)$*.*[15] *Let* x^μ *be local coordinate functions on* M*. Then we have*

$$
\Box_g x^\mu = -g^{\rho\lambda}\Gamma^\mu_{\rho\lambda} =: -\Gamma^\mu \ .
$$

We define $\Gamma_\nu := g_{\nu\mu}\Gamma^\mu$*. Then the set of coordinates* x^μ *satisfy the wave equation if, and only if,* $\Gamma_\nu = 0$ *for all* ν*.*

Going back to linearised gravity with Cartesian coordinates x^μ *we see that they satisfy the wave equation associated with* $g = \eta + \varepsilon h$ *to first order, i.e.,*

$$
\Box_g x^\mu = 0 + \mathcal{O}(\varepsilon^2) \ ,
$$

[14]One should emphasise that the first equation in (2.9) ensures that the linearised Einstein equations (2.4) hold *only if the second equation in* (2.9) *also holds!*

[15]With this terminology we have $\Box = \Box_\eta$.

if, and only if

$$
\begin{aligned}
\mathcal{O}(\varepsilon^2) \stackrel{!}{=} \Gamma_\nu &= g_{\nu\mu} g^{\rho\lambda} \Gamma^\mu_{\rho\lambda} \\
&= \eta_{\nu\mu}\eta^{\rho\lambda}\varepsilon\frac{1}{2}\eta^{\mu\sigma}(\partial_\rho \tilde{h}_{\sigma\lambda} + \partial_\lambda \tilde{h}_{\sigma\rho} - \partial_\sigma \tilde{h}_{\rho\lambda}) + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon\frac{1}{2}(\partial^\lambda \tilde{h}_{\nu\lambda} + \partial^\lambda \tilde{h}_{\nu\lambda} - \partial_\nu \tilde{h}) + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon\partial^\lambda(\tilde{h}_{\nu\lambda} - \frac{1}{2}\eta_{\lambda\nu}\tilde{h}) + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon\partial^\lambda \overline{\tilde{h}}_{\nu\lambda} + \mathcal{O}(\varepsilon^2) \ ,
\end{aligned}
$$

i.e., if, and only if, $\partial^\lambda \overline{\tilde{h}}_{\nu\lambda} = 0$.

Remark 2.11 (Analogy with gauge freedom in Maxwell's equations)**.**

Consider Maxwell's equations in Minkowski spacetime:

$$
\begin{aligned}
dF &= 0 \\
\partial^\mu F_{\mu\nu} &= 4\pi J_\nu \ ,
\end{aligned}
\tag{2.12}
$$

where J_ν *is the source. The first equation implies that we can write* $F = dA$ *for a one-form* A *(then* $dF = ddA = 0$ *is trivially satisfied). The second equation in terms of the* ***electromagnetic potential*** A *becomes* $4\pi J_\nu = \partial^\mu F_{\mu\nu} = \partial^\mu\partial_\mu A_\nu - \partial^\mu\partial_\nu A_\mu$*. Thus we can rewrite Maxwell's equations in terms of the potential as*

$$
\Box A_\nu - \partial_\nu \partial^\mu A_\mu = 4\pi J_\nu \ . \tag{2.13}
$$

Now consider $\xi \in C^\infty(\mathbb{R}^4)$ *and let* $\tilde{A} = A + d\xi$*. Note that* $\tilde{F} = d\tilde{A} = dA + \underbrace{dd\xi}_{=0} = F$*. It thus follows immediately from* (2.12) *that if* A *solves* (2.13) *then so does* $\tilde{A}$*.*[16] *The addition of* $d\xi$ *to* A *represents the gauge freedom for Maxwell's equations.*

We can also fix the gauge here. Let us choose the ***Lorentz gauge*** $\partial^\mu \tilde{A}_\mu = 0$*, which can be arranged by solving* $\Box\xi = -\partial^\mu A_\mu$*: Since then we have*

$$
\partial^\mu \tilde{A}_\mu = \partial^\mu(A_\mu + \partial_\mu \xi) = \partial^\mu A_\mu + \Box\xi = 0 \ .
$$

In the Lorentz gauge (2.13) *becomes*[17]

$$
\begin{aligned}
\Box\tilde{A}_\nu &= 4\pi J_\nu \\
\partial^\mu \tilde{A}_\mu &= 0 \ .
\end{aligned}
$$

Note that we also have the residual gauge freedom $\hat{A}_\mu = \tilde{A}_\mu + d\chi$ *with* $\chi \in C^\infty(\mathbb{R}^4)$ *satisfying* $\Box\chi = 0$*.*

Remark 2.14. *Equations* (2.7) *with* $\overset{(1)}{T} \equiv 0$ *describe a massless spin-2 field on Minkowski spacetime.*

[16] This is analogous to Proposition 2.5 and can again be alternatively verified by direct computation.
[17] Cf. (2.7).

2.3 Newtonian limit

The Newtonian theory is well-verified if

i) gravity is weak $\implies$ linearised theory

Consider as a matter source a perfect fluid with stress energy tensor $\overset{(1)}{T}_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\eta_{\mu\nu}$

ii) the relative motion of sources is much slower than the speed of light $c = 1 \implies u \simeq \partial_t$

iii) the material stresses are much smaller than the mass-energy density $\implies p \simeq 0$.

Under these assumptions we obtain $\overset{(1)}{T} \simeq \rho dt \otimes dt$, the stress-energy tensor of dust. We furthermore assume that the spacetime geometry (gravity) is slowly varying, i.e.,

iv) $\partial_t \tilde{h}_{\mu\nu} \simeq 0 \simeq \partial_t^2 \tilde{h}_{\mu\nu}$,

which is compatible with the assumption of slowly varying matter sources ii), but is not implied by it.[18]

Under these assumptions we obtain from (2.7), using $\overset{(1)}{T} = -\rho$,

$$\begin{aligned} \Delta\tilde{h}_{00} &= -2\lambda(\rho - \frac{1}{2}\rho) = -\lambda\rho \\ \Delta\tilde{h}_{0i} &= 0 \\ \Delta\tilde{h}_{ij} &= -2\lambda(0 + \frac{1}{2}\eta_{ij}\rho) = -\eta_{ij}\lambda\rho \ , \end{aligned} \tag{2.15}$$

which can be uniquely solved with the boundary conditions $\tilde{h}_{\mu\nu} \to 0$ for $r \to \infty$.[19] The only non-vanishing components are $\tilde{h}_{00} = \tilde{h}_{ii}$, which satisfy

$$\Delta\tilde{h}_{00} = -\lambda\rho \ .$$

We thus see that the whole content of gravity in this limit is encoded in just one scalar function.

Also note that $\bar{\tilde{h}}_{\mu\nu}$ satisfies $\Delta\bar{\tilde{h}}_{\mu\nu} = 0$ for $(\mu,\nu) \neq (0,0)$ and $\Delta\bar{\tilde{h}}_{00} = -2\lambda\rho$. From this it directly follows that $\bar{\tilde{h}}_{00}$ is the only non-vanishing component of the trace-reversed metric perturbations. Assumption ii) implies $\partial_t\rho = 0$ to first approximation, such that $\bar{\tilde{h}}_{00}$ is independent of time. This is self-consistent with assumptions iv) and also implies that the wave gauge $\partial^\mu\bar{\tilde{h}}_{\mu\nu} = 0$ is satisfied, cf. Remark 2.8.

To make contact with the Newtonian theory, we consider the gravitational force on a test body, which moves on a timelike geodesic

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \ .$$

Here, τ is the proper time. We consider non-relativistic motion such that

$$\left|\frac{dx^i}{d\tau}\right| \ll \frac{dt}{d\tau} \simeq 1 \ .$$

[18] E.g. there could be high frequency gravitational waves, see Section 2.5

[19] Here, Δ denotes the Laplacian $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$.

We can thus take $\tau \simeq t$ and to leading order the geodesic equation becomes

$$\frac{d^2x^i}{dt^2} \simeq \frac{d^2x^i}{d\tau^2} \simeq -\Gamma^i_{00}(x^j(\tau)) .$$

We compute

$$\Gamma^i_{00} = \frac{1}{2}\eta^{i\sigma}(2\partial_0\tilde{h}_{0\sigma} - \partial_\sigma\tilde{h}_{00}) = -\frac{1}{2}\partial_i\tilde{h}_{00} ,$$

so that we obtain

$$\frac{d^2x^i}{dt^2} \simeq \frac{1}{2}\partial_i\tilde{h}_{00} . \tag{2.16}$$

Recall the Newtonian theory

$$\Delta\Phi = 4\pi\rho \qquad \text{Poisson's equation with } G = 1 \tag{2.17}$$

$$\overrightarrow{F} = -\overrightarrow{\nabla}\Phi \qquad \Longrightarrow \quad \frac{d^2x^i}{dt^2} = -\partial_i\Phi . \tag{2.18}$$

Comparing (2.16) with (2.18) gives $\Phi = -\frac{1}{2}\tilde{h}_{00}$. Using this we obtain from (2.15) $-\lambda\rho = \Delta\tilde{h}_{00} = -2\Delta\Phi$, which we can now compare with (2.17) to obtain

$$\lambda = 8\pi .$$

Having derived the proportionality constant, the Einstein field equations now take the form $G_{ab} = 8\pi T_{ab}$.

Remark 2.19. *1. We consider the example of the gravitational field of the sun. Recall that we work in geometrised units in which $G = c = 1$. The metric (length element) $ds^2 = g = g_{\mu\nu}dx^\mu \otimes dx^\nu$ has units km^2. Here, let us give units of km to the background coordinates[20] x^μ so that the metric components $g_{\mu\nu}$ are dimensionless. The mass of the sun is $M_\odot \simeq 2 \cdot 10^{30}kg \simeq 1.5km$. Solving* (2.15) *for a spherical mass distribution with $\lambda = 8\pi$ we obtain $\tilde{h}_{00} = \frac{2M_\odot}{r}$ for $r > R_\odot$, where $r^2 = x_1^2 + x_2^2 + x_3^2$ and $R_\odot \simeq 700\,000km$ is the radius of the sun. Considering for example the effect on the orbits of planets we thus see that $\tilde{h}_{\mu\nu}$ is of order much less than $\frac{1}{700000} \ll 1$, so that the linearised theory is well justified.*

2. Note that the right hand side of (2.16) *is of order ε; a small effect. It is, however, not negligible for determining the orbits of planets, where this small force acts over a very* long *time.*

3. We did the comparison of the Newtonian limit of general relativity with the actual Newtonian theory in coordinates. However, coordinates themselves do not have any a priori physical meaning in general relativity (c.f. the instructive example in Section 2.5.1). This is a subtle point. In general relativity distances have to be determined geometrically via, for example, the radar method, measuring the time of flight of a photon. The coordinates x^μ are, however, such that the geometric determination of length and time is very close to the one given by the coordinates in

[20] Note, however, that this does *not* mean that the points $(0,0,0,0) \in \mathbb{R}^4$ and $(1,0,0,0)$ are $1km \simeq \frac{1}{300000}s$ apart. The proper time, which is accessible to measurement, has to be computed using the metric! In the linearised theory, and for short coordinate distances, these two values are, however, very close to each other.

comparison with the length scale of the orbit of a planet.[21] *This is because the time of flight of a photon across the solar system is small compared to the period of the planets.*

2.4 Far-field of stationary isolated gravitational body in linear approximation

We recall from (2.7) the linearised Einstein equations in the wave gauge

$$\Box\tilde{h}_{\mu\nu} = -16\pi \overline{\overset{(1)}{T}}_{\mu\nu} \tag{2.20}$$

$$\partial^\mu \overline{\tilde{h}}_{\mu\nu} = 0 \tag{2.21}$$

In this section we are looking for solutions with

i) $\partial^\mu \overset{(1)}{T}_{\mu\nu} = 0$

ii) $\overset{(1)}{T}_{\mu\nu}(t,\cdot)$ is compactly supported in $\mathbb{R}^3$ for all $t \in \mathbb{R}$ (i.e., isolated gravitational body)

iii) $\partial_t \overset{(1)}{T}_{\mu\nu} = 0 = \partial_t \tilde{h}_{\mu\nu}$ (i.e., time-independent solution).

Equation (2.20) thus becomes

$$\Delta\tilde{h}_{\mu\nu} = -16\pi \overline{\overset{(1)}{T}}_{\mu\nu} \ . \tag{2.22}$$

We are imposing the asymptotically flat boundary conditions $\tilde{h}_{\mu\nu}(\underline{x}) \to 0$ for $|\underline{x}| \to \infty$, where $\underline{x} = (x^1, x^2, x^3)$. As before it follows directly that $\partial^\mu \overline{\tilde{h}}_{\mu\nu}$ satisfies $\Delta\partial^\mu \overline{\tilde{h}}_{\mu\nu} = -16\pi\partial^\mu \overset{(1)}{T}_{\mu\nu} = 0$, so that the boundary conditions imply $\partial^\mu \overline{\tilde{h}}_{\mu\nu} \equiv 0$. It thus suffices to solve (2.22). We obtain[22]

$$\tilde{h}_{\mu\nu}(\underline{x}) = 4 \int\limits_{\mathbb{R}^3} \frac{\overline{\overset{(1)}{T}}_{\mu\nu}(\underline{x}')}{|\underline{x} - \underline{x}'|} \, d\underline{x}' \ . \tag{2.23}$$

In general this depends heavily on the exact form of $\overset{(1)}{T}$. Here, we are only interested in the far field. For this we will expand (2.23) in powers of $\frac{1}{r}$. We use the following identities, which have been derived on problem sheet 2:

1) $\frac{1}{|\underline{x}-\underline{x}'|} = \frac{1}{r} + \frac{1}{r^3}\underline{x} \cdot \underline{x}' + \mathcal{O}\left(\frac{1}{r^3}\right)$ as a function of $\underline{x}$, uniformly for bounded $\underline{x}'$

2) $\int_{\mathbb{R}^3} T^{ij}(t,\underline{x}) \, d\underline{x} = 0$

3) $\int_{\mathbb{R}^3} T^{0j}(t,\underline{x}) \, d\underline{x} = 0$

4) $\int_{\mathbb{R}^3} T^i{}_i(t,\underline{x}) d\underline{x} = 0$

5) $\int_{\mathbb{R}^3} (T^{0j}x^k + T^{0k}x^j)(t,\underline{x}) \, d\underline{x} = 0$

[21] In Section 2.5.1 the accuracy of the coordinates giving proper distances and proper times is of the same order as the effect one tries to measure.

[22] Recall that if $\Delta\varphi = f$ with $f \in \mathbb{C}_0^\infty(\mathbb{R}^3)$ and $\varphi(\underline{x}) \to 0$ for $|\underline{x}| \to \infty$, then $\varphi(\underline{x}) = -\int\limits_{\mathbb{R}^3} \frac{f(\underline{x}')}{4\pi|\underline{x}-\underline{x}'|} \, d\underline{x}'$.

6) $\int_{\mathbb{R}^3} T^i{}_i(t,\underline{x})x^j\,d\underline{x} = 0$.

• $\tilde{h}_{00}(\underline{x})$: We first compute

$$\overline{\overset{(1)}{T}}_{00} = \overset{(1)}{T}_{00} + \frac{1}{2}\overset{(1)}{T} = \overset{(1)}{T}_{00} + \frac{1}{2}(-\overset{(1)}{T}_{00} + \overset{(1)}{T^i}{}_i) = \frac{1}{2}(\overset{(1)}{T}_{00} + \overset{(1)}{T^i}{}_i)\ .$$

We then obtain from (2.23)

$$\begin{aligned}
\tilde{h}_{00}(\underline{x}) &= 4\int\limits_{\mathbb{R}^3} \frac{1}{r}\Big(1 + \frac{\underline{x}\cdot\underline{x}'}{r^2} + \mathcal{O}(\frac{1}{r^2})\Big)\frac{1}{2}\big(\overset{(1)}{T}_{00} + \overset{(1)}{T^i}{}_i)\big)(\underline{x}')\,d\underline{x}' \qquad \text{using 1)}\\
&= \frac{2}{r}\int\limits_{\mathbb{R}^3}\Big(\overset{(1)}{T}_{00} + \underbrace{\overset{(1)}{T^i}{}_i}_{\text{use 4)}} + \frac{x_j(x')^j}{r^2}\big(\underbrace{\overset{(1)}{T}_{00}(\underline{x}')}_{\text{use 7)}} + \underbrace{\overset{(1)}{T^i}{}_i(\underline{x}')}_{\text{use 6)}}\big) + \mathcal{O}(\frac{1}{r^2})\Big)\,d\underline{x}'\\
&= \frac{2M}{r} + \mathcal{O}(\frac{1}{r^3})\ ,
\end{aligned}$$

where $M := \int_{\mathbb{R}^3}\overset{(1)}{T}_{00}(\underline{x}')\,d\underline{x}'$ is the total mass, and we have used

7) $\int_{\mathbb{R}^3}(x')^j\overset{(1)}{T}_{00}(\underline{x}')\,d\underline{x}' = 0$

by choosing the Cartesian coordinates x^μ such that the origin is the **centre of mass**, i.e., such that 7) vanishes.

• $\tilde{h}_{0i}(\underline{x})$: We have $\overline{\overset{(1)}{T}}_{0i} = \overset{(1)}{T}_{0i}$. Thus (2.23) gives

$$\begin{aligned}
\tilde{h}_{0i}(\underline{x}) &= 4\int\limits_{\mathbb{R}^3}\frac{1}{r}\big(1 + \frac{\underline{x}\cdot\underline{x}'}{r^2} + \mathcal{O}(\frac{1}{r^2})\big)\overset{(1)}{T}_{0i}(\underline{x}')\,d\underline{x}'\\
&= \frac{4}{r^3}x^j\int\limits_{\mathbb{R}^3}\underline{x}'_j\overset{(1)}{T}_{0i}(\underline{x}')\,d\underline{x}' + \mathcal{O}(\frac{1}{r^3}) \qquad \text{using 3)}\\
&= \frac{4}{r^3}x^j\int\limits_{\mathbb{R}^3}\overset{(1)}{T}_{0[i}x'_{j]}\,d\underline{x}' + \mathcal{O}(\frac{1}{r^3}) \qquad \text{using } \overset{(1)}{T}_{0i}x'_j = \overset{(1)}{T}_{0(i}x'_{j)} + \overset{(1)}{T}_{0[i}x'_{j]} \text{ and 5).}
\end{aligned}$$

We define the **total angular momentum around the x^k-axis**

$$J_k := \int\limits_{\mathbb{R}^3}\varepsilon_{lmk}(x')^l\overset{(1)}{T^{0m}}(\underline{x}')\,d\underline{x}'\ .$$

This gives

$$\varepsilon_{ijk}J^k = \int\limits_{\mathbb{R}^3}\Big[(x')^i\overset{(1)}{T^{0j}}(\underline{x}') - (x')^j\overset{(1)}{T^{0i}}(\underline{x}')\Big]\,d\underline{x}' = 2\int\limits_{\mathbb{R}^3}\overset{(1)}{T}_{0[i}x'_{j]}\,d\underline{x}'\ .$$

Hence, we obtain

$$\tilde{h}_{0i}(\underline{x}) = \frac{2}{r^3}\varepsilon_{ijk}x^jJ^k + \mathcal{O}(\frac{1}{r^3}) = \frac{2}{r^3}(\vec{x}\times\vec{J})_i + \mathcal{O}(\frac{1}{r^3})\ .$$

- $\underline{\tilde{h}_{ij}(\underline{x})}$: We compute $\overline{\overset{(1)}{T}}_{ij} = \overset{(1)}{T}_{ij} - \frac{1}{2}\eta_{ij}(-\overset{(1)}{T}_{00} + \overset{(1)}{T^k}_k)$ Thus (2.23) gives

$$
\begin{aligned}
\tilde{h}_{ij}(\underline{x}) &= 4\int_{\mathbb{R}^3} \frac{1}{r}(1+\mathcal{O}(\frac{1}{r}))\big(\overset{(1)}{T}_{ij} - \frac{1}{2}\eta_{ij}(-\overset{(1)}{T}_{00} + \overset{(1)}{T^k}_k)\big)(\underline{x}')\, d\underline{x}' \\
&= \frac{4}{r}\int_{\mathbb{R}^3} \Big[\underbrace{\overset{(1)}{T}_{ij}(\underline{x}')}_{\text{use 2)}} + \frac{1}{2}\eta_{ij}(\overset{(1)}{T}_{00} - \underbrace{\overset{(1)}{T^k}_k}_{\text{use 4)}})(\underline{x}')\Big]\, d\underline{x}' + \mathcal{O}(\frac{1}{r^2}) \\
&= \frac{2}{r}M\eta_{ij} + \mathcal{O}(\frac{1}{r^2})\ .
\end{aligned}
$$

Collating all the terms gives the asymptotic form of the metric

$$
\begin{aligned}
g &\simeq (\eta_{\mu\nu} + \varepsilon\tilde{h}_{\mu\nu})dx^\mu \otimes dx^\nu \\
&= -\Big(1 - \frac{2\varepsilon M}{r} + \mathcal{O}(\frac{1}{r^3})\Big)\, dt^2 + \Big(\frac{2}{r^3}\varepsilon_{ijk}x^j\varepsilon J^k + \mathcal{O}(\frac{1}{r^3})\Big)(dt\otimes dx^i + dx^i \otimes dt) \\
&\quad + \Big((1 + \frac{2\varepsilon M}{r})\eta_{ij} + \mathcal{O}(\frac{1}{r^2})\Big)dx^i \otimes dx^j\ .
\end{aligned}
\tag{2.24}
$$

After a rotation of the Cartesian coordinate system we can assume that $\overrightarrow{J} = J\partial_{x^3}$. We then introduce spherical polar coordinates (r,θ,φ) on $\mathbb{R}^3$ in the standard way by $x^1 = r\cos\varphi\sin\theta$, $x^2 = r\sin\varphi\sin\theta$, $x^3 = r\cos\theta$. A simple computation then yields

$$
\varepsilon_{ijk}x^jJ^k dx^i = J(x^2dx^1 - x^1dx^2) = -Jr^2\sin^2\theta\, d\varphi\ .
$$

We can thus rewrite (2.24) in spherical coordinates as

$$
\begin{aligned}
g \simeq &-\Big(1 - \frac{2\varepsilon M}{r} + \mathcal{O}(\frac{1}{r^3})\Big)\, dt^2 - \frac{2}{r}\varepsilon J\sin^2\theta(dt\otimes d\varphi + d\varphi\otimes dt) + \mathcal{O}(\frac{1}{r^3})(dt\otimes d\underline{x} + d\underline{x}\otimes dt) \\
&+ (1 + \frac{2\varepsilon M}{r})\big(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\big) + \mathcal{O}(\frac{1}{r^2})(dx^i\otimes dx^j + dx^j \otimes dx^i)\ ,
\end{aligned}
\tag{2.25}
$$

where εM is the total mass and εJ the total angular momentum of the body.

Remark 2.26. 1. *Comparing with Section 2.3 on the Newtonian limit we see that* $\Phi = -\frac{1}{2}\tilde{h}_{00} = -\frac{1}{2}\tilde{h}_{ii} = -\frac{\varepsilon M}{r}$*, which is the Newtonian gravitational potential of a body with mass* εM*.*

2. *In contrast with Section 2.3 we allowed the relative motion of the sources to be compatible with the speed of light* ($\overset{(1)}{T}_{0i} \not\simeq 0$)*, which gives the* $\tilde{h}_{0i}$ *terms. Note that the timelike Killing vector field* ∂_t *is* not *hypersurface orthogonal (exercise) if the total angular momentum* J *is non-vanishing. Thus, in this case the solutions are only stationary – while in the case* $J = 0$ *they are static.*

3. *We allowed the material stresses to be comparable with the mass energy density* ($\overset{(1)}{T}_{ij} \not\simeq 0$)*.*

4. *The same asymptotic form of the metric can be derived for a stationary strongly gravitating isolated body in the fully non-linear theory. There, however, it does not hold any more that the mass parameter* εM *in the asymptotic expansion* (2.25) *is given by the integral* $\int_{\mathbb{R}^3} T_{00}\, d\underline{x}^3$ *of the*

mass-energy density of the matter[23] *– and similarly for the angular momentum parameter J. But we still* define *the parameters εM and εJ appearing in the asymptotic expansion of the metric for a strongly gravitating body to be the total mass and angular momentum of the spacetime.*[24]

5. *The total mass can be measured by looking at the trajectory of test particles: As in Section 2.3 we find*
$$\frac{d^2x^i}{dt^2} \simeq \frac{1}{2}\partial_i \tilde{h}_{00} = \partial_i(\frac{\varepsilon M}{r}) + \mathcal{O}(\frac{1}{r^3})\ .$$
For large r Newton's laws are thus valid and one can measure M for example from Kepler's third law $M = \omega^2 a^3$, where ω is the angular frequency and a is the semi-major axis of the elliptical orbit. This shows that our definition of the total mass as the parameter M appearing in the asymptotic expansion (2.24) *is compatible with the Newtonian concept of mass in the far field.*

6. *Similarly one can measure the total angular momentum from the precession of gyroscopes, see the textbook by Misner-Thorne-Wheeler, page 451.*

Example 2.27. *We bring the Schwarzschild metric*
$$g = -(1-\frac{2M}{r})\,dt^2 + \frac{1}{1-\frac{2M}{r}}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)$$
into the form (2.25) *to see that, with the above definitions, it indeed describes the spacetime of an isolated body with mass M and vanishing angular momentum.*

We look for a coordinate transformation $\rho(r)$ such that the metric becomes
$$g = -A(\rho)^2\,dt^2 + B(\rho)^2\big[\underbrace{d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)}_{=dx^2+dy^2+dz^2}\big]\ . \tag{2.28}$$
This gives us the conditions
$$B(\rho)^2\rho^2 = r^2 \tag{2.29}$$
$$B(\rho)^2\,d\rho^2 = \frac{1}{1-\frac{2M}{r}}\,dr^2 \tag{2.30}$$
$$A(\rho)^2 = 1-\frac{2M}{r}\ . \tag{2.31}$$
We obtain from (2.29) *and* (2.30) $(\frac{d\rho}{dr})^2 = \frac{\rho^2}{(1-\frac{2M}{r})r^2}$, *which gives*
$$\frac{d\rho}{dr} = \pm\frac{\rho}{\sqrt{1-\frac{2M}{r}}\cdot r}$$
and thus
$$\log\rho = \pm\int\frac{1}{\sqrt{r^2-2Mr}}\,dr\ .$$
We use
$$\int\frac{1}{\sqrt{y^2-a^2}}\,dy = \log(y+\sqrt{y^2-a^2}) + C\ ,$$

[23] This is because the gravitational field itself carries energy which contributes to the overall mass of the isolated body.

[24] Another approach of arriving at the definitions of the total mass and angular momentum of a strongly gravitating body is given by the ADM formalism using the Hamiltonian formulation of general relativity.

which can be obtained by trigonometric substitution, to get

$$\begin{aligned}\log\rho &= \pm\int \frac{1}{\sqrt{(r-M)^2-M^2}}\,dr \\ &= \pm\log(r-M+\sqrt{(r-M)^2-M^2})+C \\ &= \pm\log(r-M+r\sqrt{1-\frac{2M}{r}})+C\,.\end{aligned}$$

We use the positive sign so that $\rho\to\infty$ for $r\to\infty$, and so

$$\rho = C(r-M+r\sqrt{1-\frac{2M}{r}}) \qquad\qquad C>0\,.$$

Solving for r, we obtain

$$r=\frac{\rho}{2C}(1+\frac{CM}{\rho})^2\,,$$

which, together with (2.29) *gives*

$$B(\rho)^2=\frac{r^2}{\rho^2}=\frac{1}{4C^2}(1+\frac{CM}{\rho})^4\,.$$

We now choose $C=\frac{1}{2}$ such that $B(\rho)^2\to 1$ for $\rho\to\infty$, which is necessary to ensure that the metric (2.28) *agrees with* (2.25) *to leading order. Using also* (2.31) *we now obtain*

$$\begin{aligned}r &= \rho(1+\frac{M}{2\rho})^2 \\ B(\rho)^2 &= (1+\frac{M}{2\rho})^4 \\ A(\rho)^2 &= \frac{(1-\frac{M}{2\rho})^2}{(1+\frac{M}{2\rho})^2}\,,\end{aligned}$$

which finally gives the metric

$$\begin{aligned}g &= -\frac{(1-\frac{M}{2\rho})^2}{(1+\frac{M}{2\rho})^2}\,dt^2+(1+\frac{M}{2\rho})^4\left(d\rho^2+\rho^2(d\theta^2+\sin^2\theta\,d\varphi^2)\right) \\ &= -\big(1-\frac{2M}{\rho}+\mathcal{O}(\frac{1}{\rho^2})\big)\,dt^2+\big(1+\frac{2M}{\rho}+\mathcal{O}(\frac{1}{\rho^2})\big)(dx^2+dy^2+dz^2)\,,\end{aligned}$$

where we have Taylor expanded in the second line. Comparison with (2.24) *shows that M is the total mass and that $\overrightarrow{J}=0$. The coordinates (t,ρ,θ,φ) we have constructed are called* ***isotropic coordinates*** *for Schwarzschild.*

2.5 Gravitational waves

We now consider small <u>vacuum</u> perturbations of Minkowski spacetime, i.e., $T_{\mu\nu}=0$. Recall from (2.7) the linearised Einstein equations in the wave gauge

$$\begin{aligned}\Box\tilde{h}_{\mu\nu} &= 0 \\ \partial^\mu\bar{\tilde{h}}_{\mu\nu} &= 0\,.\end{aligned}$$

Also recall from Section 2.2 that we have the residual gauge freedom of choosing an infinitesimal diffeomorphism $\chi \in \mathfrak{X}^\infty(\mathbb{R}^4)$ which satisfies $\Box \chi_\mu = 0$. We can then go over to

$$\hat{h}_{\mu\nu} := \tilde{h}_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu$$

which is physically equivalent to the perturbation $\tilde{h}_{\mu\nu}$ and still in wave gauge $\partial^\mu \overline{\hat{h}}_{\mu\nu} = 0$.

<u>We show:</u> One can choose χ (depending on $\tilde{h}_{\mu\nu}$) such that $\hat{h}_{0\mu} = 0 = \hat{h}$ for $\mu = 0, 1, 2, 3$. This is called the **radiation gauge** and, as we will see, it can only be imposed in vacuum.

<u>Idea illustrated by Maxwell's equations:</u> Recall from Remark 2.11 that the vacuum Maxwell equations in Lorentz gauge are

$$\Box \tilde{A}_\mu = 0$$
$$\partial^\mu \tilde{A}_\mu = 0$$

and also recall the residual gauge freedom $\hat{A}_\mu := \tilde{A}_\mu + \partial_\mu \chi$ with $\Box \chi = 0$. We can use this freedom to set $\hat{A}_0 = 0$, which is called the **Coulomb** or **radiation gauge**:

Let χ be a solution of $\Box \chi = 0$ for which we will specify initial data at $\{t = 0\}$. Then $\hat{A}_0$ satisfies $\Box \hat{A}_0 = 0$. If we can arrange for

$$\hat{A}_0\big|_{t=0} = 0 = \partial_t \hat{A}_0\big|_{t=0} ,$$

then we obtain $\hat{A}_0 \equiv 0$ by the uniqueness of solutions to the linear wave equation. We have

$$\hat{A}_0\big|_{t=0} = \tilde{A}_0\big|_{t=0} + \partial_t \chi\big|_{t=0}$$

and

$$\partial_t \hat{A}_0\big|_{t=0} = \partial_t \tilde{A}_0\big|_{t=0} + \partial_t^2 \chi\big|_{t=0} = \partial^i \tilde{A}_i\big|_{t=0} + \Delta \chi\big|_{t=0} .$$

We can now solve Poisson's equation $\Delta \chi_0 = -\partial^i \tilde{A}_i|_{t=0}$ on $\{t = 0\}$ to obtain the initial data $\chi|_{t=0} = \chi_0$. Solving then the linear wave equation $\Box \chi = 0$ with initial data $\chi|_{t=0} = \chi_0$ and $\partial_t \chi|_{t=0} = -\tilde{A}_0|_{t=0}$ gives the wanted gauge function χ which puts $\tilde{A}$ into the radiation gauge.

We now carry out this strategy for linearised gravity: The conditions $\hat{h} = \partial_t \hat{h} = \hat{h}_{0i} = \partial_t \hat{h}_{0i} = 0$ on $\{t = 0\}$ are given by

$$(-\partial_t \chi_0 + \partial^i \chi_i) = -\tilde{h} \tag{2.32}$$
$$2(-\Delta \chi_0 + \partial^i \partial_t \chi_i) = -\partial_t \tilde{h} \tag{2.33}$$
$$\partial_t \chi_i + \partial_i \chi_0 = -\tilde{h}_{0i} \tag{2.34}$$
$$\Delta \chi_i + \partial_i \partial_t \chi_0 = -\partial_t \tilde{h}_{0i} . \tag{2.35}$$

We show that we can solve this on $\{t = 0\}$ to obtain $\chi_\mu|_{t=0}$ and $\partial_t \chi_\mu|_{t=0}$: Combining (2.34) and (2.33) gives $2(-2\Delta \chi_0 - \partial^i \tilde{h}_{0i}) = -\partial_t \tilde{h}$. Solving Poisson's equation gives $\chi_0|_{t=0}$ in terms of $\tilde{h}_{\mu\nu}$. Using (2.34) then gives $\partial_t \chi_i|_{t=0}$. Similarly, combining (2.32) and (2.35) gives a Poisson equation for $\chi_i|_{t=0}$ with a right hand side which only depends on $\tilde{h}_{\mu\nu}$. Solving it gives $\chi_i|_{t=0}$, and together with (2.32) this also determines $\partial_t\chi_0|_{t=0}$.

We now solve $\Box\chi_\mu = 0$ with the determined initial data on $\{t = 0\}$ to obtain the infinitesimal diffeomorphim $\chi \in \mathfrak{X}^\infty(\mathbb{R}^4)$ and set $\hat{h}_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_\mu\chi_\nu + \partial_\nu\chi_\mu$. Since $\hat{h}_{\mu\nu}$ is still in wave gauge, it satisfies the linearised *vacuum* Einstein equations (2.5), i.e.,

$$\Box\hat{h}_{\mu\nu} = 0 \ .$$

By construction of χ we have

$$\begin{aligned}\hat{h}|_{t=0} &= 0 = \partial_t\hat{h}|_{t=0}\\ \hat{h}_{0i}|_{t=0} &= 0 = \partial_t\hat{h}_{0i}|_{t=0}\end{aligned}$$

and thus $\hat{h} \equiv 0 \equiv \hat{h}_{0i}$.[25]

Since we assumed that $T_{\mu\nu}$ vanishes throughout the spacetime, we also obtain $\hat{h}_{00}$ as a consequence: Since $\hat{h} = 0$, we have $\hat{h}_{\mu\nu} = \bar{\hat{h}}_{\mu\nu}$, and thus, by the wave gauge property

$$0 = \partial^\mu\bar{\hat{h}}_{\mu 0} = \partial^\mu\hat{h}_{\mu 0} = \partial^0\hat{h}_{00} \ ,$$

where we used $\hat{h}_{0i} = 0$ in the last equality. It thus follows that $\partial_t\hat{h}_{00} = 0$ and thus the wave part of the linearised Einstein equations for $\hat{h}_{00}$ becomes

$$0 = \Box\hat{h}_{00} = \Delta\hat{h}_{00} \ .$$

Since we use the boundary condition $\hat{h}_{\mu\nu} \to 0$ for $|\underline{x}| \to \infty$, the unique solution is $\hat{h}_{00} \equiv 0$.

Example 2.36. *We consider plane wave solutions $\hat{h}_{\mu\nu}(x) = \mathrm{Re}\big(\hat{H}_{\mu\nu}(k)e^{ik_\rho x^\rho}\big)$, where $k = (k_0, k_1, k_2, k_3) \in \mathbb{R}^4$.*[26] *Then*

$$\begin{aligned}
&\Box\hat{h}_{\mu\nu} = 0 && \iff \eta^{\rho\sigma}k_\rho k_\sigma = 0 && \text{hence } k \text{ has to be a null vector}\\
&\hat{h} = 0 && \iff \hat{H}^\mu{}_\mu(k) = 0 && \text{(1 condition)}\\
&\hat{h}_{\mu 0} = 0 && \iff \hat{H}_{0\mu}(k) = 0 && \text{(4 conditions)}\\
&\partial^\mu\bar{\hat{h}}_{\mu\nu} = \partial^\mu\hat{h}_{\mu\nu} = 0 && \iff k^\mu\hat{H}_{\mu\nu}(k) = 0 && \underbrace{(k^\mu\hat{H}_{\mu i}(k) = 0}_{\text{3 conditions}}, \quad k^\mu\hat{H}_{\mu 0}(k) = 0 \text{ implied by above.})
\end{aligned}$$

Since $\hat{H}_{\mu\nu}(k)$ is symmetric, it has 10 degrees of freedom. The above impose $1 + 4 + 3 = 8$ constraints on $\hat{H}_{\mu\nu}(k)$, thus a gravitational plane wave has 2 degrees of freedom. For example if we consider a gravitational plane wave propagating in x_3 direction, i.e., $k = \omega(\partial_t + \partial_3)$, then

$$\hat{H}_{\mu\nu}(k) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & A & B & 0\\ 0 & B & -A & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

with $A, B \in \mathbb{R}$. Note that the distortion of the spacetime geometry is transverse to the direction of propagation.

[25] Note that this argument only works in vacuum regions.

[26] Note that $\hat{h}_{\mu\nu}$ does not satisfy the boundary conditions $\hat{h}_{\mu\nu} \to 0$ for $|\underline{x}| \to \infty$. However, as usual, physical solutions can be expressed as a superposition of such plane wave solutions.

2.5.1 Detection of gravitational waves via gravitational tidal forces

We measure the variation of "distance" between two nearby freely falling objects A & B which are initially at rest with respect to the locally inertial frame in which we conduct the measurement. Here, "distance" is measured via the radar method in terms of how long a light ray, emitted from A and reflected by B, takes to reach back to A. In this way very precise measurements are possible with interferometers. We present two ways of computing this effect:

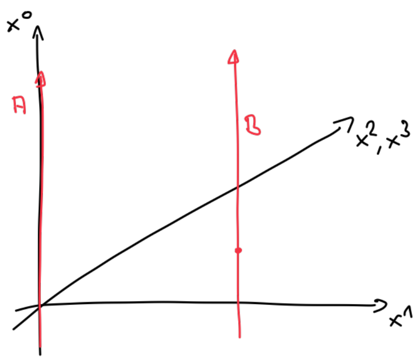
First method: Recall that $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon\hat{h}_{\mu\nu}$, where the perturbation is in the radiation gauge, i.e., $\hat{h}_{0\mu} = 0$ for $\mu = 0, 1, 2, 3$ and $\hat{h} = 0$. The worldlines of A and B are given by affinely parametrised timelike geodesics $\tau \overset{\gamma}{\mapsto} (\gamma^\mu(\tau))$ which have small velocities compared to the speed of light, i.e., $\left|\frac{d\gamma^i}{d\tau}\right| \ll \left|\frac{d\gamma^0}{d\tau}\right| \simeq 1$. We thus obtain

$$\ddot{\gamma}^\mu = -\Gamma^\mu_{\nu\kappa}\dot{\gamma}^\nu\dot{\gamma}^\kappa \simeq -\Gamma^\mu_{00} \, .$$

The radiation gauge implies that

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\sigma}(2\partial_0\hat{h}_{\sigma 0} - \partial_\sigma\hat{h}_{00}) = 0 \, ,$$

and thus we obtain that *in the radiation gauge test particles, which are initially at rest, remain at rest* ***in the coordinates*** x^μ.

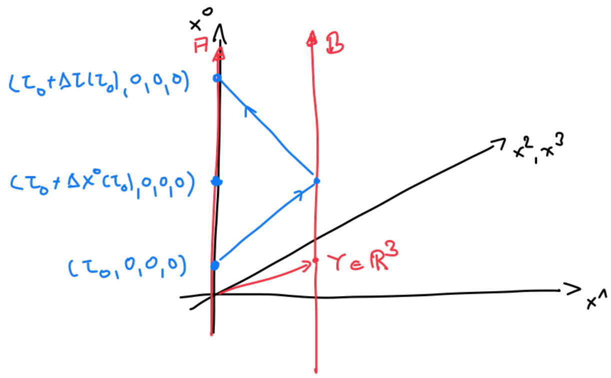


But recall that coordinates themselves (spacetime points themselves) do not have physical reality, but only in conjunction with the metric. So this result by no means implies that the particles remain at rest in any physical sense. Merely our choice of gauge (the shifting of the metric by an infinitesimal diffeomorphism) has been chosen such that in the x^μ coordinates test particles remain at rest. We infer the physical result from the geometry:

Since the 0-components of $\hat{h}_{\mu\nu}$ vanish, we obtain that $\hat{h}$ reduces to a bilinear form $\overline{\hat{h}}$ on $\mathbb{R}^3$. We can thus write

$$\begin{aligned} g &= -dx_0^2 + \underbrace{dx_1^2 + dx_2^2 + dx_3^2}_{=:\overline{\eta}} + \varepsilon\hat{h} \\ &= -dx_0^2 + (\overline{\eta} + \varepsilon\overline{\hat{h}}) \\ &=: -dx_0^2 + \overline{g} \, , \end{aligned}$$

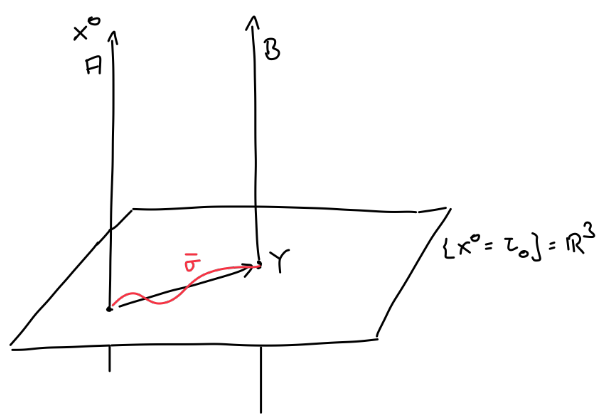
where $\overline{g}$ is a x_0-dependent Riemannian metric on $\mathbb{R}^3$. Up to corrections of order $\mathcal{O}(\varepsilon^2)$ the affinely parametrised timelike geodesic tracing out the worldline of A can be taken to be $\tau \mapsto (\tau, 0, 0, 0)$; and $\tau \mapsto (\tau, Y^1, Y^2, Y^3)$ for B, with $Y \in \mathbb{R}^3$. Let $s \overset{\sigma}{\mapsto} (\sigma^0(s), \overline{\sigma}(s))$ be the unique future directed null geodesic connecting $(\tau_0, 0, 0, 0)$ on A with B.



We now assume that the time of flight of the photon is small compared to the characteristic period of the gravitational wave, i.e., we can neglect the x^0-dependence of $\overline{g}$ and replace $\overline{g}$ by $\overline{g}|_{x^0=\tau_0}$. Thus, σ becomes the future directed null geodesic *with respect to the Lorentzian metric* $g|_{x^0=\tau_0} = -dx_0^2 + \overline{g}|_{x^0=\tau_0}$. Then $0 = g(\dot{\sigma}, \dot{\sigma}) = -(\dot{\sigma}^0)^2 + \overline{g}|_{x^0=\tau_0}(\dot{\overline{\sigma}}, \dot{\overline{\sigma}})$ and thus

$$\Delta x^0(\tau_0) \simeq \int \dot{\sigma}^0 \, ds = \int \sqrt{\overline{g}|_{x^0=\tau_0}(\dot{\overline{\sigma}}, \dot{\overline{\sigma}})} \, ds \, . \tag{2.37}$$

By Problem 8 on Sheet 2 $s \mapsto \overline{\sigma}(s)$ is a Riemannian geodesic in $(\mathbb{R}^3, \overline{g}|_{x^0=\tau_0})$, connecting 0 and Y. Up to parametrisation it is thus a small perturbation of $[0,1] \ni s \mapsto s \cdot Y$.



Since the right hand side of (2.37)[27] is independent of the parametrisation of $\overline{\sigma}$, we can reparametrise it such that

$$[0,1] \ni s \mapsto \dot{\overline{\sigma}}(s) = Y + \underbrace{\mathcal{O}(\varepsilon) Y^\perp(s)}_{\text{orthogonal perturbation}} \, ,$$

[27]Indeed, it is not difficult to see that the right hand side of (2.37) is the proper distance between 0 and Y in $(\mathbb{R}^3, \overline{g}|_{x^0=\tau_0})$.

where $\overline{\eta}(Y, Y^\perp) = 0$. Then

$$\overline{g}|_{x^0=\tau_0}(\dot{\overline{\sigma}}, \dot{\overline{\sigma}}) = \overline{\eta}(Y, Y) + 2\mathcal{O}(\varepsilon) \cdot \underbrace{\overline{\eta}(Y, Y^\perp)}_{=0} + \varepsilon \hat{h}(Y, Y) + \mathcal{O}(\varepsilon^2) \, .$$

Thus,

$$\begin{aligned}
\int \sqrt{\overline{g}|_{x^0=\tau_0}(\dot{\overline{\sigma}}, \dot{\overline{\sigma}})}\, ds &= \int_0^1 \sqrt{\overline{\eta}(Y, Y) + \varepsilon \hat{h}|_{(\tau_0, \overline{\sigma}(s))}(Y, Y)}\, ds + \mathcal{O}(\varepsilon^2) \\
&= \sqrt{\overline{\eta}(Y, Y)} \int_0^1 \sqrt{1 + \frac{\varepsilon \hat{h}|_{(\tau_0, \overline{\sigma}(s))}(Y, Y)}{\overline{\eta}(Y, Y)}}\, ds + \mathcal{O}(\varepsilon^2) \\
&= \sqrt{\overline{\eta}(Y, Y)} + \frac{\varepsilon}{2\sqrt{\overline{\eta}(Y, Y)}} \cdot \int_0^1 \overline{\hat{h}}|_{(\tau_0, \overline{\sigma}(s))}(Y, Y)\, ds + \mathcal{O}(\varepsilon^2) \, ,
\end{aligned}$$

where in the last step we have expanded the square root $\sqrt{1+x} = 1 + \frac{1}{2}x + \mathcal{O}(x^2)$.

Assuming now that the separation of the particles A and B is small compared to the wavelength of the gravitational wave we obtain

$$\int \sqrt{\overline{g}|_{x^0=\tau_0}(\dot{\overline{\sigma}}, \dot{\overline{\sigma}})}\, ds \simeq \sqrt{\overline{\eta}(Y, Y)} + \frac{\varepsilon}{2\sqrt{\overline{\eta}(Y, Y)}} \hat{h}|_{(\tau_0, 0,0,0)}(Y, Y) + \mathcal{O}(\varepsilon^2) \, . \tag{2.38}$$

Taking into account also the reflected light ray and also that, by virtue of the radiation gauge, coordinate time along the worldline of A is proper time, we finally obtain

$$\Delta\tau(\tau_0) = 2\Delta x^0(\tau_0) \simeq 2\sqrt{\overline{\eta}(Y, Y)} + \frac{\varepsilon}{\sqrt{\overline{\eta}(Y, Y)}} \hat{h}|_{(\tau_0, 0,0,0)}(Y, Y) \, ,$$

which can be measured.

In the LIGO interferometers the test masses are $4km = \sqrt{\overline{\eta}(Y, Y)}$ apart. A typical frequency of an observed gravitational wave is of order $f \simeq 100Hz$, so that the first assumption we made, namely that the time of flight of the photon is small compared to the characteristic period of the gravitational wave, is satisfied. We obtain for the wavelength of such a gravitational wave $\lambda \simeq 1000km$, so that the second assumption we made, namely that the separation of the test masses is small compared to the wavelength of the gravitational wave, is also satisfied. The gravitational waves observed by LIGO induce a change of proper distance of order $\frac{1}{1000} \times$ (diameter of proton) $= 10^{-18}m$ in the interferometer arms of $4km$ length.

Second method: We change the coordinates x^μ, which are not directly associated with measurements, to locally inertial coordinates y^μ along A's worldline, which are locally associated to our familiar special relativistic measurements. We choose $e_0 = \frac{\partial}{\partial x^0}$ and $e_i = \frac{\partial}{\partial x^i} + \mathcal{O}(\varepsilon)$ as an orthonormal frame field along A's worldline, which induces locally inertial coordinates y^α in a small neighbourhood with

$$\begin{aligned}
\frac{\partial}{\partial t} := \frac{\partial}{\partial y^0} &= \frac{\partial}{\partial x^0} \\
\frac{\partial}{\partial y^i} &= \frac{\partial}{\partial x^i} + \mathcal{O}(\varepsilon)
\end{aligned} \tag{2.39}$$

along the worldline of A, see Problem 7 on Problem sheet 1.

Assuming that the distance of A and B is sufficiently small we can use the Jacobi equation

$$D_t^2 Y = R(\partial_t, Y)\partial_t$$

along A to describe the spatial separation of A and B, where Y is the geodesic deviation vector along A. Using that we have $\Gamma^{y^\mu}_{y^\nu y^\sigma} = 0$ along A we obtain *in the locally inertial coordinates*

$$\frac{d^2}{dt^2} Y^{y^\mu} = R^{y^\mu}{}_{y^0 y^0 y^\nu} Y^{y^\nu} \ .$$

Because of $R^{y^0}{}_{y^0 y^0 y^\nu} = 0$ and $R^{y^\mu}{}_{y^0 y^0 y^0} = 0$ this reduces to

$$\begin{aligned} \frac{d^2}{dt^2} Y^{y^0} &= 0 \\ \frac{d^2}{dt^2} Y^{y^i} &= R^{y^i}{}_{y^0 y^0 y^j} Y^{y^j} \ . \end{aligned} \tag{2.40}$$

We now compute $R^{y^i}{}_{y^0 y^0 y^j}$ and show that if the tensor $\hat{h}$ is in the radiation gauge, then (2.40) can be easily integrated. Using (2.39) we obtain

$$R^{y^i}{}_{y^0 y^0 y^j} = R^{x^i}{}_{x^0 x^0 x^j} + \mathcal{O}(\varepsilon^2) \ . \tag{2.41}$$

In other words, if the change of coordinates is the identity up to order ε, then the components of the Riemann curvature tensor are invariant in linearised gravity, since they are already of order ε.

Recall from (2.3) that <u>in the fixed background coordinates</u> x^μ we have

$$R^\mu{}_{\kappa\rho\nu} \simeq \varepsilon \frac{1}{2} \eta^{\mu\sigma} (\partial_\rho \partial_\kappa \hat{h}_{\nu\sigma} - \partial_\rho \partial_\sigma \hat{h}_{\nu\kappa} - \partial_\nu \partial_\kappa \hat{h}_{\rho\sigma} + \partial_\nu \partial_\sigma \hat{h}_{\rho\kappa}) \ .$$

In the radiation gauge we have $\hat{h}_{\mu 0} = 0$, and thus we obtain

$$\begin{aligned} R^i{}_{00j} &\simeq \varepsilon \frac{1}{2} \eta^{i\sigma} \big(\partial_0 \partial_0 \hat{h}_{j\sigma} - \partial_0 \partial_\sigma \hat{h}_{j0} - \partial_j \partial_0 \hat{h}_{0\sigma} + \partial_j \partial_\sigma \hat{h}_{00}\big) \\ &= \varepsilon \frac{1}{2} \partial_0 \partial_0 \hat{h}_{ji} \ . \end{aligned}$$

Using again (2.39) we thus obtain $R^{x^i}{}_{x^0 x^0 x^j} \simeq \varepsilon \frac{1}{2} \partial_{x^0} \partial_{x^0} \hat{h}_{x^j x^i} \simeq \varepsilon \frac{1}{2} \partial_{y^0} \partial_{y^0} \hat{h}_{y^j y^i}$, so that (2.40) becomes

$$\begin{aligned} \frac{d^2}{dt^2} Y^{y^0} &= 0 \\ \frac{d^2}{dt^2} Y^{y^i} &\simeq \frac{1}{2} \varepsilon \big(\frac{\partial^2}{\partial t^2} \hat{h}_{y^i y^j}\big) Y^{y^j} \ . \end{aligned} \tag{2.42}$$

We now assume that the geodesic deviation vector Y satisfies initially

$$Y^{y^0}(0) = 0 \qquad \text{and} \qquad \frac{d}{dt} Y^{y^0}(0) = 0 \ , \tag{2.43}$$

i.e., the internal clocks of the two test bodies A and B are initially synchronised in the locally inertial frame of A[28], and

$$\frac{d}{dt} Y^{y^i}(0) = 0 \ , \tag{2.44}$$

[28]This is not an important point.

i.e., the test body B is initially at rest in the locally inertial frame of A. (2.43) and (2.42) together directly give $Y^0 \equiv 0$. Since the right hand side of (2.42) is of order ε and is integrated over a finite time, and by (2.44), we obtain $Y^{y^i}(t) = Y^{y^i}(0) + \mathcal{O}(\varepsilon)$. Inserting this into (2.42) we obtain to leading order

$$\frac{d^2}{dt^2} Y^{y^i} \simeq \frac{1}{2}\varepsilon\big(\frac{\partial^2}{\partial t^2}\hat{h}_{y^i y^j}\big) Y^{y^j}(0) \, .$$

Integrating and using (2.44) we finally obtain

$$Y^{y^i}(t) \simeq Y^{y^i}(0) + \frac{\varepsilon}{2}\hat{h}_{y^i y^j}(t,0,0,0) Y^{y^j}(0) \, . \tag{2.45}$$

Note that this formula is only valid if the metric perturbation $\hat{h}$ is in the radiation gauge. Also note that the distance to B in the locally inertial coordinate system of A is given by

$$\begin{aligned}
\sqrt{\eta(Y(t),Y(t))} &= \sqrt{\overline{\eta}(Y(t),Y(t))} \\
&= \sqrt{\overline{\eta}(Y(0),Y(0)) + \varepsilon\hat{h}(t)(Y(0),Y(0)) + \mathcal{O}(\varepsilon^2)} \\
&= \sqrt{\overline{\eta}(Y(0),Y(0))}\Big(\sqrt{1 + \varepsilon\frac{\hat{h}(t)(Y(0),Y(0))}{\overline{\eta}(Y(0),Y(0))}} + \mathcal{O}(\varepsilon^2)\Big) \\
&= \sqrt{\overline{\eta}(Y(0),Y(0))}\Big(1 + \frac{\varepsilon}{2}\frac{\hat{h}(t)(Y(0),Y(0))}{\overline{\eta}(Y(0),Y(0))}\Big) + \mathcal{O}(\varepsilon^2) \, ,
\end{aligned}$$

which agrees with (2.38) from method 1. Again we can measure the distance using the radar method – and using that the speed of light in the locally inertial coordinate system can be taken to be equal to 1 for small distances we obtain the same result as before if we again make the assumption that the time of flight of the photon is small compared to the period of the gravitational wave.

Example 2.46. *We evaluate* (2.45) *for the gravitational plane wave travelling in x_3-direction from Example 2.36, i.e., for*

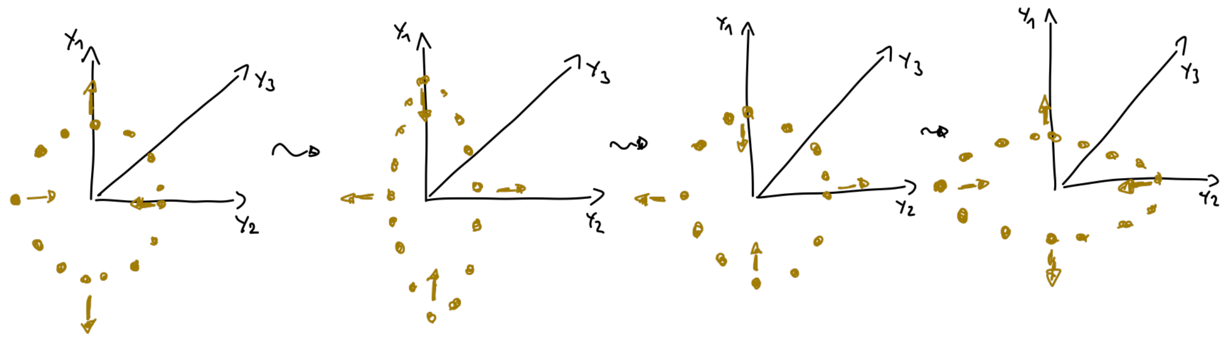
$$\hat{h}_{\mu\nu}(t,\underline{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & B & -A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \mathrm{Re}\big(e^{-i\omega(t-x_3)}\big) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & B & -A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \cos\big(\omega(t-x_3)\big) \, .$$

We distinguish the following two linear polarisations:

i) $B = 0$, the "+" polarisation. Then

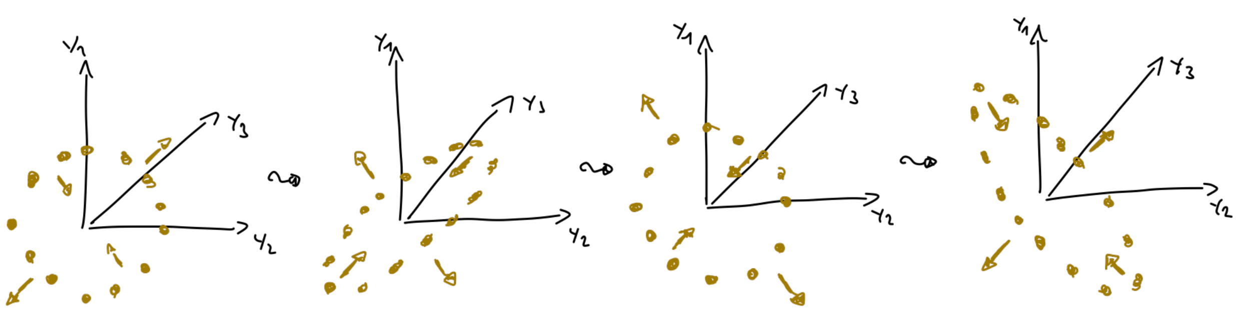
$$\begin{aligned}
Y^1(t) &\simeq Y^1(0) + \frac{\varepsilon}{2} A \cdot \cos\big(\omega(t-x_3)\big) Y^1(0) \\
Y^2(t) &\simeq Y^2(0) - \frac{\varepsilon}{2} A \cdot \cos\big(\omega(t-x_3)\big) Y^2(0) \\
Y^3(t) &\simeq Y^3(0) \, .
\end{aligned}$$

Note that the coordinate components are here with respect to the locally inertial coordinates y^μ and that x_3 is constant along the worldline of A.



ii) $A=0$, the "$\times$" polarisation. Then

$$\begin{aligned}
Y^1(t) &\simeq Y^1(0) + \frac{\varepsilon}{2} B \cdot \cos\big(\omega(t-x_3)\big) Y^2(0) \\
Y^2(t) &\simeq Y^2(0) + \frac{\varepsilon}{2} B \cdot \cos\big(\omega(t-x_3)\big) Y^1(0) \\
Y^3(t) &\simeq Y^3(0)\ .
\end{aligned}$$



For circular polarisations see Misner-Thorne-Wheeler p.952.

2.5.2 Generation of gravitational waves

The gravitational waves produced in the collision of two black holes is outside the validity of linearised gravity. Here, we derive the **quadrupole formula**, which describes the generation of gravitational waves under the following assumptions:

1) The assumptions of linearised gravity are met, i.e., the gravitational field is weak, the mass-energy density and the material stresses are small.

2) The system is isolated i.e., $\overset{(1)}{T}_{\mu\nu}(t,\underline{x})$ is compactly supported for all $t \in \mathbb{R}$.

3) The ε^2 terms in $\nabla_\mu T^{\mu\nu} = \varepsilon \partial_\mu \overset{(1)}{T}{}^{\mu\nu} + \mathcal{O}(\varepsilon^2)$ are indeed negligible, i.e., the system is *non-self-gravitating.*[29] Examples are spinning rods and rotating (inhomogeneous) stars.

[29] Note that this assumption is not compatible with, for example, a binary system where the two objects move on curved trajectories due to the influence of each other's gravity. If $\partial_\mu T^{\mu\nu} = 0$ held, they would be moving on straight lines (problem sheet 2). Nevertheless, there is evidence that the quadrupole formula is still a good approximation in those cases. However, those cases cannot be dealt with within linearised gravity.

4) The system has a characteristic frequency of change ω. If R_0 is the radius of its support, then $R_0 \cdot \omega \ll 1$ holds.[30]

5) We restrict to the far field.

It follows from 3) together with problem sheet 2 that the total momentum $P^i = \int_{\mathbb{R}^3} T^{0i}(t, \underline{x})\, d\underline{x}$ is conserved and, after a boost of the background coordinates, can be assumed to vanish. After a translation of the background coordinates we can assume that the centre of mass $D^i(t) = \int_{\mathbb{R}^3} T^{00}(t, \underline{x}) x^i\, d\underline{x}$ is at the origin.

Recall the linearised Einstein equations in wave gauge, (2.9):

$$\begin{aligned} \Box \bar{\tilde{h}}_{\mu\nu} &= -16\pi \overset{(1)}{T}_{\mu\nu} \\ \partial^\mu \bar{\tilde{h}}_{\mu\nu} &= 0\ . \end{aligned}$$

We solve the first equation using the retarded solution (i.e., no incoming radiation):

$$\bar{\tilde{h}}_{\mu\nu}(t, \underline{x}) = 4 \int_{\mathbb{R}^3} \frac{\overset{(1)}{T}_{\mu\nu}(t - |\underline{x}' - \underline{x}|, \underline{x}')}{|\underline{x}' - \underline{x}|}\, d\underline{x}'\ . \tag{2.47}$$

Recall from Remark 2.8 that the second equation is then automatically satisfied by our assumption 3).

Firstly, we only set out to describe the far field, so we expand $\frac{1}{|\underline{x}' - \underline{x}|}$ in powers of $\frac{1}{r}$. To leading order we have $\frac{1}{|\underline{x}' - \underline{x}|} \simeq \frac{1}{r}$. As will transpire, the gravitational wave part is visible at this order already so that we can neglect all higher order corrections.

Secondly, the dependency on $t - |\underline{x}' - \underline{x}|$ in the first argument of $\overset{(1)}{T}_{\mu\nu}$ still prevents an evaluation of $\bar{\tilde{h}}_{\mu\nu}(t, \underline{x})$ in terms of the moments of the stress-energy tensor. Thus we expand $\overset{(1)}{T}_{\mu\nu}(t - r|\frac{\underline{x}'}{r} - \frac{\underline{x}}{r}|, \underline{x}')$ for $\underline{y}' := \frac{\underline{x}'}{r}$ small, which is justified since $|\underline{x}|$ is large and $|\underline{x}'| \leq R_0$.

Let $f(\underline{y}') := t - r\big|\underline{y}' - \frac{\underline{x}}{r}\big|$. We begin by computing

$$\begin{aligned} \partial_j f(\underline{y}') &= -r \frac{[y'_j - \frac{x_j}{r}]}{|\underline{y}' - \frac{\underline{x}}{r}|} = \mathcal{O}(r) & \partial_j f(0) &= x_j \\ \partial_k \partial_j f(\underline{y}') &= -r \frac{\delta_{jk}}{|\underline{y}' - \frac{\underline{x}}{r}|} + r \frac{[y'_j - \frac{x_j}{r}][y'_k - \frac{x_k}{r}]}{|\underline{y}' - \frac{\underline{x}}{r}|^3} = \mathcal{O}(r) & \partial_k \partial_j f(0) &= -r\delta_{jk} + \frac{x_j x_k}{r} \\ \partial_l \partial_k \partial_j f(\underline{y}') &= \mathcal{O}(r)\ . \end{aligned}$$

We now Taylor-expand $\overset{(1)}{T}_{\mu\nu}\big(f(\underline{y}'), \underline{x}'\big)$ in $\underline{y}'$ around 0. We compute

$$\begin{aligned} \partial_j \overset{(1)}{T}_{\mu\nu}\big(f(\underline{y}'), \underline{x}'\big) &= \partial_0 \overset{(1)}{T}_{\mu\nu}\big(f(\underline{y}'), \underline{x}'\big) \cdot \partial_j f(\underline{y}') \\ \partial_k \partial_j \overset{(1)}{T}_{\mu\nu}\big(f(\underline{y}'), \underline{x}'\big) &= \partial_0^2 \overset{(1)}{T}_{\mu\nu}\big(f(\underline{y}'), \underline{x}'\big) \cdot \partial_j f(\underline{y}') \cdot \partial_k f(\underline{y}') + \partial_0 \overset{(1)}{T}_{\mu\nu}\big(f(\underline{y}'), \underline{x}'\big) \partial_k \partial_j f(\underline{y}')\ , \end{aligned}$$

[30] Note that this assumption is equivalent to the typical wavelength of the radiation being much larger than the extent of the source.

which gives

$$\overset{(1)}{T}_{\mu\nu}\big(f(\underline{y}'),\underline{x}'\big) = \overset{(1)}{T}_{\mu\nu}\big(f(0),\underline{x}'\big) + \partial_0\overset{(1)}{T}_{\mu\nu}\big(f(0),\underline{x}'\big)\cdot\partial_j f(0)\cdot y'_j$$
$$+\frac{1}{2}\Big[\partial_0^2\overset{(1)}{T}_{\mu\nu}\big(f(0),\underline{x}'\big)\cdot\underbrace{\partial_k f(0)\cdot\partial_j f(0)}_{=\mathcal{O}(r^2)}+\partial_0\overset{(1)}{T}_{\mu\nu}\big(f(0),\underline{x}'\big)\cdot\underbrace{\partial_k\partial_j f(0)}_{=\mathcal{O}(r)}\Big]\cdot\underbrace{y'_j y'_k}_{=\mathcal{O}(\frac{1}{r^2})}$$
$$+\frac{1}{6}\Big[\partial_0^3\overset{(1)}{T}_{\mu\nu}\big(f(\underline{\xi}'),\underline{x}'\big)\cdot\underbrace{\partial_j f(\underline{\xi}')\cdot\partial_k f(\underline{\xi}')\cdot\partial_l f(\underline{\xi}')}_{=\mathcal{O}(r^3)}+\mathcal{O}(r^2)\Big]\cdot\underbrace{y'_j y'_k y'_l}_{=\mathcal{O}(\frac{1}{r^3})}\,,$$

for some $\underline{\xi}' = \alpha\underline{y}'$ with $\alpha\in(0,1)$.

The system having a characteristic frequency ω implies $|\partial_0^k\overset{(1)}{T}_{\mu\nu}|\sim\omega^k|\overset{(1)}{T}_{\mu\nu}|$. Reinstating $\underline{y}'=\frac{\underline{x}'}{r}$ in the above Taylor expansion and noticing that $|\omega\underline{x}'|\lesssim\omega\cdot R_0$, we keep the terms of order 0 in the small value of $\frac{1}{r}$ and terms of order up to 2 in the small value of ωR_0, c.f. assumption 4):

$$\begin{aligned}\overset{(1)}{T}_{\mu\nu}\big(t-|\underline{x}'-\underline{x}|,\underline{x}'\big) &= \overset{(1)}{T}_{\mu\nu}(t-r,\underline{x}') + \underbrace{\partial_0\overset{(1)}{T}_{\mu\nu}(t-r,\underline{x}')\cdot x_j\frac{x'_j}{r}}_{=\mathcal{O}(\omega R_0)}\\ &\quad+\underbrace{\frac{1}{2}\partial_0^2\overset{(1)}{T}_{\mu\nu}(t-r,\underline{x}')x_j x_k\frac{x'_j}{r}\frac{x'_k}{r}}_{=\mathcal{O}\big((\omega R_0)^2\big)}+\mathcal{O}(\frac{1}{r})+\mathcal{O}\big((\omega R_0)^3\big)\ .\end{aligned}\tag{2.48}$$

Using (2.48) in (2.47) together with $\frac{1}{|\underline{x}'-\underline{x}|}=\frac{1}{r}+\mathcal{O}(\frac{1}{r^2})$, we compute the trace-reversed metric perturbations:

$$\begin{aligned}\bar{\tilde{h}}_{ij}(t,\underline{x}) &= \frac{4}{r}\int_{\mathbb{R}^3}\overset{(1)}{T}_{ij}(t-r,\underline{x}')\,d\underline{x}'+h.o.t.\\ &= \frac{2}{r}\frac{d^2}{dt^2}\overset{(1)}{Q}_{ij}(t-r)+h.o.t.\ ,\end{aligned}$$

where $\overset{(1)}{Q}_{ij}(t)=\int_{\mathbb{R}^3}\overset{(1)}{T}_{00}(t,\underline{x}')x'_i x'_j\,d\underline{x}'$ is the **quadrupole moment**. Here we have used Problem 6 on problem sheet 2.

$$\begin{aligned}\bar{\tilde{h}}_{0i}(t,\underline{x}) &= \underbrace{\frac{4}{r}\int_{\mathbb{R}^3}\overset{(1)}{T}_{0i}(t-r,\underline{x}')\,d\underline{x}'}_{=-\frac{4}{r}P^i(t-r)=0}+\frac{4}{r}\frac{x_j}{r}\int_{\mathbb{R}^3}\partial_0\overset{(1)}{T}_{0i}(t-r,\underline{x}')x'_j\,d\underline{x}'+h.o.t.\\ &= \frac{4}{r}\frac{x_j}{r}\int_{\mathbb{R}^3}\partial_k\overset{(1)}{T}_{ki}(t-r,\underline{x}')x'_j\,d\underline{x}'+h.o.t. \qquad\text{(using } \partial_\mu\overset{(1)}{T}{}^{\mu\nu}=0\text{)}\\ &= -\frac{4}{r}\frac{x_j}{r}\int_{\mathbb{R}^3}\overset{(1)}{T}_{ji}(t-r,\underline{x}')\,d\underline{x}'+h.o.t.\\ &= -\frac{2}{r}\frac{x_j}{r}\frac{d^2}{dt^2}\overset{(1)}{Q}_{ij}(t-r)+h.o.t.\ .\end{aligned}$$

And finally

$$
\begin{aligned}
\bar{\tilde{h}}_{00}(t,\underline{x}) &= \frac{4}{r}\int\limits_{\mathbb{R}^3} \overset{(1)}{T}_{00}(t-r,\underline{x}')\,d\underline{x}' + \frac{4}{r}\frac{x_i}{r}\underbrace{\int\limits_{\mathbb{R}^3} \partial_0 \overset{(1)}{T}_{00}(t-r,\underline{x}')x_i'\,d\underline{x}'}_{=\frac{d}{dt}D^i(t-r)=P^i(t-r)=0} \\
&\quad + \frac{4}{r}\cdot\frac{1}{2}\frac{x_i}{r}\frac{x_k}{r}\int\limits_{\mathbb{R}^3} \partial_0^2 \overset{(1)}{T}_{00}(t-r,\underline{x}')x_i'x_k'\,d\underline{x}' + h.o.t. \\
&= \frac{4M}{r} + \frac{2x_ix_k}{r^3}\frac{d^2}{dt^2}\overset{(1)}{Q}_{ik}(t-r) + h.o.t.\ ,
\end{aligned}
$$

where we used the definition of the total mass M in the linearised theory and again Problem 6 on problem sheet 2. Note that the first term is time-independent, it is a non-radiating contribution. Collating the expressions for the trace reversed metric perturbations we obtain the **quadrupole formula**:

$$
\begin{aligned}
\bar{\tilde{h}}_{00} &\simeq \frac{4M}{r} + \frac{2x_ix_k}{r^3}\frac{d^2}{dt^2}\overset{(1)}{Q}_{ik}(t-r) \\
\bar{\tilde{h}}_{0i} &\simeq -\frac{2x_j}{r^2}\frac{d^2}{dt^2}\overset{(1)}{Q}_{ij}(t-r) \\
\bar{\tilde{h}}_{ij} &\simeq \frac{2}{r}\frac{d^2}{dt^2}\overset{(1)}{Q}_{ij}(t-r)\ .
\end{aligned}
\tag{2.49}
$$

Remark 2.50. *1. Note that the wave gauge is indeed satisfied to highest order in $\frac{1}{r}$: Using $\partial_i r = \frac{x_i}{r}$ we have*

$$
\begin{aligned}
\partial^i\bar{\tilde{h}}_{ij} &\simeq -\frac{2}{r}\frac{x_i}{r}\frac{d^3}{dt^3}\overset{(1)}{Q}_{ij}(t-r) & \qquad \partial^0\bar{\tilde{h}}_{0j} &\simeq \frac{2x_i}{r^2}\frac{d^3}{dt^3}\overset{(1)}{Q}_{ij}(t-r) \\
\partial^j\bar{\tilde{h}}_{0j} &\simeq \frac{2x_i}{r^2}\frac{x_j}{r}\frac{d^3}{dt^3}\overset{(1)}{Q}_{ij}(t-r) & \qquad \partial^0\bar{\tilde{h}}_{00} &\simeq -\frac{2x_ix_j}{r^3}\frac{d^3}{dt^3}\overset{(1)}{Q}_{ij}(t-r)\ .
\end{aligned}
$$

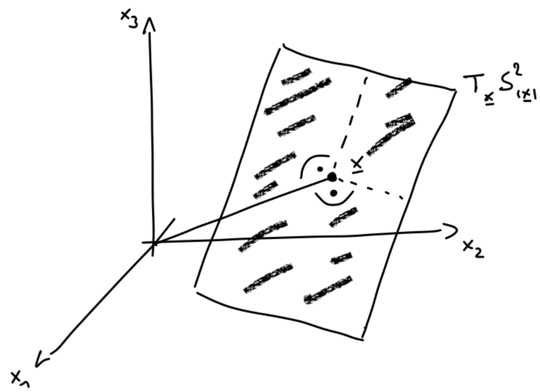
2. Note that the monopole moment $\int\limits_{\mathbb{R}^3}\overset{(1)}{T}_{00}(t,\underline{x}')\,d\underline{x}' = M$*, i.e. the mass, is independent of time in linearised gravity. Also the* dipole moment $\int\limits_{\mathbb{R}^3}\overset{(1)}{T}_{00}(t,\underline{x}')x_i'\,d\underline{x}' = D^i$*, i.e. the centre of mass, is independent of time due to our choice of coordinates*[31]*. Thus, the lowest moment that radiates in linearised general relativity is the quadrupole moment. This is in contrast to electromagnetism, where the dipole moment gives the leading order of radiation, see problem sheet 3.*

Using $\tilde{h}_{\mu\nu} = \bar{\tilde{h}}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\tilde{h}}$ we can now compute $\tilde{h}_{\mu\nu}$ from (2.49) and then the components $R^i{}_{00j}$ of the curvature tensor of $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \varepsilon\tilde{h}_{\mu\nu}$ to leading order in ε and $\frac{1}{r}$. We find (see Problem 2 on problem sheet 3)

$$
R^i{}_{00j} \simeq \frac{\varepsilon}{r}\Big[\Pi^m{}_i\Pi^n{}_j - \frac{1}{2}\Pi^{mn}\Pi_{ij}\Big]\frac{d^4}{dt^4}\overset{(1)}{Q}_{mn}(t-r)\ , \tag{2.51}
$$

where $\Pi^{mn}(\underline{x}) := \delta^{mn} - \frac{x_n}{r}\frac{x_m}{r}$. Note that $\Pi(\underline{x}) : T_{\underline{x}}\mathbb{R}^3 \to T_{\underline{x}}\mathbb{S}^2_{|\underline{x}|}$ is the orthogonal projection in $\mathbb{R}^3$ from the tangent space at $\underline{x}$ onto the tangent space of the sphere of radius $|\underline{x}|$ at the point $\underline{x}$.

[31] Recall that we chose the background coordinates x^μ such that the total momentum vanishes. Otherwise the dipole moment would be a linear function of time.



As in the second method in Section 2.5.1 for computing the gravitational tidal effects of gravitational waves we can set up a locally inertial coordinate system y^μ for a freely falling observer such that

$$\begin{aligned} \frac{\partial}{\partial\tau} := \frac{\partial}{\partial y^0} &= \frac{\partial}{\partial x^0} + \mathcal{O}(\varepsilon) = \frac{\partial}{\partial t} + \mathcal{O}(\varepsilon) \\ \frac{\partial}{\partial y^i} &= \frac{\partial}{\partial x^i} + \mathcal{O}(\varepsilon) \ . \end{aligned} \tag{2.52}$$

holds along the worldline of the freely falling observer. As in the second method, using (2.52), we obtain[32]

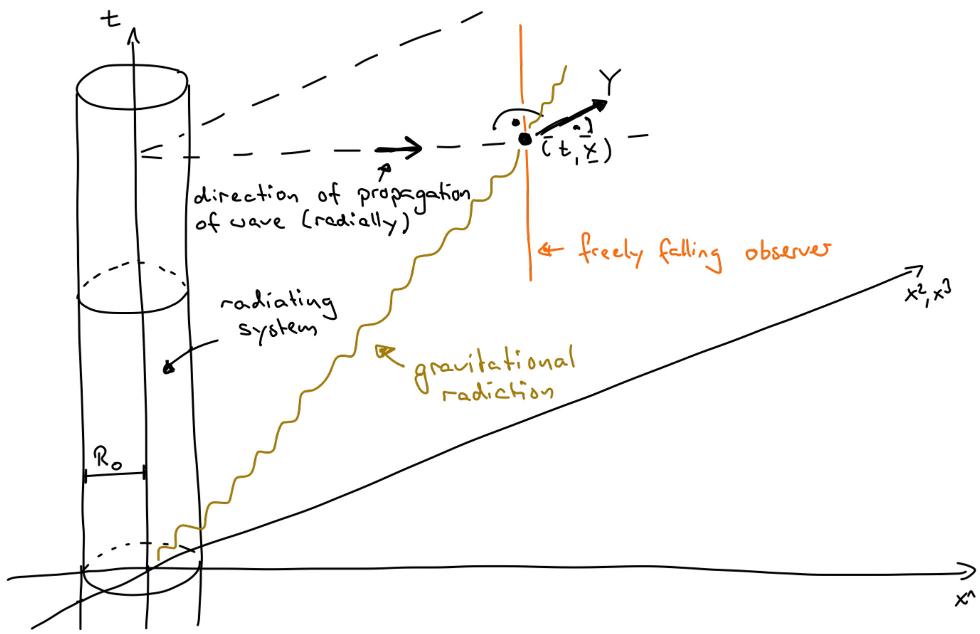
$$\frac{d^2}{d\tau^2}Y^i \simeq R^i{}_{00j}Y^j \simeq \frac{\varepsilon}{r}\Big[\Pi^m{}_i\Pi^n{}_j - \frac{1}{2}\Pi^{mn}\Pi_{ij}\Big]\frac{d^4}{d\tau^4}\overset{(1)}{Q}{}_{mn}(t(\tau)-r(\tau))Y^j \ , \tag{2.53}$$

where the indices here and below are now with respect to the locally inertial coordinates y^μ. Assuming again the initial condition $\frac{d}{d\tau}Y^i(0) = 0$ we obtain again $Y^i(\tau) = Y^i(0) + \mathcal{O}(\varepsilon)$ which we insert into (2.53). Using that $\frac{\partial}{\partial\tau}x_i = \mathcal{O}(\varepsilon)$, we can pull the total derivatives on the right hand side past the projections and then integrate to obtain our final result

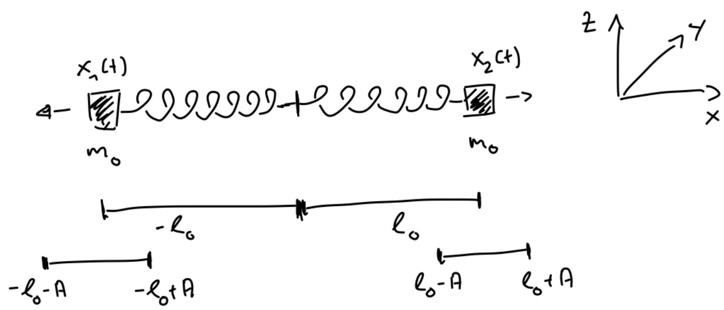
$$Y^i(\tau) \simeq Y^i(0) + \frac{\varepsilon}{r}\Big[\Pi^m{}_i\Pi^n{}_j - \frac{1}{2}\Pi^{mn}\Pi_{ij}\Big]\frac{d^2}{d\tau^2}\overset{(1)}{Q}{}_{mn}(t(\tau)-r(\tau))Y^j(0) \ . \tag{2.54}$$

Note that r is constant up to $\mathcal{O}(\varepsilon)$ in τ and that we have $t(\tau) = \tau + \mathcal{O}(\varepsilon)$. Also note that the gravitational waves described by the quadrupole formula (2.49) propagate radially. It follows directly from $\Pi_{ij}(\underline{x})x^j = 0$ (coordinates with respect to x^μ) and (2.54) that test masses experience only an acceleration in the locally inertial coordinate system if they are in spatial directions orthogonal to the propagation direction of the wave.

[32]Note that the metric perturbation is not in radiation gauge, so the wordline is not of constant spatial coordinates $\underline{x}$. However, (2.52) is sufficient for following the argument in the second method up to (2.41). Then, the radiation gauge was used crucially to integrate the Jacobi equation. Here, we will integrate the Jacobi equation directly.



Example 2.55 (Laboratory gravitational wave generator)**.** *Consider two masses m_0 on elastic springs oscillating with angular frequency ω and amplitude A around positions $\pm l_0$ on the x-axis.*



The positions are $x_1(t) = -l_0 - A\cos(\omega t)$ and $x_2(t) = l_0 + A\cos(\omega t)$. Assuming that the velocities are small compared to the speed of light, the only non-negligible component of the stress energy tensor is

$$T_{00}(t, x, y, z) = m_0\delta^3(x - x_1(t)) + m_0\delta^3(x - x_2(t)) \; .$$

Thus

$$\begin{aligned} Q_{xx}(t) &= \int_{\mathbb{R}^3} T_{00}(t, x', y', z')x'x' \, dx'dy'dz' \\ &= m_0\left[x_1^2(t) + x_2^2(t)\right] \\ &= 2m_0\left[l_0 + A\cos(\omega t)\right]^2 \\ &= 2m_0\left[l_0^2 + 2l_0A\cos(\omega t) + A^2\cos^2(\omega t)\right] \; . \end{aligned}$$

All other components of the quadrupole moment are zero. First note that it then follows from $\Pi^{ix}(x, 0, 0) = 0$ and (2.51) (or (2.54)) that there is no radiation in the x-direction (but of course in the y and z-

directions). We now estimate the strength of the gravitational radiation: we have

$$\frac{d^2}{dt^2}Q_{xx}(t) \sim m_0 l_0 A\omega^2 + m_0 A^2\omega^2 \ . \tag{2.56}$$

Take

$$\begin{aligned} m_0 &= 10^6 g \simeq 0.75 \cdot 10^{-22} cm \\ l_0 &= 10^2 cm \\ A &= 10^{-1} cm \\ \omega &= 10^4 s^{-1} \simeq 0.3 \cdot 10^{-6} cm^{-1} \ . \end{aligned}$$

Note that the system is non-self-gravitating and that we have indeed $l_0 \cdot \omega \ll 1$*, so that the quadrupole formula is valid. Since we have* $l_0 \gg A$ *the first term in* (2.56) *is dominant. We obtain* $\frac{d^2}{dt^2}Q_{xx}(t) \sim 6.75 \cdot 10^{-35} cm$*, and thus if we set up test masses at distance* r *away from the gravitational wave generator we obtain from* (2.54)

$$\Delta Y^i \simeq \frac{1}{r} 6.75 \cdot 10^{-35} \cdot Y^i(0) cm \ .$$

If the test masses are $1km = 10^5 cm$ *apart and* $r = 10km = 10^6 cm$*, then the displacement of the test masses is of order*

$$\Delta Y^i = 6.75 \cdot 10^{-31} cm \ ,$$

which is far too small to be detectable (recall that the displacement measured with LIGO is about $10^{-16} cm$*).*

Remark 2.57. *Recall that the derivation of the quadrupole formula was only valid for non-self-gravitating systems, but that is is expected to be still a good approximation for example for orbiting binaries. Indeed, Hulse and Taylor observed in 1975 the increase of orbiting frequency of a binary system which contains a pulsar at a rate compatible with the loss of energy due to emission of gravitational waves predicted by the quadrupole formula. This first indirect observation of gravitational waves received the Nobel prize in physics in 1993.*

Remark 2.58. *Einstein, who derived the quadrupole formula, noted that one would expect that on small scales (quantum scales) one has to modify general relativity because otherwise the hydrogen atom would be unstable (even if over extremely large time-scales) due to the emission of gravitational radiation.*

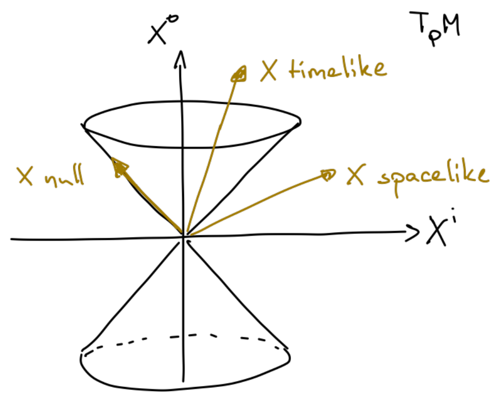
3 Causality & Penrose diagrams

3.1 Lorentzian causality

Let (M, g) be a Lorentzian manifold, let $p \in M$ and $X \in T_pM$ a tangent vector. We then classify X according to the sign of its Lorentzian inner product:

$$g(X,X) = \begin{cases} < 0 & \iff: \quad X \quad \textbf{timelike} \\ = 0 & \iff: \quad X \quad \textbf{null} \text{ (or } \textbf{lightlike}\text{)} \\ > 0 & \iff: \quad X \quad \textbf{spacelike} \end{cases} .$$

We also refer to X being **causal** iff it is timelike or null. We can choose local coordinates x^μ such that at $p \in M$ the metric takes the Minkowski form $g_{\mu\nu}|_p = \mathrm{diag}(-1, 1, 1, 1)$.



The set of timelike vectors in T_pM forms the disconnected double cone

$$C_p = \{X = X^\mu \partial_\mu \mid X^0 > \sqrt{(X^1)^2 + \ldots + (X^n)^2}\} \cup \{X \mid X^0 < -\sqrt{(X^1)^2 + \ldots + (X^n)^2}\} .$$

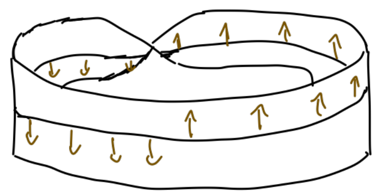
If we can single out one of those components throughout M in a continuous way, then we say that (M, g) is **time-orientable**. This is equivalent to the existence of a continuous timelike vector field on (M, g). Making such a continuous choice determines a **time-orientation**. Timelike vectors in this component are called **future-directed**, timelike vectors in the other component are called **past-directed**. These notions extend by continuity to non-vanishing null vectors. A time-oriented Lorentzian manifold is also called a **spacetime**.

Example 3.1.

$M = \mathbb{R}^4$, $\eta = \mathrm{diag}(-1, 1, 1, 1)$. *Then* ∂_t *provides a time-orientation.*

$M = \mathbb{R} \times (2m, \infty) \times \mathbb{S}^2$, $g = -(1 - \frac{2m}{r})\, dt^2 + \frac{1}{1-\frac{2m}{r}}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2)$. *Then* ∂_t *provides a time orientation.*

$M = \underbrace{(-1, 1)}_{\ni t} \times \underbrace{[-10, 10]}_{\ni x}$ *with the identification* $(t, -10) \sim (-t, 10)$ *(i.e., the Möbius strip),* $g = -dt^2 + dx^2$ *is not time-orientable.*



A smooth curve $\gamma : I \to M$ in a Lorentzian manifold (M, g) is called **timelike/null/causal/spacelike** iff its tangent vector $\dot{\gamma}(s)$ is timelike/null/causal/spacelike for all $s \in I$. If (M, g) is time-oriented, then γ is called **future directed timelike/null/causal** iff $\dot{\gamma}(s)$ is future directed timelike/null/causal for all $s \in I$. Similarly for past directed.

As in special relativity, massive particles can only move along timelike curves, light rays follow null geodesics, and nothing moves along spacelike curves.

Let (M, g) be a spacetime and $A \subseteq M$. We define the **timelike future of A in M**

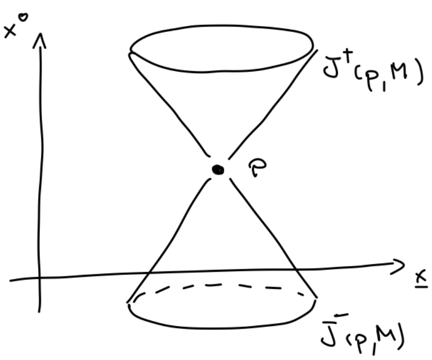
$$I^+(A, M) := \{q \in M \mid \text{ there exists a future directed timelike curve from some point } p \in A \text{ to } q\}$$

and the **causal future of A in M**

$$J^+(A, M) := \{q \in M \mid \text{ there exists a future directed causal curve from some point } p \in A \text{ to } q\} \, .$$

The **timelike past of A in M**, $I^-(A, M)$, and the **causal past of A in M**, $J^-(A, M)$, are defined analogously.

Example 3.2. *Consider the Minkowski spacetime $M = \mathbb{R}^4$ and $g = \operatorname{diag}(-1, 1, 1, 1)$.*



The sets $J^{\pm}(A, M)$ are of fundamental importance since they determine the causal relations: $J^+(A, M)$ is the set of all points which can be causally influenced from A, and $J^-(A, M)$ is the set of all points which can causally influence A.

Penrose diagrams are an easy way to visualise the causal structure of (spherically symmetric) spacetimes, i.e., to visualise sets of the form $J^{\pm}(A, M)$.

3.2 Penrose diagrams

Let (M, g) be a Lorentzian manifold. Another Lorentzian metric $\tilde{g}$ on M is called **conformal to** g iff there exists a smooth (positive) function $\Omega \in C^\infty(M)$ such that $\tilde{g} = \Omega^2 \cdot g$. Note that for $X \in T_pM$

we have $\tilde{g}(X,X) = \Omega^2 \cdot g(X,X)$. Thus, X is $\tilde{g}$-timelike/null/causal/spacelike if, and only if, X is g-timelike/null/causal/spacelike. Hence, we obtain

$$J^\pm_{\tilde{g}}(A,M) = J^\pm_g(A,M)$$

and similarly for the timelike future/past. Thus, *conformal metrics have the same causal structure.*

Idea of Penrose diagrams:

1) We want to understand the *global* structure of a spacetime (M,g). By a suitable coordinate transformation we bring in the infinities of (M,g) to a finite coordinate range. As a consequence the metric components $g_{\mu\nu}$ blow up in these coordinates at the infinites.

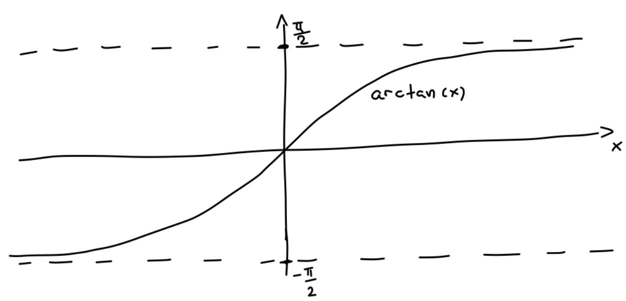
2) Choose a conformal factor Ω to make $\tilde{g}_{\mu\nu} = \Omega^2 \cdot g_{\mu\nu}$ regular at the infinities.

3) We can add the infinities as boundaries to the spacetime to create a **conformal compactification**.

4) If needed we drop some (spherically symmetric) dimensions and draw a 2-dimensional diagram with the causality of $1+1$-dimensional Minkowski spacetime.

$1+1$-dimensional Minkowski spacetime: $M = \mathbb{R}^2$ with $g = -dt^2 + dx^2$.

1) We introduce null coordinates $v := t + x$, $u := t - x$, which have the range $u, v \in \mathbb{R}$. The metric becomes $g = -\frac{1}{2}\big(dv \otimes du + du \otimes dv\big)$. We now bring the infinities to finite coordinate range by setting $\tilde{u} := \arctan u$ and $\tilde{v} := \arctan v$.



We have $u = \tan\tilde{u} = \frac{\sin\tilde{u}}{\cos\tilde{u}}$ and thus $du = \frac{1}{\cos^2\tilde{u}} d\tilde{u}$. Hence the metric becomes

$$g = -\frac{1}{2\cos^2\tilde{u}\cdot\cos^2\tilde{v}}\big(d\tilde{v}\otimes d\tilde{u} + d\tilde{u}\otimes d\tilde{v}\big)$$

with the coordinate range $\tilde{u}, \tilde{v} \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Note that the metric diverges at the infinities $\tilde{u}, \tilde{v} \to \pm\frac{\pi}{2}$.

2) We choose the conformal factor $\Omega^2 = \cos^2\tilde{u}\cdot\cos^2\tilde{v}$. Then

$$\tilde{g} = \Omega^2 \cdot g = -\frac{1}{2}(d\tilde{v}\otimes d\tilde{u} + d\tilde{u}\otimes d\tilde{v})$$

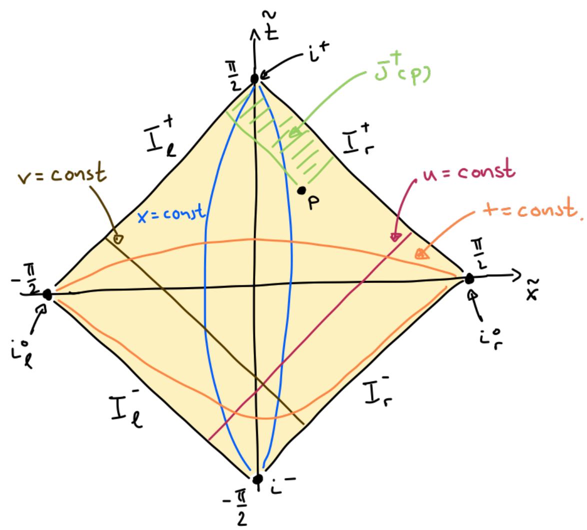
is regular for $\tilde{u}, \tilde{v} \to \pm\frac{\pi}{2}$.

3) We can now add the boundaries to create the conformal compactification $\tilde{g} = -\frac{1}{2}(d\tilde{v}\otimes d\tilde{u} + d\tilde{u}\otimes d\tilde{v})$ on the manifold $(\tilde{u}, \tilde{v}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2$.

4) Set $\tilde{t} := \frac{1}{2}(\tilde{v} + \tilde{u})$ and $\tilde{x} := \frac{1}{2}(\tilde{v} - \tilde{u})$. Then the coordinate range is

$$\big\{(\tilde{t}, \tilde{x}) \in \mathbb{R}^2 \mid -\frac{\pi}{2} \leq \underbrace{\tilde{t} + \tilde{x}}_{=\tilde{v}} \leq \frac{\pi}{2}\,,\ -\frac{\pi}{2} \leq \underbrace{\tilde{t} - \tilde{x}}_{=\tilde{u}} \leq \frac{\pi}{2}\big\} =: \tilde{M}$$

and the metric takes on the standard form $\tilde{g} = -d\tilde{t}^2 + d\tilde{x}^2$ of $1+1$-dimensional Minkowski spacetime. We now draw the resulting compact spacetime, which is the **Penrose diagram of $1+1$-dimensional Minkowski spacetime**:



The shape of the level sets of t and x in terms of the $\tilde{t}, \tilde{x}$ coordinates follows directly from

$$t = \frac{1}{2}(v+u) = \frac{1}{2}(\tan\tilde{v} + \tan\tilde{u}) = \frac{1}{2}\big(\tan(\tilde{t}+\tilde{x}) + \tan(\tilde{t}-\tilde{x})\big)$$
$$x = \frac{1}{2}(v-u) = \frac{1}{2}(\tan\tilde{v} - \tan\tilde{u}) = \frac{1}{2}\big(\tan(\tilde{t}+\tilde{x}) - \tan(\tilde{t}-\tilde{x})\big)$$

Note that since $\tilde{g}$ and g are conformal, we have $J^+_{\tilde{g}}(p) = J^+_g(p)$. So the global causality of (M, g) can be easily read off from the Penrose diagram.

We have labelled the following infinities:

- $\mathcal{I}^+_r/\mathcal{I}^+_l$ are called right/left **future null infinity**. They form the asymptotic endpoints of all future directed right/left going null geodesics.

- $\mathcal{I}^-_r/\mathcal{I}^-_l$ are called right/left **past null infinity**. They form the asymptotic endpoints of all past directed right/left going null geodesics.

- i^+ is called **future timelike infinity**. It is the endpoint of all future directed timelike geodesics.

- i^- is called **past timelike infinity**. It is the endpoint of all past directed timelike geodesics.

- i^0_r/i^0_l are called right/left **spacelike infinity**. They form the endpoints of all right/left going spacelike geodesics.

Note that there are timelike and spacelike curves going to $\mathcal{I}^\pm_{r/l}$ – but they are not geodesics.

<u>$3+1$**-dimensional Minkowski spacetime:**</u> $M = \mathbb{R}^4$ with $g = -dt^2 + dr^2 + r^2 d\sigma^2$, where $d\sigma^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the standard metric on $\mathbb{S}^2$.

1) We introduce spherically symmetric null coordinates $v := t + r$ and $u := t - r$. We have $r \geq 0 \iff v - u \geq 0$ and thus the domain of the new coordinates is $\infty > v \geq u > -\infty$. The metric becomes

$$g = -\frac{1}{2}\big(dv \otimes du + du \otimes dv\big) + \frac{1}{4}(v-u)^2 d\sigma^2 \ .$$

We compactify again by setting $\tilde{u} := \arctan u$ and $\tilde{v} := \arctan v$. The new coordinate range is $\frac{\pi}{2} > \tilde{v} \geq \tilde{u} > -\frac{\pi}{2}$ and the metric becomes[33]

$$g = \frac{1}{\cos^2\tilde{u}\cos^2\tilde{v}}\Big(-\frac{1}{2}\big(d\tilde{v} \otimes d\tilde{u} + d\tilde{u} \otimes d\tilde{v}\big) + \frac{1}{4}\sin^2(\tilde{v}-\tilde{u}) d\sigma^2\Big) \ .$$

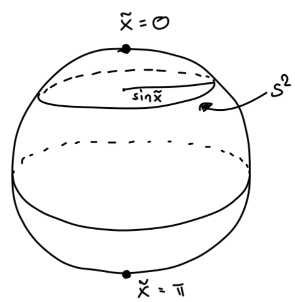
2) We choose the conformal factor $\Omega^2 = 4\cos^2\tilde{u}\cos^2\tilde{v}$ and make the coordinate transformation $\tilde{t} := \tilde{v} + \tilde{u}$ and $\tilde{x} := \tilde{v} - \tilde{u}$. The domain of the new coordinates is

$$-\frac{\pi}{2} < \frac{1}{2}(\tilde{t} \pm \tilde{x}) < \frac{\pi}{2} \qquad \text{and} \qquad \pi > \tilde{x} \geq 0$$

and the conformal metric is

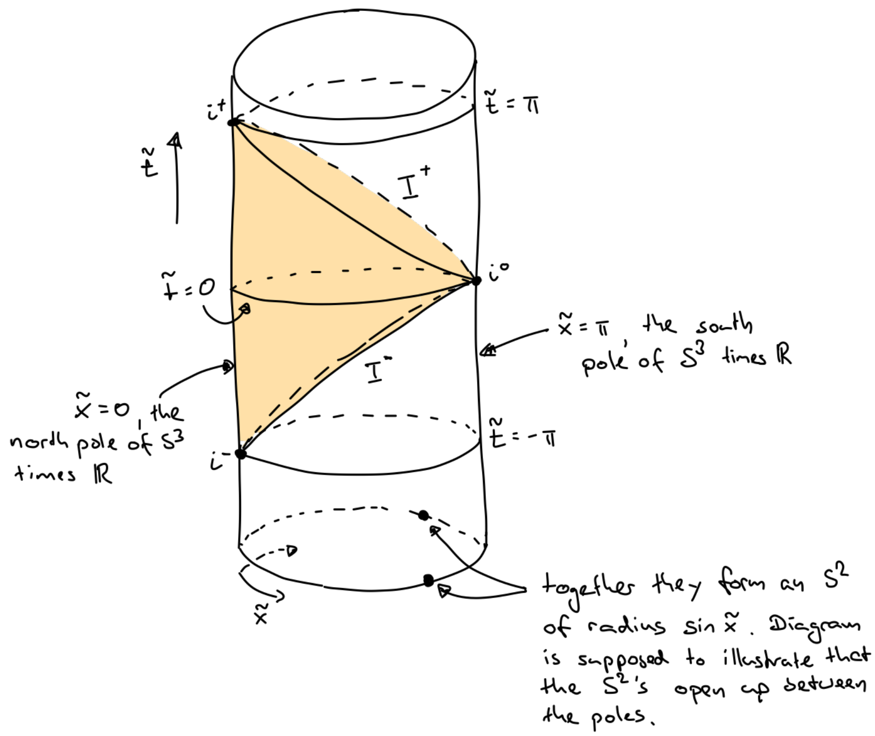
$$\tilde{g} = \Omega^2 \cdot g = -d\tilde{t}^2 + d\tilde{x}^2 + \sin^2\tilde{x}\, d\sigma^2 \ .$$

3) Observe that $d\tilde{x}^2 + \sin^2\tilde{x}\, d\sigma^2$ is the standard metric on $\mathbb{S}^3$, where $\tilde{x} = 0, \pi$ are the poles of $\mathbb{S}^3$ and $\tilde{x} = \text{const} \neq 0, \pi$ are 2-spheres of radius $\sin\tilde{x}$.



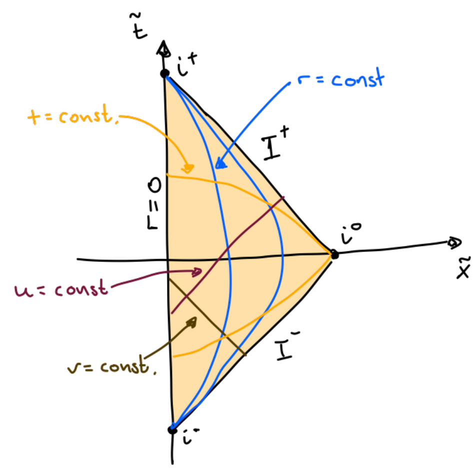
The spacetime $(\tilde{M}, \tilde{g})$ with $\tilde{M} = \mathbb{R} \times \mathbb{S}^3$, $\tilde{g} = -d\tilde{t}^2 + d\tilde{x}^2 + \sin^2\tilde{x}\, d\sigma^2$ is known as the **Einstein static universe**. We have thus mapped $3+1$-dimensional Minkowski spacetime conformally into a portion of the Einstein static universe!

[33] Here we use
$$v - u = \tan\tilde{v} - \tan\tilde{u} = \frac{\sin\tilde{v}}{\cos\tilde{v}} - \frac{\sin\tilde{u}}{\cos\tilde{u}} = \frac{\sin\tilde{v}\cos\tilde{u} - \sin\tilde{u}\cos\tilde{v}}{\cos\tilde{v}\cos\tilde{u}} = \frac{\sin(\tilde{v}-\tilde{u})}{\cos\tilde{v}\cos\tilde{u}} \ .$$



We can now add future/past timelike infinity i^+/i^- and spacelike infinity i^0, which are all points in $\tilde{M}$, and future/past null infinity $\mathcal{I}^+/\mathcal{I}^-$, which have topology $(0, \pi) \times \mathbb{S}^2$, to create a conformal compactification of $3+1$-dimensional Minkowski spacetime.

4) We quotient out the spheres of symmetry and then draw the quotient, the **Penrose diagram of $3+1$-dimensional Minkowski spacetime**:



Note that

- Every point corresponds to an $\mathbb{S}^2$ except $\{r = 0\}$, i^+, i^-, and i^0.
- The form of the level sets of t and r follows from $t = \frac{1}{2}(v+u) = \frac{1}{2}\big(\tan(\tilde{t}+\tilde{x}) + \tan(\tilde{t}-\tilde{x})\big)$ and $r = \frac{1}{2}(v-u) = \frac{1}{2}\big(\tan(\tilde{t}+\tilde{x}) - \tan(\tilde{t}-\tilde{x})\big)$.

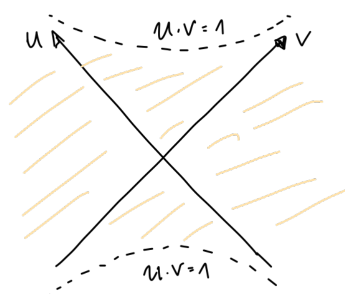
- $i^\pm$ is the future/past endpoint of all future/past directed timelike geodesics.
- i^0 is the endpoint of all spacelike geodesics.
- $\mathcal{I}^\pm$ are the future/past endpoints of all future/past directed null geodesics.

For example, radial null geodesics in $3+1$-dimensional Minkowski spacetime are lines of 45 degrees in the above Penrose diagram and $t \mapsto (t, r_0, \theta_0, \varphi_0)$ are the lines of constant r.

Maximal analytic Schwarzschild spacetime: Recall the Kruskal coordinates (U, V, θ, φ) for maximal analytic Schwarzschild, where $\{(U,V) \in \mathbb{R}^2 \mid U \cdot V < 1\}$, and the metric takes the form

$$g = -\frac{16M^3}{r} e^{-\frac{r}{2M}} (dU \otimes dV + dV \otimes dU) + r^2\, d\sigma^2 \; .$$

Here, r is implicitly defined by $U \cdot V = (1 - \frac{r}{2M}) e^{\frac{r}{2M}}$ and the Schwarzschild coordinate t is given by $\frac{V}{U} = -e^{\frac{t}{2M}}$.



1) Let $\tilde{u} := \arctan U$ and $\tilde{v} := \arctan V$. The range of the new coordinates is then[34]

$$\big\{(\tilde{u}, \tilde{v}) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \mid -\frac{\pi}{2} < \tilde{u} + \tilde{v} < \frac{\pi}{2}\big\}$$

and the metric takes the form

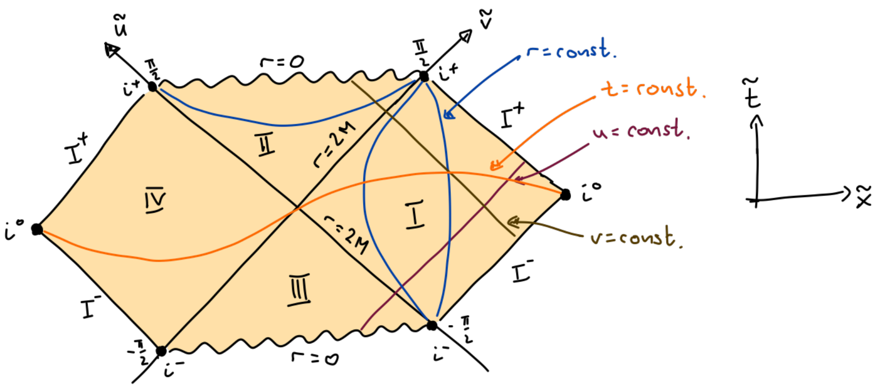
$$g = \frac{1}{\cos^2\tilde{u} \cdot \cos^2\tilde{v}} \Big(-\frac{16M^3}{r} e^{-\frac{r}{2M}} (d\tilde{u} \otimes d\tilde{v} + d\tilde{v} \otimes d\tilde{u}) + r^2 \cdot \cos^2\tilde{u} \cdot \cos^2\tilde{v}\, d\sigma^2 \Big) \; .$$

2) We choose the conformal factor $\Omega^2 = \cos^2\tilde{u} \cdot \cos^2\tilde{v}$ so that the metric becomes

$$\tilde{g} = -\frac{16M^3}{r} e^{-\frac{r}{2M}} (d\tilde{u} \otimes d\tilde{v} + d\tilde{v} \otimes d\tilde{u}) + r^2 \cdot \cos^2\tilde{u} \cdot \cos^2\tilde{v}\, d\sigma^2 \; .$$

3 & 4) Let $\tilde{t} := \tilde{v} + \tilde{u}$ and $\tilde{x} := \tilde{v} - \tilde{u}$. We again quotient out the spheres of symmetry and draw the quotient, the **Penrose diagram of the maximal analytic Schwarzschild spacetime**:

[34] This follows from $\tan\tilde{u} \cdot \tan\tilde{v} = U \cdot V < 1 \iff \sin\tilde{u} \cdot \sin\tilde{v} < \cos\tilde{u} \cdot \cos\tilde{v} \iff \cos(\tilde{u} + \tilde{v}) > 0$.



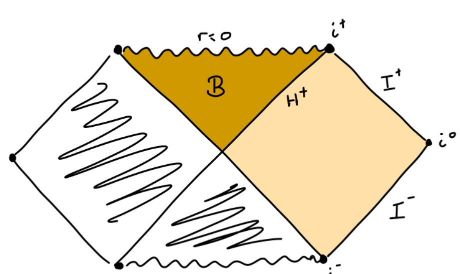
Note that

- At $r = 0 \iff U \cdot V = 1 \iff \tilde{u} + \tilde{v} = \pm\frac{\pi}{2}$ we have a curvature singularity.
- The form of the level sets of t and r is left as an exercise.
- The metric $\tilde{g}$ extends continuously to future/past null infinity $\mathcal{I}^+/\mathcal{I}^-$ and to spacelike infinity i^0; so these infinities can again be added as conformal boundaries. However, the metric does not extend continuously to future/past timelike infinity i^+/i^-.[35]
- Timelike geodesics asymptote either to $\{r = 0\}$ or to i^+/i^-.
- Null geodesics asymptote to $\{r = 0\}$, to i^+/i^- (if they asymptote towards the photon sphere at $\{r = 3M\}$ or towards the horizons $\{r = 2M\}$), or to $\mathcal{I}^+/\mathcal{I}^-$.

4 Black holes

4.1 The concept of a black hole

Let (M, g) be the maximal analytic Schwarzschild spacetime. We now define the **black hole region** B by $B := M \setminus J^-(\mathcal{I}^+)$, where we use the right $\mathcal{I}^+$.[36]



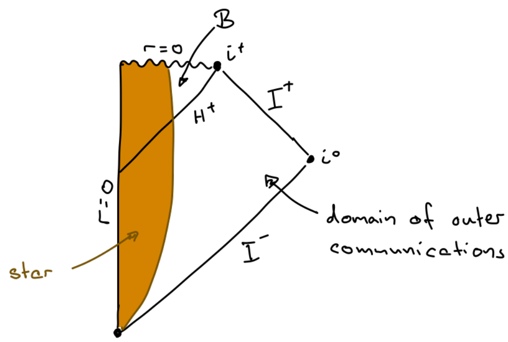
By definition this is the set of spacetime points from which one cannot send future directed signals to $\mathcal{I}^+$. The boundary $\mathcal{H}^+ := \partial\big(J^-(\mathcal{I}^+)\big)$ of $J^-(\mathcal{I}^+)$ in M is called the **(future) event horizon**.

[35] Think: because they meet the singularity at $\{r = 0\}$ in the Penrose diagram.

[36] The scribbled regions in the Penrose diagram, i.e., regions III and IV, are non-physical.

The region $J^+(\mathcal{I}^-) \cap J^-(\mathcal{I}^+)$ is called the **domain of outer communications**. The event horizon separates the black hole region from the domain of outer communications.

The maximal analytic Schwarzschild black hole is not a realistic model for a black hole arising from the gravitational collapse of a star: it has two asymptotic flat ends and also the white hole region III. A better model is given by the collapse of a spherically symmetric homogeneous dust cloud (the star) in an asymptotically flat spacetime with only one end.[37] The Penrose diagram is depicted below. The white vacuum region outside the star is given by the corresponding region in the maximal analytic Schwarzschild spacetime.



4.2 Hypersurfaces

Let (M, g) be a $n+1$-dimensional Lorentzian manifold and $\Sigma \subseteq M$ a hypersurface. Recall that this means that for every $p \in \Sigma$ there exists local coordinates $(x^0, \ldots, x^n)$ on a neighbourhood $U \subseteq M$ of p such that $\Sigma \cap U = \{x^0 = 0\}$. The tangent space $T\Sigma$ of Σ is locally given by $\text{span}\{\partial_1, \ldots, \partial_n\} \subseteq TM$ in these coordinates, i.e., for all $p \in \Sigma$, $T_p\Sigma$ is an n-dimensional subspace of T_pM.

$$\text{We say that } \Sigma \text{ is a } \begin{cases} \textbf{spacelike hypersurface} & :\iff g|_{T_p\Sigma} \text{ is positive definite (Riemannian) for all } p \in \Sigma \\ \textbf{timelike hypersurface} & :\iff g|_{T_p\Sigma} \text{ is Lorentzian for all } p \in \Sigma \\ \textbf{null hypersurface} & :\iff g|_{T_p\Sigma} \text{ is degenerate for all } p \in \Sigma \end{cases}$$

Since $T_p\Sigma$ is an n-dimensional subspace of T_pM there exists a covector $n \in T_p^*M$ such that $\ker n = T_p\Sigma$. This covector is unique up to multiplication by $\lambda \neq 0$ and is called a **normal covector** to Σ at p. We have $n(X) = 0$ for all $X \in T_p\Sigma$. In the local coordinates we have $n = \lambda dx^0$, $\lambda \neq 0$. We can also define $N := n^\sharp$, a **normal vector** to Σ at p. We have $g(N, X) = 0$ for all $X \in T_p\Sigma$. Again, N is unique up to multiplication by $\lambda \neq 0$.

Proposition 4.1.

$$\Sigma \text{ is a } \quad \begin{cases} \text{spacelike hypersurface} & \iff N \text{ is timelike } \forall p \in \Sigma \\ \text{timelike hypersurface} & \iff N \text{ is spacelike } \forall p \in \Sigma \\ \text{null hypersurface} & \iff N \text{ is null } \forall p \in \Sigma\,. \end{cases}$$

Proof. Let Σ be a hypersurface and let N be a normal vector at p. We distinguish the two cases that $N \in T_p\Sigma$ and $N \notin T_p\Sigma$.

[37] This model has been constructed by Oppenheimer and Snyder in 1939, in 'On Continued Gravitational Contraction'.

i) $N \notin T_p\Sigma$. Then let $E_1, \ldots, E_n$ (e.g. $E_i = \partial_i$) be a basis of $T_p\Sigma$. Thus $\{N, E_1, \ldots, E_n\}$ is a basis of T_pM. With respect to this basis g has the matrix

$$\begin{pmatrix} g(N,N) & 0 & \cdots & 0 \\ 0 & g(E_1,E_1) & \cdots & g(E_1,E_n) \\ \vdots & \vdots & g(E_i,E_j) & \vdots \\ 0 & g(E_n,E_1) & \cdots & g(E_n,E_n) \end{pmatrix}$$

It thus follows that $g(N,N) \neq 0$, since otherwise $g|_{T_pM}$ would be degenerate. It also follows that $g|_{T_p\Sigma}$ is non-degenerate.

ii) $N \in T_p\Sigma$. Thus $\forall X \in T_p\Sigma$ we have $g|_{T_p\Sigma}(N,X) = 0$, i.e., $g|_{T_p\Sigma}$ is degenerate and choosing $X = N$ gives $g(N,N) = 0$.

The proof now follows easily from this. For example let Σ be a spacelike hypersurface. Then $g|_{T_p\Sigma}$ is non-degenerate and thus we are in case i). Using that $g|_{T_pM}$ is Lorentzian and $g|_{T_p\Sigma}$ is Riemannian gives $g(N,N) < 0$. For the reverse let N be timelike. Then we must be in case i) and $g|_{T_p\Sigma}$ is positive definite. Similarly for the other cases. □

In particular we have seen that if Σ is a null hypersurface, then the normal vector field N is *tangent* to Σ.

Proposition 4.2. *Let Σ be a null hypersurface and N a normal vector field. Then the integral curves of N are null geodesics, but not necessarily affinely parametrised. They are called the* ***generators*** *of the null hypersurface.*

Proof. Let N be a normal vector field. Locally Σ is given as the level set of a function f (e.g. the x^0-coordinate). Then $df = n$ is a normal covector field on Σ and we have $N = \lambda(df)^\sharp$, $\lambda \neq 0$. Since the integral curves of N and $\lambda^{-1}N$ are the same up to parametrisation, we can without loss of generality assume $N = (df)^\sharp$. Now using that the second covariant derivative of a scalar function is symmetric we compute

$$(\nabla_N n)_a = N^b \nabla_b \nabla_a f = N^b \nabla_a \nabla_b f = N^b \nabla_a n_b = \frac{1}{2}\partial_a (N^b n_b) \ .$$

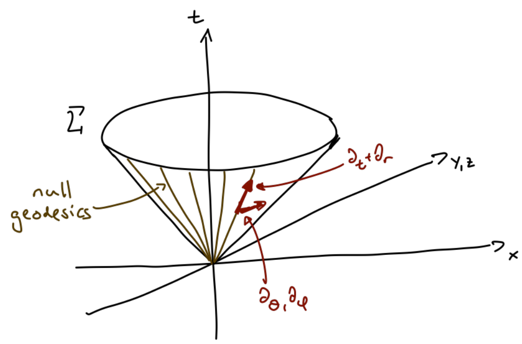
But $N^b n_b = g(N,N)$ is constant on Σ (equal to zero) and thus $d(N^b n_b) = \mu \cdot n$ with μ a smooth function. This implies $(\nabla_N n)_a = \frac{1}{2}\mu n_a$, which, after raising the index, reads $\nabla_N N = \frac{1}{2}\mu N$. This shows the claim. □

Example 4.3. *Consider $3+1$-dimensional Minkowski spacetime.*

a) $t = \text{const}$ are spacelike hypersurfaces, since $\eta^{-1}(dt, dt) = -1$.

b) $x_i = \text{const}$ are timelike hypersurfaces, since $\eta^{-1}(dx_i, dx_i) = 1$

c) Let Σ be the future light cone of the origin in Minkowski spacetime with the origin removed, i.e., $\Sigma = J^+(0) \setminus \{I^+(0) \cup \{0\}\}$



Then $\partial_t + \partial_r$ and ∂_θ, ∂_φ span the tangent space. Clearly $\partial_t + \partial_r$ is null and it is orthogonal to ∂_θ, ∂_φ. This shows that $\partial_t + \partial_r$ is the normal of Σ. Thus Σ is a null hypersurface. It is generated *by the null geodesics which are the straight lines in the cone.*

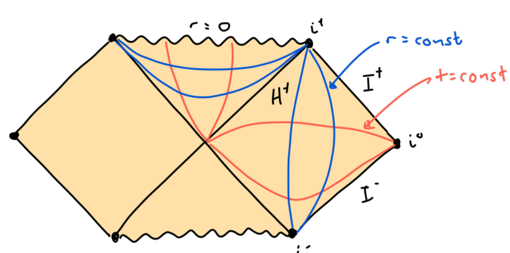
Example 4.4. *Consider the Schwarzschild spacetime in (t, r, θ, φ) coordinates with metric* $g = -(1 - \frac{2M}{r})\, dt^2 + \frac{1}{1-\frac{2M}{r}}\, dr^2 + r^2\, d\sigma^2$.

a) $t = \text{const}$ *are* $\begin{cases} \text{spacelike hypersurfaces for } r > 2M \\ \text{timelike hypersurfaces for } r < 2M \end{cases}$.

This follows from $g^{-1}(dt, dt) = -\frac{1}{1-\frac{2M}{r}} = \begin{cases} < 0 & \text{for } r > 2M \\ > 0 & \text{for } r < 2M \end{cases}$.

b) $r = \text{const}$ *are* $\begin{cases} \text{timelike hypersurfaces for } r > 2M \\ \text{spacelike hypersurfaces for } r < 2M \end{cases}$.

This follows from $g^{-1}(dr, dr) = 1 - \frac{2M}{r} = \begin{cases} > 0 & \text{for } r > 2M \\ < 0 & \text{for } r < 2M \end{cases}$.



*c) Let $v = t + r^*$ with $r^* = r + 2M \log\left(\frac{r-2M}{2M}\right)$ for $r > 2M$. Then (v, r, θ, φ) are **ingoing Eddington-Finkelstein coordinates**, they cover regions I and II in the Penrose diagram of maximal analytic Schwarzschild. The metric becomes*

$$g = -(1 - \frac{2M}{r})\, dv^2 + dv \otimes dr + dr \otimes dv + r^2\, d\sigma^2$$

and the inverse metric is

$$g^{-1} = \partial_v \otimes \partial_r + \partial_r \otimes \partial_v + (1 - \frac{2M}{r})\, \partial_r \otimes \partial_r + \frac{1}{r^2}(\partial_\theta \otimes \partial_\theta + \frac{1}{\sin^2\theta}\, \partial_\varphi \otimes \partial_\varphi)\ .$$

We thus have $g^{-1}(dr, dr)|_{r=2M} = 0$ and hence the event horizon $\{r = 2M\} = \mathcal{H}^+$ is a null hypersurface.

A choice of normal vector field is $N = \partial_v = (dr)^\sharp$. Thus the integral curves of ∂_v are null geodesics by Proposition 4.2. Indeed, we have

$$\Gamma^v_{vv}|_{r=2M} = \frac{1}{2} g^{vr}(-\partial_r g_{vv})|_{r=2M} = \frac{1}{2}\partial_r(1 - \frac{2M}{r})|_{r=2M} = \frac{1}{4M}\ ,$$

and $\Gamma^r_{vv}|_{r=2M} = \Gamma^\theta_{vv}|_{r=2M} = \Gamma^\varphi_{vv}|_{r=2M} = 0$ as is easily seen from the form of the metric and its inverse above. Thus

$$\nabla_{\partial_v}\partial_v = \frac{1}{4M}\partial_v\ . \tag{4.5}$$

Hence, the integral curves of ∂_v, the null geodesics, are not affinely parametrised. But let $\partial_V := e^{-\frac{1}{4M}v}\partial_v$. Then

$$\nabla_{\partial_V}\partial_V = e^{-\frac{1}{4M}v}\nabla_{\partial_v}\big(e^{-\frac{1}{4M}v}\partial_v\big) = e^{-\frac{1}{2M}v}\nabla_{\partial_v}\partial_v - \frac{1}{4M}e^{-\frac{1}{2M}v}\partial_v = 0\ .$$

Thus, the integral curves of ∂_V are affinely parametrised. Note/recall that ∂_V is exactly the coordinate vector field in Kruskal coordinates (V, U, θ, φ).

4.3 Killing horizons & surface gravity

Let (M, g) be a Lorentzian manifold with Killing vector field T. A null hypersurface Σ is a **Killing horizon** of T iff T is normal to Σ on Σ. Since T is a normal to Σ, Proposition 4.2 implies $\nabla_T T|_\Sigma = \kappa T|_\Sigma$ for some function κ on Σ. κ is called the **surface gravity** of Σ with respect to the Killing vector field T.

Remark 4.6. *1) If Σ is a Killing horizon of T, then it is also a Killing horizon of $\tilde{T} := cT$ with $c \in \mathbb{R} \setminus \{0\}$. Then $\nabla_{\tilde{T}}\tilde{T}|_\Sigma = \tilde{\kappa}\tilde{T}|_\Sigma$ with $\tilde{\kappa} = c\kappa$. Hence, the surface gravity depends on the normalisation of the Killing vector field T. For asymptotically flat spacetimes we normalise T at infinity. For example if T is a time translation then we require that $g(T, T) \to -1$ for $r \to \infty$ and we fix the sign of κ by requiring T to be future directed.*

2) Using Killing's equation $\nabla_\mu T_\nu + \nabla_\nu T_\mu = 0$ from Proposition 1.24, we obtain

$$(\nabla_T T)_\mu = T^\nu \nabla_\nu T_\mu = -T^\nu \nabla_\mu T_\nu = -\frac{1}{2}\partial_\mu\big(g(T, T)\big)$$

and thus

$$d\big(g(T, T)\big)|_\Sigma = -2\kappa T^\flat|_\Sigma\ . \tag{4.7}$$

3) Note that we have $\mathcal{L}_T(g(T, T)) = 0$, since $\mathcal{L}_T g = 0$ and $\mathcal{L}_T T = 0$. Using Proposition 1.20 vi) we thus obtain

$$\mathcal{L}_T\big(d(g(T, T))\big) = d\big(\mathcal{L}_T(g(T, T))\big) = 0\ .$$

Moreover, we have

$$-2\mathcal{L}_T(\kappa T^\flat) = -2T(\kappa) \cdot T^\flat \,,$$

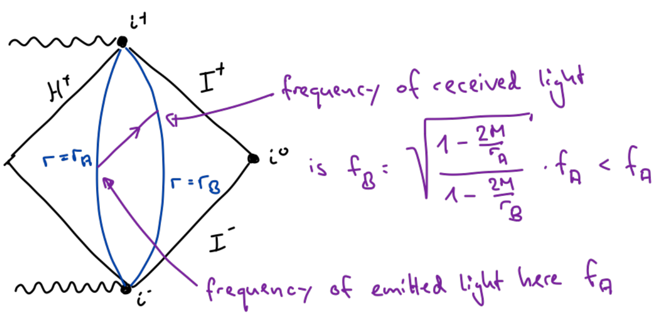
where we have again used that $\mathcal{L}_T g = 0$ and $\mathcal{L}_T T = 0$. Also note that since T is tangent to Σ we have for any tensor field E that $(\mathcal{L}_T E)|_\Sigma$ only depends on $E|_\Sigma$.[38] *We thus obtain from* (4.7) *$0 = -2T(\kappa) \cdot T^\flat|_\Sigma$ and thus $T(\kappa) = 0$. Thus,* ***the surface gravity κ is constant along the generators of Σ.***

4) *Indeed, one can strengthen the above result and show that if (M, g) is a solution of $G_{ab} = 8\pi T_{ab}$ where the matter T_{ab} satisfies that so-called dominant energy condition*[39] *and if Σ is a Killing horizon of a Killing vector field T, then the surface gravity κ is constant on all of Σ.*

5) *The event horizon $\{r = 2M\}$ in Schwarzschild is a Killing horizon of the Killing vector field $\frac{\partial}{\partial t} = \frac{\partial}{\partial v}$, see Example 4.4 c), where $\frac{\partial}{\partial v}$ is with respect to ingoing Eddington-Finkelstein coordinates. It follows from* (4.5) *that the surface gravity is $\kappa = \frac{1}{4M}$. Note that $\partial_t = \partial_v$ is normalised at infinity.*

4.3.1 Physical interpretation of the surface gravity

We consider the maximal analytic Schwarzschild black hole. We are already familiar with the gravitational redshift in the exterior of the black hole (see also problem sheet 4).

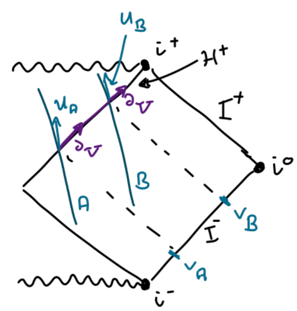


A positive surface gravity implies that there is also a gravitational redshift *at the surface of the black hole*, i.e., along the event horizon: Recall from Example 4.4 c) that the integral curves of $\partial_V = e^{-\kappa v}\partial_v$ are affinely parametrised null geodesics, light rays, along the event horizon. From Problem 4 on the fourth problem sheet we know that ∂_V, the affine velocity vector of the null geodesic, corresponds to the wave vector of the light ray. Moreover, an observer with 4-velocity U measures the frequency of the light ray as $-\frac{1}{2\pi}g(U, \partial_V)$.

Consider now an observer A crossing the event horizon at $(v_A, 2M, \theta_0, \varphi_0)$ in ingoing Eddington-Finkelstein coordinates with 4-velocity $U_A = \frac{1}{\sqrt{2}}(\partial_v - \partial_r)$ and sending a light signal along the event horizon that is received by another observer B crossing $\mathcal{H}^+$ at $(v_B, 2M, \theta_0, \varphi_0)$ with $v_B > v_A$ and with 4-velocity $U_B = \frac{1}{\sqrt{2}}(\partial_v - \partial_r)$.

[38] We have $\mathcal{L}_T E = \lim_{t \to 0} \frac{\Phi_t^* E - E}{t}$, and since the flow Φ_t of T maps points on Σ to points on Σ, it is clear that $\mathcal{L}_T E$ only depends on $E|_\Sigma$.

[39] I.e., $T^{ab}W_aW_b \geq 0$ for all W timelike and $T^{ab}W_b$ is a causal vector.



Recall that $g = -(1 - \frac{2M}{r})\, dv^2 + dv \otimes dr + dr \otimes dv + r^2\, d\sigma^2$. Thus the frequency f_A of the light ray given by ∂_V as observed by A is

$$f_A = -\frac{1}{2\pi} g(\partial_V, U_A) = \frac{1}{2\sqrt{2}\pi} e^{-\kappa v_A}$$

and the frequency f_B observed by B is

$$f_B = -\frac{1}{2\pi} g(\partial_V, U_B) = \frac{1}{2\sqrt{2}\pi} e^{-\kappa v_B} = e^{-\kappa\cdot(v_B - v_A)} \cdot f_A\,.$$

Thus, it follows that **the light is red-shifted by a factor** $e^{-\kappa(v_B - v_A)}$, where $\kappa = \frac{1}{4M}$ is the surface gravity. We thus see an exponential redshift in advanced time v along the event horizon, where the exponential factor is given by the surface gravity.

Remark 4.8. *1. The observed frequency depends of course on the 4-velocity of the observer. However, note that* $[\partial_v - \partial_r, \underbrace{\partial_v}_{=\partial_t}] = 0$*, so the observer B arises from Lie-transporting the observer A to some later time along the flow-lines of the stationary Killing vector field ∂_t. So A and B are 'the same observers, just at different times'. If the observer B was boosted with respect to A, then one would of course pick up an additional Doppler contribution.*

2. *The above argument generalises to other black hole spacetimes. Black holes with $\kappa = 0$ are called* ***extremal black holes****. There is no red-shift along the event horizon of such black holes.*

3. *Another interpretation of the surface gravity is via Hawking radiation. $T_H = \frac{\kappa}{2\pi}$ is the temperature of the black hole.*

4.4 The Kerr black hole

Consider

$$g = g_{tt}\, dt^2 + g_{t\varphi}\,(dt \otimes d\varphi + d\varphi \otimes dt) + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2\, d\theta^2 + g_{\varphi\varphi}\, d\varphi^2 \tag{4.9}$$

where

$$\begin{aligned} g_{tt} &:= -1 + \frac{2Mr}{\rho^2} \\ g_{t\varphi} &:= -\frac{2Mra\sin^2\theta}{\rho^2} \\ g_{\varphi\varphi} &:= \big[r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\big]\sin^2\theta \end{aligned}$$

with

$$\rho^2 := r^2 + a^2 \cos^2\theta \qquad \text{and} \qquad \Delta := r^2 - 2Mr + a^2 \; .$$

Assume $0 < a < M$.[40] Then Δ has the two roots $r_\pm = M \pm \sqrt{M^2 - a^2}$ and we have $\Delta > 0$ for $r > r_+$. We first define (4.9) on the manifold $M := \underset{t}{\mathbb{R}} \times \underset{r}{(r_+, \infty)} \times \underset{\theta,\varphi}{\mathbb{S}^2}$. In (t, φ, r, θ) coordinates[41] the matrix of g becomes

$$g = \begin{pmatrix} g_{tt} & g_{t\varphi} & 0 & 0 \\ g_{t\varphi} & g_{\varphi\varphi} & 0 & 0 \\ 0 & 0 & \frac{\rho^2}{\Delta} & 0 \\ 0 & 0 & 0 & \rho^2 \end{pmatrix} \; .$$

A direct computation gives $g_{tt}g_{\varphi\varphi} - (g_{t\varphi})^2 = -\Delta \sin^2\theta$. Thus g is a Lorentzian metric on M. One can show that it is a solution of the vacuum Einstein equations. It is called the **Kerr solution**. The coordinates (t, φ, r, θ) are called **Boyer-Lindquist coordinates**. We also record the form of the inverse metric for later:

$$g^{-1} = \begin{pmatrix} -\frac{g_{\varphi\varphi}}{\Delta \sin^2\theta} & \frac{g_{t\varphi}}{\Delta \sin^2\theta} & 0 & 0 \\ \frac{g_{t\varphi}}{\Delta \sin^2\theta} & -\frac{g_{tt}}{\Delta \sin^2\theta} & 0 & 0 \\ 0 & 0 & \frac{\Delta}{\rho^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\rho^2} \end{pmatrix} \; . \tag{4.10}$$

__**The far field**__: We use $\tilde{r} := \frac{1}{2}\big(r - M + r(1 - \frac{2M}{r})^{1/2}\big)$ as a new radial coordinate as in Example 2.27 for the Schwarzschild metric and define the asymptotically Euclidean coordinates $x = \tilde{r} \sin\theta \cos\varphi$, $y = \tilde{r} \sin\theta \sin\varphi$, $z = \tilde{r}\cos\theta$. Then a computation shows that the metric (4.9) becomes[42]

$$\begin{aligned} g = &-\big(1 - \frac{2M}{\tilde{r}} + \mathcal{O}(\frac{1}{\tilde{r}^2})\big)\, dt^2 - \frac{4Ma}{\tilde{r}^3}\, dt[-ydx + xdy] + \mathcal{O}(\frac{1}{\tilde{r}^3}) dt(dx, dy, dz) \\ &+ (1 + \frac{2M}{\tilde{r}})(dx^2 + dy^2 + dz^2) + \mathcal{O}(\frac{1}{\tilde{r}^2})(dx, dy, dz)(dx, dy, dz) \; . \end{aligned}$$

Comparison with (2.24) shows that M is the total mass and $J = Ma$ the total angular momentum in z-direction. The parameter $a = \frac{J}{M}$ is the **angular momentum per unit mass**. We thus see that the Kerr metric describes the spacetime of a rotating body.

Remark 4.11. *One can show that for $a = 0$ the Kerr metric* (4.9) *reduces to the Schwarzschild metric with mass M. Also, when $M = 0$ (but not necessarilyi $a = 0$), the Kerr metric* (4.9) *equals the Minkowski metric in spheroidal coordinates. See problem sheet 4.*

The Kerr solution has two Killing vector fields ∂_t, ∂_φ (the metric components are independent of t and φ) which commute $[\partial_t, \partial_\varphi] = 0$. The Killing vector field ∂_t is timelike for large r. Thus Kerr is *stationary* (but not static, see problem sheet 3). The Killing vector field ∂_φ asymptotically generates rotations around the z-axis. We say that Kerr is *axisymmetric*.

[40] The case $a = M$ corresponds to an extremal black hole and $a > M$ is a naked singularity. Neither of those cases is being discussed in this course.

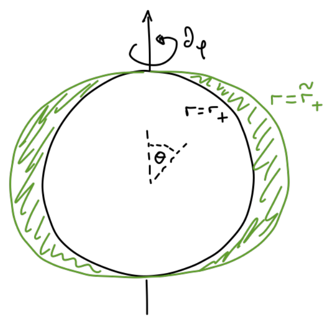
[41] Note the ordering of the coordinates.

[42] We use the notation $dtdx := \frac{1}{2}(dt \otimes dx + dx \otimes dt)$ etc.

Note that

$$g(\partial_t, \partial_t) = -1 + \frac{2Mr}{\rho^2} = \begin{cases} < 0 & \text{for } 2Mr < \rho^2 \\ = 0 & \text{for } 2Mr = \rho^2 \iff r = \tilde{r}_\pm := M \pm \sqrt{M^2 - a^2 \cos^2\theta} \\ > 0 & \text{for } 2Mr > \rho^2 \, . \end{cases}$$

The region $r_+ < r < \tilde{r}_+$, in which ∂_t is spacelike, is called the **ergoregion**.



Given a stationary observer A with 4-velocity $\sim (\partial_t + \Omega \cdot \partial_\varphi)$, then Ω is the angular frequency of the observer as seen by an observer B with velocity ∂_t at infinity (see problem sheet 4). Thus, A appears static to B if, and only if, $\Omega = 0$.

In order for an observer with 4-velocity $\sim (\partial_t + \Omega\partial_\varphi)$ to exist at radius r and latitude θ, we need

$$0 > g(\partial_t + \Omega\partial_\varphi, \partial_t + \Omega\partial_\varphi) = g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi} \, .$$

Thus we need $\Omega \in (\Omega_{\min}, \Omega_{\max})$ with

$$\Omega_{\min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \qquad \text{and} \qquad \Omega_{\max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \, ,$$

where $\omega = \frac{1}{2}(\Omega_{\min} + \Omega_{\max}) = -\frac{g_{\varphi t}}{g_{\varphi\varphi}}$.

i) Since $g_{\varphi t}^2 - g_{tt} g_{\varphi\varphi} = \Delta \sin^2\theta > 0$ for all $r > r_+$, we indeed have two roots $\Omega_{\min} < \Omega_{\max}$. Since $g_{tt} < 0$ for large r and $g_{\varphi\varphi} > 0$, we have $\Omega_{\min} < 0 < \Omega_{\max}$ for large r.

ii) At the boundary of the ergoregion $r = \tilde{r}_+$ we have $g_{tt} = 0$, and thus $\Omega_{\min} = 0$. In the ergoregion $g_{tt} > 0$ and hence $0 < \Omega_{\min} < \Omega_{\max}$. Thus, in the ergoregion stationary observers have to rotate in the φ-direction as seen from infinity. This is an extreme manifestation of the **gravitational dragging of frames by rotating bodies** in general relativity.

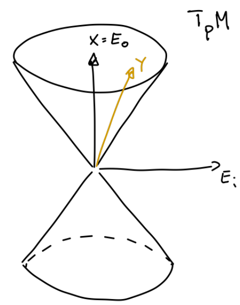
4.4.1 Global structure

We start with the following

Lemma 4.12. *Let (M, g) be a spacetime and let $X \in T_pM$ be future directed timelike and $Y \in T_pM$ future directed causal. Then $g(X, Y) < 0$.*

If X is future directed timelike and Y past directed causal, then $g(X, Y) > 0$.

Proof. Without loss of generality assume $g(X,X) = -1$ and let $E_0 := X, E_1, \ldots, E_n$ be an orthonormal basis. Then $Y = a(E_0 + \sum_{i=1}^n b^i E_i)$ with $\sum_{i=1}^n (b^i)^2 \leq 1$ (since Y is causal) and $a > 0$ (since Y is future directed). This gives $g(X,Y) = -a < 0$.



The case that Y is past directed causal follows analogously. $\square$

The metric (4.9) degenerates at $r = r_+$. We show that this is a coordinate singularity similar to the one at $r = 2M$ for Schwarzschild. Let $r^*(r)$ and $\overline{r}(r)$ be two functions on (r^+, ∞) which satisfy

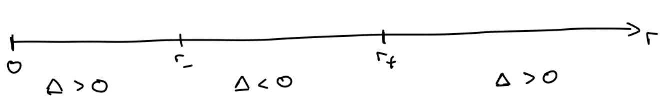
$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta} \qquad \text{and} \qquad \frac{d\overline{r}}{dr} = \frac{a}{\Delta} .$$

We then define $v_\pm := t \pm r^*$ and $\varphi_\pm := \varphi \pm \overline{r} \mod 2\pi$. $(v_\pm, r, \theta, \varphi_\pm)$ are Eddington-Finkelstein-like coordinates ("+" for ingoing, "−" for outgoing). In $(v_+, r, \theta, \varphi_+)$ coordinates the metric (4.9) takes the form

$$\begin{aligned} g = g_{tt}\, dv_+^2 + g_{t\varphi}\,(dv_+ \otimes d\varphi_+ + d\varphi_+ \otimes dv_+) + g_{\varphi\varphi}\, d\varphi_+^2 + \rho^2\, d\theta^2 \\ + (dv_+ \otimes dr + dr \otimes dv_+) - a \sin^2\theta\,(d\varphi_+ \otimes dr + dr \otimes d\varphi_+) \end{aligned} \tag{4.13}$$

which is a Lorentzian metric on $\tilde{M} := \underset{v_+}{\mathbb{R}} \times \underset{r}{(0,\infty)} \times \underset{\theta,\varphi_+}{\mathbb{S}^2}$. Note that $-\frac{\partial}{\partial r}$ in these coordinates is a continuous non-vanishing null vector field. It thus fixes a time-orientation on $(\tilde{M}, g)$. Also note that[43] $\frac{\partial}{\partial t}\Big|_{\mathrm{BL}} = \frac{\partial}{\partial v_+}$ and that $g(-\partial_r, \partial_{v_+}) = -1$, thus $-\partial_r$ determines the same time-orientation as $\partial_{v_+} = \partial_t|_{\mathrm{BL}}$ for large r by Lemma 4.12.

In the following we investigate the causal structure of this spacetime.



For $r \in (r_-, r_+)$ choose a functions $r^*_{\mathrm{int}}(r)$ and $\overline{r}_{\mathrm{int}}(r)$ with $\frac{dr^*_{\mathrm{int}}}{dr} = \frac{r^2+a^2}{\Delta}$ and $\frac{d\overline{r}_{\mathrm{int}}}{dr} = \frac{a}{\Delta}$ and set $t = v_+ - r^*_{\mathrm{int}}$ and $\varphi = \varphi_+ - \overline{r}_{\mathrm{int}} \mod 2\pi$ to obtain again the form (4.9) of the metric (4.13) in the region $\mathbb{R} \times (r_-, r_+) \times \mathbb{S}^2$ in Boyer-Lindquist coordinates (t, r, θ, φ).

We want to compute $g^{-1}(dr, dr)$. For $r \in (r_-, r_+)$ and $r \in (r_+, \infty)$ we can use Boyer-Lindquist coordinates and (4.10) to easily obtain $g^{-1}(dr, dr) = \frac{\Delta}{\rho^2}$ in $(r_-, r_+) \cup (r_+, \infty)$. By continuity we thus

[43] Here, and in the following, we will mark coordinate vector fields with respect to the Boyer-Lindquist coordinates by BL. If no subscript is given, they are coordinate vector fields with respect to the $(v_+, r, \theta, \varphi_+)$ coordinates.

infer[44]

$$g^{-1}(dr, dr) = \frac{\Delta}{\rho^2} \qquad \text{for } r_- < r < \infty\ . \tag{4.14}$$

Hence

$$\{r = r_0\} \text{ is a } \begin{cases} \text{timelike hypersurface for } r_+ < r_0 < \infty \\ \text{null hypersurface for } r_0 = r_+ \\ \text{spacelike hypersurface for } r_- < r_0 < r_+\ . \end{cases}$$

Proposition 4.15. *The hypersurface $\{r = r_+\}$ is a black hole event horizon.*

Proof. Consider the vector field $(dr)^\sharp$ for $r_- < r \leq r_+$. By (4.14) it is a causal vector field. Moreover we have $g(-\frac{\partial}{\partial r}, (dr)^\sharp) = -dr(\frac{\partial}{\partial r}) = -1$ and thus it is future directed causal by Lemma 4.12. Let now $\gamma : I \to \tilde{M}$ be a future directed timelike curve in $r_- < r \leq r_+$. Then by Lemma 4.12 we have

$$\dot{\gamma}^r = \dot{\gamma}(r) = dr(\dot{\gamma}) = g((dr)^\sharp, \dot{\gamma}) < 0\ ,$$

and thus once a future directed timelike curve has entered the region $\{r_- < r \leq r_+\}$, its r-coordinate value can never increase beyond r_+. By continuity the above argument extends to future directed *causal* curves γ. Thus the region $r \leq r_+$ lies inside the black hole region.

On the other hand it is easy to see that for every $r_0 > r_+$ there are future directed causal curves starting from r_0 which reach into the asymptotically flat region $r \gg r_+$ and thus to future null infinity. For example we have shown in the last section that for each $r > r_+$ there is $\Omega(r)$ such that $\partial_t|_{\mathrm{BL}} + \Omega(r)\partial_\varphi|_{\mathrm{BL}}$ is future directed timelike. Since the cone of timelike vectors is open we can add a bit of $\partial_r|_{\mathrm{BL}}$ so that it stays timelike, i.e., $\partial_t|_{\mathrm{BL}} + \Omega(r)\partial_\varphi|_{\mathrm{BL}} + \varepsilon(r)\partial_r|_{\mathrm{BL}}$ with $\varepsilon(r) > 0$. The integral curves then reach the asymptotically flat region $r \gg r_+$. This shows that $r > r_+$ lies in the past of future null infinity and thus $\{r = r_+\}$ is indeed a black hole event horizon. □

Proposition 4.16. *$\{r = r_+\}$ is a Killing horizon of the Killing vector field $T_H = \partial_{v_+} + \frac{a}{r_+^2 + a^2}\partial_{\varphi_+}$ with surface gravity $\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)}$.*

Proof. First observe that $\partial_t|_{\mathrm{BL}} = \partial_{v_+}$ and $\partial_\varphi|_{\mathrm{BL}} = \partial_{\varphi_+}$. Thus T_H is indeed a Killing vector field.

The normal of $\{r = r_+\}$ is given by $(dr)^\sharp$. We first compute for $r > r_+$ in Boyer-Lindquist coordinates $(dr)^\sharp = g^{rr}\frac{\partial}{\partial r}\Big|_{\mathrm{BL}} = \frac{\Delta}{\rho^2}\frac{\partial}{\partial r}\Big|_{\mathrm{BL}}$ and

$$\frac{\partial}{\partial r}\Big|_{\mathrm{BL}} = \frac{\partial v_+}{\partial r}\Big|_{\mathrm{BL}}\frac{\partial}{\partial v_+} + \frac{\partial r}{\partial r}\Big|_{\mathrm{BL}}\frac{\partial}{\partial r} + \frac{\partial \varphi_+}{\partial r}\Big|_{\mathrm{BL}}\frac{\partial}{\partial \varphi_+} = \frac{r^2 + a^2}{\Delta}\frac{\partial}{\partial v_+} + \frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial \varphi_+}\ .$$

Thus by continuity we obtain

$$\begin{aligned} (dr)^\sharp|_{r=r_+} &= \Big[\frac{r^2 + a^2}{\rho^2}\frac{\partial}{\partial v_+} + \frac{\Delta}{\rho^2}\frac{\partial}{\partial r} + \frac{a}{\rho^2}\frac{\partial}{\partial \varphi_+}\Big]\Big|_{r=r_+} \\ &= \frac{r_+^2 + a^2}{r_+^2 + a^2\cos^2\theta}\Big[\frac{\partial}{\partial v_+} + \frac{a}{r_+^2 + a^2}\frac{\partial}{\partial \varphi_+}\Big]\ , \end{aligned}$$

[44] The region $r \leq r_-$ is not discussed in this course – it is not physically relevant. The region $r_- < r < \infty$ in the $(v_+, r, \theta, \varphi_+)$ coordinates is isometric to the regions I and II in the Penrose diagram in the next section.

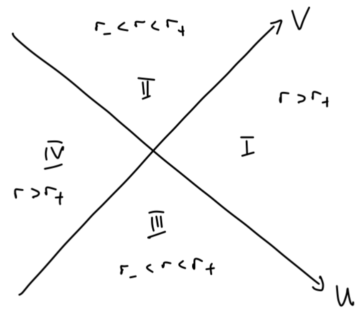
which shows that T_H is indeed normal to $\{r = r_+\}$.

To compute the surface gravity we use the formula $\partial_a\big((T_H)^b(T_H)_b\big) = -2\kappa_+(T_H)_a$ from (4.7), which is left as an exercise. $\square$

4.4.2 Penrose diagram

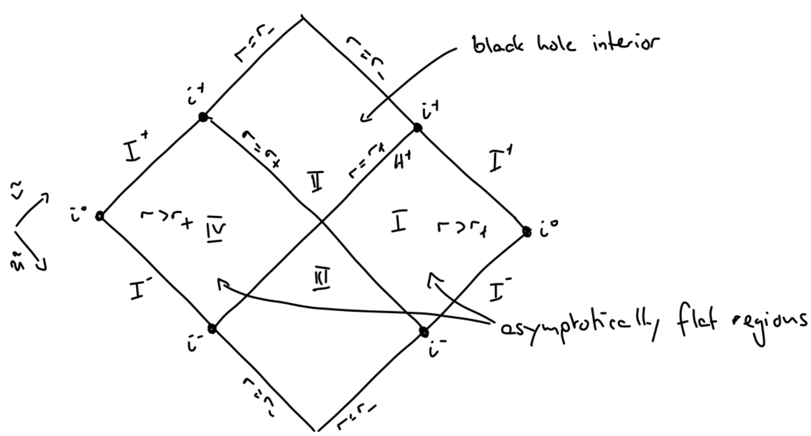
A Penrose-like diagram for Kerr is more difficult to draw since Kerr is not spherically symmetric and so one loses more information if one quotients out the spheres. Here, we restrict to the two-dimensional axis $\theta = 0, \pi$.

We define Kruskal-like coordinates $U := e^{-\kappa_+ v_-}$ and $V := e^{\kappa_+ v_+}$, and $\Phi := \varphi - \frac{at}{r_+^2+a^2} \mod 2\pi$ in the region $r_+ < r < \infty$.[45] Then one can show that in the coordinates (U, V, θ, Φ) the metric extends analytically to $\mathbb{R}^2 \times \mathbb{S}^2$, where r ranges from $r_- < r < \infty$.[46]



Each region in this diagram is isometric to (4.9) in Boyer-Lindquist coordinates where the r-coordinate is restricted to the corresponding range.

For $\theta = 0, \pi$ the metric takes the form $g = F(r)dUdV$, where $F(r)$ is an analytic function in r and $r = r(UV)$. We can now compactify by setting $\tilde{u} := \arctan U$ and $\tilde{v} := \arctan V$ and draw the following **Penrose-like diagram for Kerr**:



The hypersurfaces $\{r = r_-\}$ in the black hole interior are called the **Cauchy horizon**. The metric can be extended past it but the extension is not expected to correspond to anything physical. One

[45] Recall that $v_\pm = t \pm r^*$, see Section 4.4.1.

[46] See for example the book 'The geometry of Kerr black holes' by O'Neill.

expects that for small gravitational perturbations the Cauchy horizon turns into a singularity.