1 Equations of motion

1.1 Introduction

In this section we will derive the equations of motion for an *inviscid* fluid, that is a fluid with zero viscosity. We begin by setting up the basic concepts that are needed to describe the motion of a continuous medium in three dimensions, and the fundamental *kinematic* equations relating the density to the deformation and the velocity. Next we derive the momentum equation for an inviscid fluid. By assuming that the fluid has constant density, we obtain a closed system of equations for the velocity and pressure, known as the *Euler equations.* We introduce the concepts of vorticity and circulation, and explain why it is reasonable to assume that most flows are *irrotational.* Finally, we will show that incompressible irrotational flow is governed by Laplace's equation.

It is worth emphasising that this course relies on familiarity with various concepts from Mods, in particular vector calculus and manipulations of multidimensional integrals. The basic material relevant to this section is collected in §1.6.

1.2 Kinematics

1.2.1 Preliminaries

The term *kinematics* refers to "the science of pure motion, considered without reference to the matter or objects moved or the force producing or changing the motion."[1] In this section, we will examine what can be said about the motion of *any* continuous medium, although we will often use the word "fluid" to help fix ideas. We will later restrict our attention to inviscid fluids in §1.3.

In a continuous medium, all state variables, such as density, velocity and pressure, are assumed to be continuous functions of position $\boldsymbol{x}$ and time t; in fact in this course we will assume that all dependent variables are continuously differentiable.

1.2.2 Eulerian and Lagragian variables

We can describe the motion of a fluid by tracking the position of each material "particle" or "element" as the medium deforms. Suppose that the fluid starts in some *reference state* at time zero before being subsequently deformed, so that a material point initially at position $\boldsymbol{X} = (X, Y, Z)$ is displaced to a new position $\boldsymbol{x} = (x, y, z)$ at time t, as illustrated in Figure 1.1. The initial position vector $\boldsymbol{X}$ defines the *Lagrangian* coordinates

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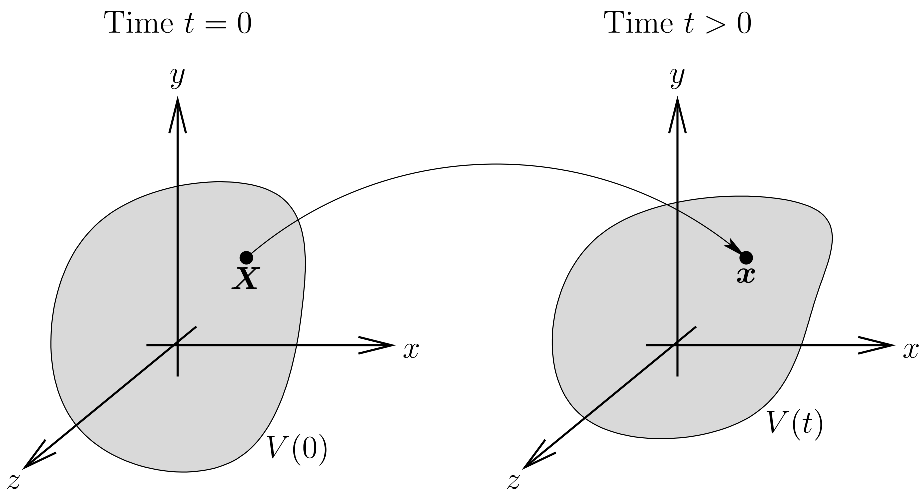

Figure 1.1: Schematic of the deformation of a fluid occupying a volume $V(t)$. The highlighted particle has *Eulerian* coordinate $\boldsymbol{x}$ and *Lagrangian* coordinate $\boldsymbol{X}$.

of each material element, while the current position vector $\boldsymbol{x}$ gives its *Eulerian* coordinates. A deformation of the medium corresponds to a mapping from each element's initial position to its current position at time t, that is a vector-valued transformation $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$. Our assumption that the medium is continuous implies that this mapping should be continuous and one-to-one, so that each element in the reference configuration is displaced continuously to a unique element in the deformed state. A sufficient condition for this to be true is[2]

$$0 < J < \infty, \tag{1.1}$$

where J is the *Jacobian* of the transformation, that is

$$J = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \begin{vmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{vmatrix}. \tag{1.2}$$

1.2.3 The convective derivative

In fluid dynamics, it is usually more convenient to use Eulerian variables, so we would write any property of the fluid (for example density, pressure, temperature, *etc.*) as a function of current position vector $\boldsymbol{x}$ and time t, say $f(\boldsymbol{x}, t)$. Then fixing $\boldsymbol{x}$ and letting t increase corresponds to following the time variation of f at a fixed point in space. Alternatively, we could write the same property as a function of Lagrangian variables by defining

$$F(\boldsymbol{X}, t) \equiv f\big(\boldsymbol{x}(\boldsymbol{X}, t), t\big). \tag{1.3}$$

[2] *cf* Part A Multivariable Calculus

Now, fixing $\boldsymbol{X}$ corresponds to tracking how f varies for a particular material element that moves with the deforming fluid.

These two viewpoints prompt us to define two different time derivatives. We use the usual partial derivative notation to denote the Eulerian time derivative, at a fixed position $\boldsymbol{x}$ in space, that is

$$\frac{\partial}{\partial t} := \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} = \text{rate of change with } \boldsymbol{x} \text{ held constant.} \tag{1.4}$$

On the other hand, we introduce the notation

$$\frac{\mathrm{D}}{\mathrm{D}t} := \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}} = \text{rate of change with } \boldsymbol{X} \text{ held constant,} \tag{1.5}$$

to denote the Lagrangian time derivative (the notation $\mathrm{d}/\mathrm{d}t$ is also often employed). With $\boldsymbol{X}$ held constant, $\mathrm{D}/\mathrm{D}t$ corresponds to the rate of change following an element that convects with the fluid, and it is referred to as the *convective derivative* or the *material derivative* or sometimes the *derivative following the flow.*

1.2.4 Velocity and acceleration

The *velocity* $\boldsymbol{u}$ of the fluid is simply the rate of change of the position vector $\boldsymbol{x}$ for a material element, that is

$$\boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t}. \tag{1.6}$$

In general, the velocity will be a function of both position and time, so that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$. If $\boldsymbol{u}$ does *not* depend on t, then the we say that the flow is *steady.*

We can now relate the Eulerian and Lagrangian time derivatives in terms of the velocity $\boldsymbol{u}$. The chain rule implies that

$$\frac{\mathrm{D}f}{\mathrm{D}t}(\boldsymbol{x}, t) = \frac{\mathrm{D}t}{\mathrm{D}t}\frac{\partial f}{\partial t}(\boldsymbol{x}, t) + \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t} \cdot \boldsymbol{\nabla} f(\boldsymbol{x}, t) \tag{1.7}$$

and hence

$$\frac{\mathrm{D}f}{\mathrm{D}t} \equiv \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} f, \tag{1.8}$$

for any differentiable function f. We can write this in operator form as

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}), \tag{1.9}$$

where $(\boldsymbol{u} \cdot \boldsymbol{\nabla})$ is shorthand for the directional derivative

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla}) \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{1.10}$$

and $\boldsymbol{u} = (u, v, w)$ are the components of the velocity.

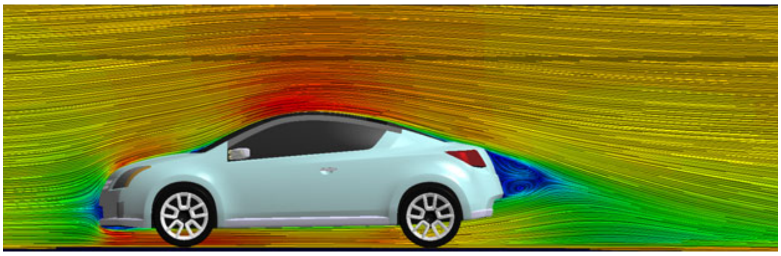

Figure 1.2: Streamlines for air flow over a car.

In particular, we can now compute the *acceleration* of the fluid, namely the material rate of change of the velocity:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \equiv \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}. \tag{1.11}$$

Note the way that the second term is grouped: $(\boldsymbol{u}\cdot\boldsymbol{\nabla})$ is a linear scalar differential operator which can easily be applied to the vector $\boldsymbol{u}$. Had we instead written this term as $\boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{u})$ we would have faced the problem of defining the grad of a vector. This can be done, but is to be avoided throughout this course.

1.2.5 Flow visualisation

If we know the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, there are several ways of trying to visualise it. One is to plot the *streamlines*. This corresponds to taking a snapshot of the flow at a fixed time t, then plotting curves that are everywhere parallel to the velocity field $\boldsymbol{u}$. This results in the sort of plots that we often see in car commercials, for example: see Figure 1.2.

With t held constant, we can construct a curve that is everywhere parallel to $\boldsymbol{u}$ by solving the simultaneous differential equations

$$\frac{\mathrm{d}\boldsymbol{x}(s)}{\mathrm{d}s} = \boldsymbol{u}\big(\boldsymbol{x}(s),t\big).$$

By solving these differential equations with different initial conditions, we will obtain a family of streamlines.

Example 1.1 *The two-dimensional velocity field* $\boldsymbol{u} = (x,-y,0)$ *is called a* stagnation point flow. *We find the streamlines by solving the differental equations*

$$\frac{\mathrm{d}x}{\mathrm{d}s} = x, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = -y, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = 0.$$

The solution is easily found to be

$$x = A\mathrm{e}^{s}, \qquad y = B\mathrm{e}^{-s}, \qquad z = C,$$

where A, B *and* C *are integration constants. These parametrise the hyperbolae* $xy = AB = \mathrm{const}$ *in the* (x,y)*-plane, as illustrated in* Figure 1.3.

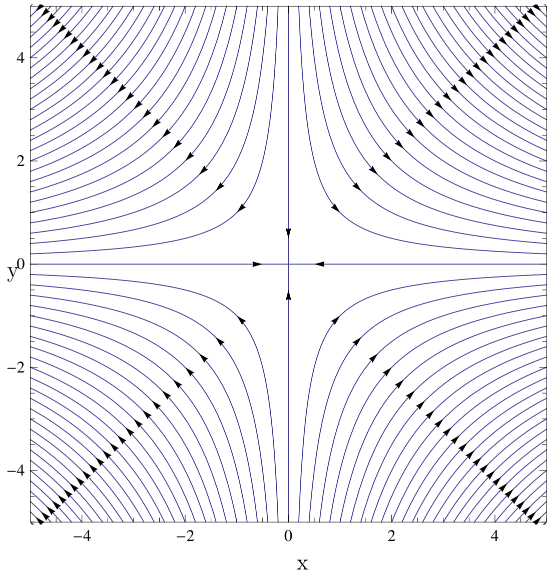

Figure 1.3: Streamlines for a stagnation point flow.

Plotting streamlines is similar to plotting phase plane trajectories for plane autonomous differential equations. The streamlines have a unique tangent vector equal to $\boldsymbol{u}$ at each point in the flow. Hence they can only cross at so-called *stagnation points*, where $\boldsymbol{u} = \boldsymbol{0}$ and the fluid is locally stationary. In Example 1.1, there is just one stagnation point at the origin, and it resembles a saddle point in a phase plane.

An alternative flow visualisation strategy, often used in experiments, is to insert tiny tracer particles into the flow and follow their trajectories. Assuming that each particle moves with the local flow velocity, its position vector $\boldsymbol{x}(t)$ must satisfy the differential equation

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{u}\big(\boldsymbol{x}(t), t\big). \tag{1.12}$$

Solutions of this equation are called *particle paths* for the flow $\boldsymbol{u}(\boldsymbol{x}, t)$.

If the flow is steady (*i.e.* $\boldsymbol{u}$ is independent of t), then we see that the streamline pattern will not vary with time, and that the streamlines and particle paths will coincide. For an unsteady flow, though, the streamline pattern will in general vary with time and not coincide with the particle paths.

Example 1.2 *For the two-dimensional unsteady flow $\boldsymbol{u} = (\cos t, \sin t, 0)$, the streamlines satisfy*

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos t, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin t, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = 0, \tag{1.13}$$

with t held fixed, and hence are given by

$$x = A + s\cos t, \qquad y = B + s\sin t, \qquad z = C,$$

where A, B and C are integration constants. These give a family of parallel straight lines in the (x, y)-plane whose direction rotates as t varies, as shown in Figure 1.4.

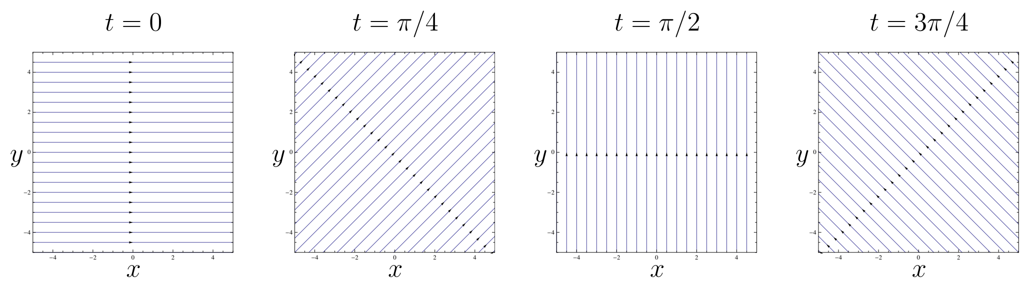

Figure 1.4: Streamlines for the unsteady flow (1.13).

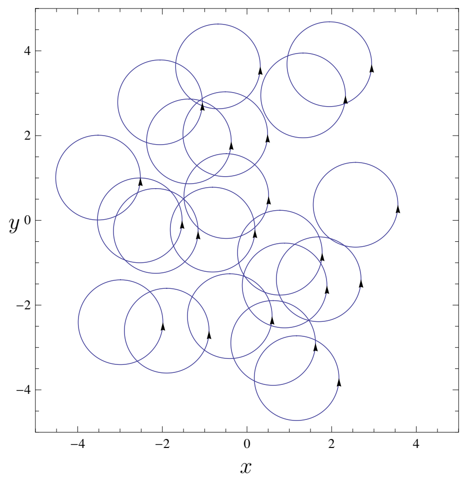

Figure 1.5: Particle paths for the unsteady flow (1.13).

The particle paths satisfy the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \sin t, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 0, \tag{1.14}$$

and hence are given by

$$x = A + \sin t, \qquad y = B - \cos t, \qquad z = C,$$

for some constants A, B, C. Particles therefore trace out unit circles in the (x, y)-plane; some examples are shown in Figure 1.5.

1.2.6 Euler's identity

Now we derive an important result showing how the divergence of the velocity is related to expansion or contraction of the fluid. The derivation is lengthy and non-examinable but included here for completeness. We start by differentiating the definition (1.2) of J

with respect to t, differentiating each row of the determinant in term to obtain

$$\frac{\mathrm{D}J}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} = \begin{vmatrix} \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x}{\partial X}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x}{\partial Y}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x}{\partial Z}\right) \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}$$
$$+ \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial y}{\partial X}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial y}{\partial Y}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial y}{\partial Z}\right) \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial z}{\partial X}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial z}{\partial Y}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial z}{\partial Z}\right) \end{vmatrix}. \tag{1.15}$$

For convenience we denote the three determinants on the right-hand side of (1.15) by Δ_1, Δ_2 and Δ_3 respectively. Since the convective derivative is taken with $\boldsymbol{X}$ fixed, it commutes with X-, Y- and Z-derivatives. Recalling also that $\mathrm{D}x/\mathrm{D}t = u$, we can rewrite Δ_1 as

$$\Delta_1 = \begin{vmatrix} \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}. \tag{1.16}$$

We apply the chain rule to each of the derivatives in the first row to obtain

$$\Delta_1 = \frac{\partial u}{\partial x}\begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} + \frac{\partial u}{\partial y}\begin{vmatrix} \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} + \frac{\partial u}{\partial z}\begin{vmatrix} \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}. \tag{1.17}$$

The final two determinants in (1.17) have repeated rows and are therefore identically zero. It follows that

$$\Delta_1 = \frac{\partial u}{\partial x}J, \tag{1.18}$$

and analogous manipulations lead to

$$\Delta_2 = \frac{\partial v}{\partial y}J, \qquad \Delta_3 = \frac{\partial w}{\partial z}J. \tag{1.19}$$

By substituting these into (1.15), we obtain *Euler's identity*

$$\frac{\mathrm{D}J}{\mathrm{D}t} = J\boldsymbol{\nabla}\cdot\boldsymbol{u}. \tag{1.20}$$

Recall that the Jacobian relates infinitesimal volumes in the Eulerian and Lagrangian frames, via $\mathrm{d}x\mathrm{d}y\mathrm{d}z = J\mathrm{d}X\mathrm{d}Y\mathrm{d}Z$. Hence we can interpret J as measuring the local expansion or contraction: the fluid is expanding if J is increasing with time or contracting if J decreases with time. The identity (1.20) shows how this local expansion or contraction of the medium is related to the divergence of the velocity. A flow is said to be *incompressible* or *volume-preserving* if it preserves infinitesimal volumes, that is if $\mathrm{D}J/\mathrm{D}t \equiv 0$. From (1.20), we see that

$$\text{flow is incompressible} \quad \Leftrightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} \equiv 0. \tag{1.21}$$

1.2.7 Reynolds' transport theorem

Consider a time-dependent volume $V(t)$ that is convected by the fluid, so that it always consists of the same fluid particles. Then, for any function $f(\boldsymbol{x}, t)$ that is continuously differentiable with respect to all of its arguments,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{V(t)} \frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\boldsymbol{u})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.22}$$

To prove this important result, we transform the integral on the left-hand side into Lagrangian variables to obtain

$$I(t) := \iiint_{V(t)} f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{V(0)} fJ\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z, \tag{1.23}$$

where J again denotes the Jacobian (1.2). In (1.23), the Lagrangian integral is over the fixed initial domain $V(0)$ corresponding to the moving volume $V(t)$. We can therefore differentiate through the integral to obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \iiint_{V(0)} \frac{\mathrm{D}}{\mathrm{D}t}(fJ)\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z, \tag{1.24}$$

where the time derivative is taken with the integration variables (X, Y, Z) held fixed.

Now we expand out the derivative in (1.24) and use Euler's identity (1.20) to obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \iiint_{V(0)} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) J\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z = \iiint_{V(t)} \frac{\mathrm{D}f}{\mathrm{D}t} + f\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.25}$$

The definition (1.9) of the convective derivative then leads to Reynolds' Transport Theorem (1.22).

1.2.8 Conservation of mass

It is instructive to consider mass conservation from both Eulerian and Lagrangian viewpoints. To begin with, consider a volume D which is fixed in space, so that fluid flows in and out through its boundary ∂D. The net mass of fluid inside D is given by

$$\iiint_D \rho(\boldsymbol{x}, t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

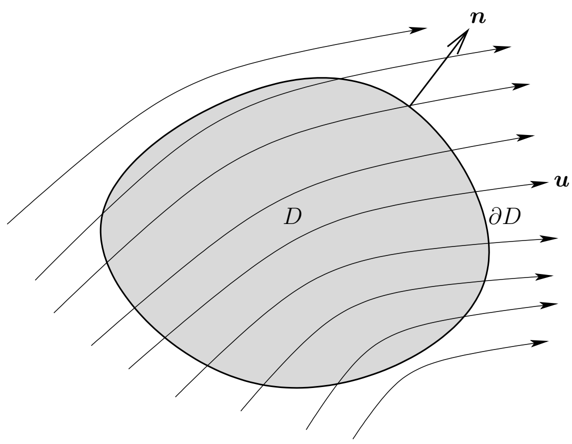

Figure 1.6: Schematic of a region D, fixed in space, with fluid flowing in and out through its boundary ∂D.

where ρ is the *density*, which may in general vary with both position and time. The net rate at which mass flows *out* of V is given by

$$\iint_{\partial D} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S,$$

where $\boldsymbol{n}$ is the unit outward-pointing normal to ∂D. Since mass can neither be created nor destroyed inside D, we must have

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_D \rho \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = - \iint_{\partial D} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{1.26}$$

We can commute the differentiation and integration on the left-hand side to write

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_D \rho(\boldsymbol{x}, t) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_D \frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.27}$$

Note that, when we differentiate through the integral, the time derivative $\partial/\partial t$ is performed while holding the integration variables (x, y, z) constant. Applying the Divergence Theorem to the right-hand side of (1.26), we therefore obtain

$$\iiint_D \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0. \tag{1.28}$$

This result must hold for *any* fixed volume D, and it follows that (assuming it is continuous) the integrand must be zero. We therefore deduce the equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \tag{1.29}$$

relating the density and velocity in any continuous medium. We can expand out the divergence here to write (1.29) in the equivalent form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{1.30}$$

which demonstrates how the rate of change of the density and the divergence of the velocity field are intimately related.

Using a Lagrangian approach, we would instead consider the mass of a material volume $V(t)$ that is convected by the flow, so that it always consists of the same fluid elements. As above, we can write the net mass inside V in the form

$$\iiint_{V(t)} \rho \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Now the integration region V varies with t, so we cannot directly differentiate through the integral. Instead, we can use the Transport Theorem (1.22) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_{V(t)} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.31}$$

Since the volume $V(t)$ is defined to consist always of the same fluid elements, its mass cannot change with time. This must be true for all material volumes, and, as above, we deduce that the integrand on the right-hand side of (1.31) must be zero (assuming it is continuous). Hence we reproduce the mass conservation equation (1.29).

We can use (1.29) to deduce the following useful corollary of the transport theorem. If $f = \rho h$ in (1.22), where h is any continuously differentiable function, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho h \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_{V(t)} \rho \frac{\mathrm{D}h}{\mathrm{D}t} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.32}$$

1.3 The Euler equations

1.3.1 Conservation of momentum

Thus far we have been concerned just with describing the motion of a continuous medium, without considering what sort of medium it is (*e.g.* solid, liquid, gas, *etc.*) or what is causing it to move. Next we derive an equation linking the velocity of the fluid to the applied forces by applying Newton's second law, namely "force equals rate of change of momentum" to a material volume $V(t)$. The net momentum of such a volume is

$$\iiint_V \rho\boldsymbol{u} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

while the applied force has two ingredients. First there is the *external* body force $\boldsymbol{g}$ per unit mass, which contributes a net force

$$\iiint_V \rho\boldsymbol{g} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

In this course, we will ususally think of $\boldsymbol{g}$ as being the acceleration due to gravity, although it might also incorporate other effects such as electromagnetic forces on a liquid metal.

Second there is the *internal* force exerted on each volume V by the surrounding fluid. We suppose that this may be accounted for by a *pressure*, p, which acts in the inward normal direction at each point, so the net internal force on V is

$$\iint_{\partial V} -p\boldsymbol{n}\,\mathrm{d}S = \iiint_V -\boldsymbol{\nabla} p\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

using a well-known corollary of the divergence theorem. It is at this stage that we have restricted ourselves to considering *inviscid fluids*. Other continuous media, such as viscous fluids[3] or elastic solids,[4] can transmit tangential as well as normal internal forces.

Now we can formulate Newton's second law in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\boldsymbol{u}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_V -\boldsymbol{\nabla} p\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_V \rho\boldsymbol{g}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.33}$$

To calculate the left-hand side, we apply the transport theorem corollary (1.32) and hence obtain

$$\iiint_V \left(\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p - \rho\boldsymbol{g}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \boldsymbol{0}, \tag{1.34}$$

which must hold for all material volumes V. It follows that (assuming it is continuous) the integrand must be zero, and we therefore obtain the *momentum equation*

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \rho\boldsymbol{g}. \tag{1.35}$$

1.3.2 Incompressible flow

We have shown that conservation of mass and momentum for an inviscid fluid leads to the scalar equation (1.29) and the vector equation (1.35). In total, we therefore have four scalar equations for five unknowns: ρ, p and the three components of $\boldsymbol{u}$. We therefore need more information to close the system. One possibility is to try and obtain a relation between the pressure and the density. This is the focus of *compressible* fluid dynamics, which describes such phenomena as sound waves and shock waves in gases.[5] However, it is an empirical observation that, in most liquids, the density varies by only a few per cent under typical variations in temperature and pressure. It is therefore common to assume that liquids have *constant density*, and we will see this approximation allows many familiar and important flows to be described.

With $\rho = \text{constant}$, we deduce from (1.30) that

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{1.36}$$

and we recall from (1.21) that the flow is therefore incompressible. The implication does not quite go the other way. If the flow is incompressible, then (1.30) reduces to

[3] *cf* B6a Viscous Flow
[4] *cf* C6.1a Solid Mechanics, C6.2b Elasticity and Plasticity
[5] *cf* B6b Waves & Compressible Flow

$\mathrm{D}\rho/\mathrm{D}t = 0$, so that ρ is preserved following the flow, but need not be constant (this can occur for example in *stratified* fluids). However, the term *incompressible* is often slightly abused to refer to constant-density fluids, and we will assume that ρ is constant throughout the remainder of this course.

By expanding out the convective derivative, we can write the momentum equation (1.35) in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{g}. \tag{1.37}$$

Now (1.36) and (1.37) amount to a closed system of four scalar equations for p and the three components of $\boldsymbol{u}$, known as the *Euler equations*.

If we assume that the body force is *conservative*, then it may be written as $\boldsymbol{g} = -\boldsymbol{\nabla}\chi$ in terms of a potential χ. For example, a constant gravitational acceleration in the $-z$-direction corresponds to $\boldsymbol{g} = -g\boldsymbol{e}_z$ and hence $\chi = gz$. We also note the vector identity

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} \equiv \boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{u}|^2\right) + (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u}, \tag{1.38}$$

whose proof is a straightforward exercise.

We can therefore rearrange (1.37) to the alternative form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi\right). \tag{1.39}$$

For steady flow, the first term on the left-hand side is zero. If we dot the whole equation with $\boldsymbol{u}$, then the second term also disappears, since we end up with a triple scalar product $[\boldsymbol{u}, \boldsymbol{\nabla}\times\boldsymbol{u}, \boldsymbol{u}]$ with two repeated entries. It follows that

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi\right) = 0 \tag{1.40}$$

when $\partial\boldsymbol{u}/\partial t = \boldsymbol{0}$, and from this we deduce that

$$\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi \text{ is constant along streamlines in steady flow.} \tag{1.41}$$

This is known as **Bernoulli's Theorem for steady flow**. We will see shortly that various different versions of Bernoulli's Theorem may apply when the flow is not steady.

1.3.3 Boundary conditions

If the fluid is in contact with a fixed rigid boundary B, then the normal velocity of the fluid there must be zero, that is

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } B, \tag{1.42}$$

where $\boldsymbol{n}$ denotes the unit normal to B. This condition states that the fluid can neither flow through B nor separate from B, leaving behind a vacuum. However, it says nothing about the *tangential* velocity.[6]

Free boundaries will be introduced later in the course when we study water waves.

[6] Notice the contrast with a *viscous* fluid, in which *all* the velocity components are zero on a fixed boundary: $\boldsymbol{u} = \boldsymbol{0}$ on B. While a viscous fluid "sticks" to B, an inviscid fluid may slide past.

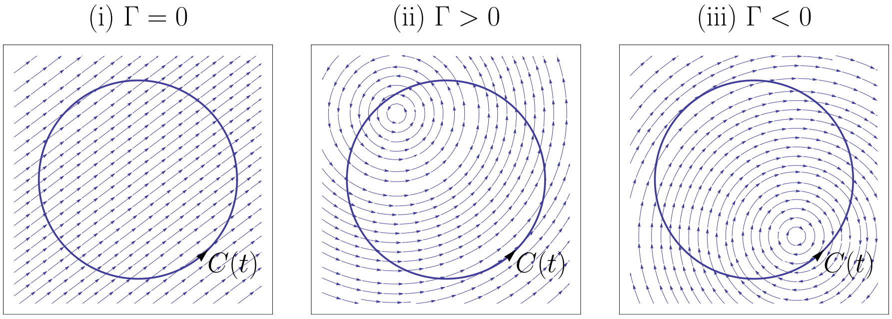

Figure 1.7: A closed curve $C(t)$ in flows with (i) zero circulation, (ii) positive circulation, (iii) negative circulation.

1.4 Vorticity and circulation

1.4.1 The vorticity equation

The *vorticity* $\boldsymbol{\omega}$ is defined to be the curl of the velocity field:

$$\boldsymbol{\omega} := \boldsymbol{\nabla} \times \boldsymbol{u}. \tag{1.43}$$

The vorticity is a measure of the local rotation of the flow. We can obtain an equation for $\boldsymbol{\omega}$ by taking the curl of the momentum equation in the form (1.39), recalling that curl grad $\equiv \mathbf{0}$, so that

$$\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \mathbf{0}. \tag{1.44}$$

The partial derivative $\partial/\partial t$ commutes with the curl operator, since it is taken with the Eulerian coordinates (x, y, z) held constant, and we can expand out the second term in (1.44) by using the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{v}) \equiv (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v}. \tag{1.45}$$

We recall that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ for incompressible flow, and $\boldsymbol{\omega}$ is likewise divergence-free, since div curl $\equiv 0$. Hence (1.44) may be rearranged to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u}, \tag{1.46}$$

which is known as the *vorticity equation.*

We see that $\boldsymbol{\omega}$ is not in general preserved following the flow. However, equation (1.46) suggests that, if $\boldsymbol{\omega}$ is initially zero, then it will remain zero for all time. To establish this fact, it is helpful first to introduce the concept of *circulation.*

1.4.2 Kelvin's Circulation Theorem

Consider a closed curve $C(t)$ that is convected by the flow, for example a smoke ring. We define the *circulation* around such a curve by

$$\Gamma(t) = \oint_{C(t)} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}. \tag{1.47}$$

The circulation is thus the net flow along the closed curve $C(t)$. Figure 1.7 shows schematically how circulation is related to rotation in the flow. In diagram (i) there is no rotation. The net clockwise and anticlockwise flows around C will cancel, resulting in a net circulation of zero. In diagram (ii), there is an anti clockwise rotation in the flow, resulting in a positive circulation about C. Finally, in diagram (iii) we see that a clockwise-rotating flow leads to a negative circulation about C.

We can also relate circulation to vorticity, since Stokes' Theorem implies that

$$\Gamma(t) = \iint_{S(t)} (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}S = \iint_{S(t)} \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{1.48}$$

where S is any surface spanning C. This reinforces the connection between vorticity and rotation in the flow alluded to in §1.4.1.

Kelvin's Circulation Theorem states that Γ is independent of t, and we will prove it by showing that $\mathrm{d}\Gamma/\mathrm{d}t$ is zero. To differentiate Γ, it is helpful to transform the integral to Lagrangian variables, using the chain rule:

$$\Gamma = \oint_{C(t)} \sum_i u_i \, \mathrm{d}x_i = \oint_{C(0)} \sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} \, \mathrm{d}X_j. \tag{1.49}$$

With respect to Lagrangian variables, the integral is taken around the fixed initial curve $C(0)$. We can therefore now differentiate through the integral to obtain

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \oint_{C(0)} \sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} \, \mathrm{d}X_j = \oint_{C(0)} \frac{\mathrm{D}}{\mathrm{D}t} \left(\sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} \right) \mathrm{d}X_j, \tag{1.50}$$

holding the integration variables $\boldsymbol{X}$ constant when performing the time derivative $\mathrm{D}/\mathrm{D}t$. We expand out the derivative in the integrand, using the fact that $\mathrm{D}/\mathrm{D}t$ commutes with $\partial/\partial X_j$, to obtain

$$\begin{aligned} \frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= \oint_{C(0)} \sum_{i,j} \left(\frac{\mathrm{D}u_i}{\mathrm{D}t} \frac{\partial x_i}{\partial X_j} + u_i \frac{\partial u_i}{\partial X_j} \right) \mathrm{d}X_j \\ &= \oint_{C(t)} \sum_i \frac{\mathrm{D}u_i}{\mathrm{D}t} \, \mathrm{d}x_i + \oint_{C(t)} \sum_{i,j} u_i \frac{\partial u_i}{\partial x_j} \, \mathrm{d}x_j. \end{aligned} \tag{1.51}$$

The second integrand here is an exact derivative, and we use (1.37) to substitute for the acceleration in the first integral:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_{C(t)} \sum_i \frac{\partial}{\partial x_i} \left(-\frac{p}{\rho} - \chi + \frac{1}{2} |\boldsymbol{u}|^2 \right) \mathrm{d}x_i = \left[-\frac{p}{\rho} - \chi + \frac{1}{2} |\boldsymbol{u}|^2 \right]_{C(t)}, \tag{1.52}$$

where $[\cdot]_{C(t)}$ denotes the change in $\cdot$ as the closed loop C is traversed. Since p, χ and $\boldsymbol{u}$ are all single-valued functions of position, we deduce that the right-hand side is zero and, hence, that Γ is constant.

Now, we can use this property to show that, if the vorticity is initially zero, then it remains zero for all time. Suppose for contradiction that $\boldsymbol{\nabla} \times \boldsymbol{u} = \mathbf{0}$ at $t = 0$ but that $\boldsymbol{\nabla} \times \boldsymbol{u}$ is nonzero at some later time t. By (1.48), we can thus find a closed loop $C(t)$ such that the circulation $\Gamma(t)$ is nonzero. Since Γ is independent of t, $\Gamma(0)$ must likewise be nonzero, which is impossible because $\boldsymbol{\nabla} \times \boldsymbol{u}$ was supposed to be zero initially.

1.5 Potential flow

1.5.1 Irrotational flow

A flow is said to be *irrotational* if the vorticity is identically zero:

$$\text{flow is irrotational} \quad \Leftrightarrow \quad \boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \boldsymbol{u} \equiv \mathbf{0}. \tag{1.53}$$

At first glance, this might seem like a far-fetched assumption. However, we note that the trivial solution $\boldsymbol{\omega} \equiv \mathbf{0}$ is consistent with the vorticity equation (1.46). Furthermore, we have just argued from Kelvin's Circulation Theorem that an initially irrotational flow must remain irrotational for all time. Therefore, it is difficult to create vorticity in an inviscid fluid, and actually rather likely that a flow will be irrotational.

For an irrotational flow, the momentum equation (1.39) simplifies to

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi\right). \tag{1.54}$$

If the flow is steady, so the left-hand side is zero, we see that the bracketed term on the right-hand side must be constant and therefore deduce that

$$\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi \text{ is constant \textit{everywhere} in steady irrotational flow.} \tag{1.55}$$

This is **Bernoulli's Theorem for steady irrotational flow**, and should be compared with the weaker version (1.41) that holds for general steady flow.

1.5.2 The velocity potential

The Euler equations (1.36) and (1.37) are very difficult to solve in general, largely because of the nonlinear term $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ in the momentum equation. (In fact, the question of whether solutions of the Euler equations can develop singularities in finite time is hotly debated.) The problem becomes much more straightforward if the flow is irrotational, and we will assume henceforth that this is the case.

If $\boldsymbol{\nabla} \times \boldsymbol{u} \equiv \mathbf{0}$, there must exist a *velocity potential* $\phi(\boldsymbol{x}, t)$ such that

$$\boldsymbol{u} \equiv \boldsymbol{\nabla}\phi. \tag{1.56}$$

To prove this, we define

$$\phi(\boldsymbol{x}, t) := \phi_0(t) + \int_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}, \tag{1.57}$$

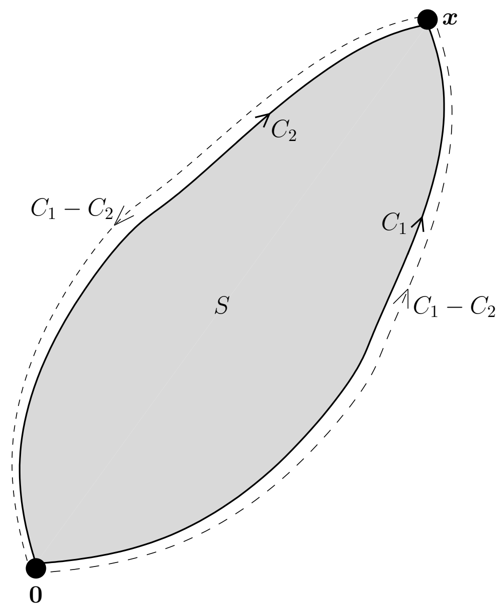

Figure 1.8: Schematic of two paths C_1 and C_2 joining the origin $\mathbf{0}$ to a point $\boldsymbol{x}$, along with the closed path $C_1 - C_2$ formed by joining them, spanned by a surface S.

where C is any curve joining the origin to the point $\boldsymbol{x}$.

Note that ϕ is *not* unique: we can choose the scalar function $\phi_0(t)$ arbitrarily and (1.56) will still be satisfied. However, we will now show that the definition (1.57) is independent of the choice of the curve C. Let us consider two alternative paths C_1 and C_2 joining the origin to $\boldsymbol{x}$. Then

$$\int_{C_1} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} - \int_{C_1} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \equiv \oint_{C_1 - C_2} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}, \tag{1.58}$$

where $C_1 - C_2$ is the closed circuit formed by joining C_1 and C_2 together, as illustrated in Figure 1.8. Now Stokes' Theorem gives

$$\oint_{C_1 - C_2} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \equiv \iint_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{1.59}$$

where S is any surface spanning $C_1 - C_2$, and this is zero since the flow is assumed to be irrotational.

Hence ϕ is well defined by (1.57), up to the arbitrary function $\phi_0(t)$, and it is a simple exercise to show that ϕ then satisfies (1.56). Then the incompressibility condition (1.36) gives us

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} \equiv \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \phi) \equiv \nabla^2 \phi = 0, \tag{1.60}$$

so that ϕ satisfies *Laplace's equation*. This is very much easier to solve than the nonlinear Euler equations: given suitable boundary conditions, all the standard techniques such as separation of variables, transforms, *etc.* can be used to solve for ϕ and hence the velocity field $\boldsymbol{u}$.

The pressure may be found *a posteriori* from the momentum equation (1.35). This final step may be simplified as follows. With $\boldsymbol{\nabla} \times \boldsymbol{u} \equiv 0$ and $\boldsymbol{u} \equiv \boldsymbol{\nabla}\phi$, equation (1.39) becomes

$$\frac{\partial \boldsymbol{\nabla}\phi}{\partial t} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \chi\right). \tag{1.61}$$

Since the t-derivative commutes with $\boldsymbol{\nabla}$, we can rearrange this to

$$\boldsymbol{\nabla}\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{p}{\rho} + \chi\right\} = \mathbf{0}. \tag{1.62}$$

It follows that the quantity in braces can be a function only of t, that is

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \chi = F(t) \text{ in irrotational flow.} \tag{1.63}$$

This generalisation of (1.55) is yet another version of Bernoulli's Theorem, namely **Bernoulli's Theorem for irrotational flow**.

Now we recall that the velocity potential is only defined up to an arbitrary function of t; if we define

$$\tilde{\phi} = \phi + f(t), \tag{1.64}$$

then $\tilde{\phi}$ is a potential corresponding to exactly the same velocity field through (1.56). In terms of $\tilde{\phi}$, Bernoulli's equation (1.63) becomes

$$\frac{\partial \tilde{\phi}}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{\nabla}\tilde{\phi}|^2 + \chi = F(t) - f'(t). \tag{1.65}$$

Hence the function $F(t)$ may be chosen arbitrarily by simply absorbing a suitable function of t into ϕ. For example, we can obtain (1.63) with $F(t) \equiv 0$ by choosing $f'(t) = F(t)$.

1.6 Background material

Here we list some of the standard results from Mods with which you should be familiar before starting this section of the course.

1.6.1 Vector calculus

Here we use the notation $\phi((x))$ to represent any differentiable scalar function, $\boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{v}(\boldsymbol{x})$ any differentiable vector functions of the position vector $\boldsymbol{x} = (x, y, z)$. It is also sometimes handy to introduce suffix notation, so that $\boldsymbol{x}$ may be denoted by

$$\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = \sum_k x_k\boldsymbol{e}_k, \tag{1.66}$$

where we define $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $\boldsymbol{e}_1 \equiv \boldsymbol{i}$, $\boldsymbol{e}_2 \equiv \boldsymbol{j}$, $\boldsymbol{e}_3 \equiv \boldsymbol{k}$.

First we recall that the **grad** of a scalar function ϕ is defined by

$$\operatorname{grad}\phi \equiv \boldsymbol{\nabla}\phi := \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z} \equiv \sum_k \boldsymbol{e}_k\frac{\partial\phi}{\partial x_k}; \tag{1.67}$$

the **divergence** and **curl** of a vector field $\boldsymbol{u}$ are defined by

$$\operatorname{div}\boldsymbol{u} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{u} := \frac{\partial(\boldsymbol{i}\cdot\boldsymbol{u})}{\partial x} + \frac{\partial(\boldsymbol{j}\cdot\boldsymbol{u})}{\partial y} + \frac{\partial(\boldsymbol{k}\cdot\boldsymbol{u})}{\partial z} \equiv \sum_k \boldsymbol{e}_k\cdot\frac{\partial\boldsymbol{u}}{\partial x_k}; \tag{1.68}$$

$$\operatorname{curl}\boldsymbol{u} \equiv \boldsymbol{\nabla}\times\boldsymbol{u} := \frac{\partial(\boldsymbol{i}\times\boldsymbol{u})}{\partial x} + \frac{\partial(\boldsymbol{j}\times\boldsymbol{u})}{\partial y} + \frac{\partial(\boldsymbol{k}\times\boldsymbol{u})}{\partial z} \equiv \sum_k \boldsymbol{e}_k\times\frac{\partial\boldsymbol{u}}{\partial x_k}. \tag{1.69}$$

Curl grad and div curl are zero

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) \equiv \boldsymbol{0}, \qquad\qquad \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}) \equiv 0. \tag{1.70}$$

Orthogonality of grad to level surfaces

$$\boldsymbol{\nabla}\phi \text{ is normal to the surface is given by the equation } \phi(x,y,z) = \text{constant}. \tag{1.71}$$

Directional derivative The directional derivative of a scalar function $\phi(\boldsymbol{x})$ along the vector $\boldsymbol{u}$ is given by

$$\boldsymbol{u}\cdot(\boldsymbol{\nabla}\phi) \equiv (\boldsymbol{u}\cdot\boldsymbol{\nabla})\phi, \tag{1.72}$$

where $(\boldsymbol{u}\cdot\boldsymbol{\nabla})$ denotes the differential operator

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla}) := (\boldsymbol{i}\cdot\boldsymbol{u})\frac{\partial}{\partial x} + (\boldsymbol{j}\cdot\boldsymbol{u})\frac{\partial}{\partial y} + (\boldsymbol{k}\cdot\boldsymbol{u})\frac{\partial}{\partial z} \equiv \sum_k (\boldsymbol{e}_k\cdot\boldsymbol{u})\frac{\partial}{\partial x_k}. \tag{1.73}$$

In this way, we can make sense of the directional derivative of a vector field $\boldsymbol{v}(\boldsymbol{x})$ without addressing the problem of defining the grad of a vector:

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} := \sum_k (\boldsymbol{e}_k\cdot\boldsymbol{u})\frac{\partial\boldsymbol{v}}{\partial x_k}. \tag{1.74}$$

Vector forms of the product rule

$$\boldsymbol{\nabla}\cdot(\phi\boldsymbol{u}) \equiv (\boldsymbol{\nabla}\phi)\cdot\boldsymbol{u} + \phi(\boldsymbol{\nabla}\cdot\boldsymbol{u}), \tag{1.75a}$$

$$\boldsymbol{\nabla}\times(\phi\boldsymbol{u}) \equiv (\boldsymbol{\nabla}\phi)\times\boldsymbol{u} + \phi(\boldsymbol{\nabla}\times\boldsymbol{u}), \tag{1.75b}$$

$$\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{v}) \equiv (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{u}), \tag{1.75c}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{v}) \equiv (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{u} - (\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v}. \tag{1.75d}$$

1.6.2 One-dimensional integrals

Differentiation under the integral Given a function $f(x,t)$, the integral

$$I(t) = \int_a^b f(x,t)\,\mathrm{d}x$$

is a function of t alone. When computing the derivative of $I(t)$, the order of integration and differentation may be reversed, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x \equiv \int_a^b \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x. \tag{1.76}$$

Note that the partial derivative $\partial f/\partial t$ is performed while *holding the integration variable* x *constant.*

Equation (1.76) is a special case of **Leibnitz' rule**: if the limits a and b also depend on t, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x \equiv \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x + f\big(b(t),t\big)\dot{b}(t) - f\big(a(t),t\big)\dot{a}(t). \tag{1.77}$$

1.6.3 Multi-dimensional integrals

Line integrals A space curve C may be described using a single parameter, ξ say, by $\boldsymbol{x} = \boldsymbol{x}(\xi)$ (i.e. $x = x(\xi)$, $y = y(\xi)$, $z = z(\xi)$), where ξ lies in some interval, say $[a,b]$. Then, given a scalar field $\phi(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$, we define

$$\int_C \phi\,\mathrm{d}\boldsymbol{x} := \int_a^b \phi\big(\boldsymbol{x}(\xi)\big)\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi, \tag{1.78a}$$

$$\int_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x} := \int_a^b \boldsymbol{u}\big(\boldsymbol{x}(\xi)\big)\cdot\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi, \tag{1.78b}$$

$$\int_C \phi\,\mathrm{d}s := \int_a^b \phi\big(\boldsymbol{x}(\xi)\big)\left|\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi}\right|\mathrm{d}\xi. \tag{1.78c}$$

The notation $\oint$ is sometimes used to distinguish integrals around closed curves, i.e. those where $\boldsymbol{x}(a) = \boldsymbol{x}(b)$.

Surface integrals A surface S in $\mathbb{R}^3$ may be described using two parameters, say $\boldsymbol{x} = \boldsymbol{x}(\xi,\eta)$, where (ξ,η) occupies some region $R \subseteq \mathbb{R}^2$. Then, given a scalar field $\phi(\boldsymbol{x})$

and vector field $\boldsymbol{u}(\boldsymbol{x})$, we define

$$\iint_S \phi \,\mathrm{d}S := \iint_R \phi\big(\boldsymbol{x}(\xi,\eta)\big) \left|\frac{\partial \boldsymbol{x}}{\partial \xi}(\xi,\eta) \times \frac{\partial \boldsymbol{x}}{\partial \eta}(\xi,\eta)\right| \mathrm{d}\xi\mathrm{d}\eta, \tag{1.79a}$$

$$\iint_S \phi \,\mathrm{d}\boldsymbol{S} \equiv \iint_S \phi\boldsymbol{n} \,\mathrm{d}S := \iint_R \phi\big(\boldsymbol{x}(\xi,\eta)\big) \left(\frac{\partial \boldsymbol{x}}{\partial \xi}(\xi,\eta) \times \frac{\partial \boldsymbol{x}}{\partial \eta}(\xi,\eta)\right) \mathrm{d}\xi\mathrm{d}\eta, \tag{1.79b}$$

$$\iint_S \boldsymbol{u} \,\mathrm{d}S := \iint_R \boldsymbol{u}\big(\boldsymbol{x}(\xi,\eta)\big) \left|\frac{\partial \boldsymbol{x}}{\partial \xi}(\xi,\eta) \times \frac{\partial \boldsymbol{x}}{\partial \eta}(\xi,\eta)\right| \mathrm{d}\xi\mathrm{d}\eta, \tag{1.79c}$$

$$\iint_S \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} \equiv \iint_S \boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}S := \iint_R \boldsymbol{u}\big(\boldsymbol{x}(\xi,\eta)\big) \cdot \left(\frac{\partial \boldsymbol{x}}{\partial \xi}(\xi,\eta) \times \frac{\partial \boldsymbol{x}}{\partial \eta}(\xi,\eta)\right) \mathrm{d}\xi\mathrm{d}\eta, \tag{1.79d}$$

where $\boldsymbol{n}$ denotes the unit normal to S.

Volume integrals A volume V in $\mathbb{R}^3$ may be described using three parameters, say $\boldsymbol{x} = \boldsymbol{x}(\xi,\eta,\zeta)$, where (ξ,η,ζ) occupies some region $R \subseteq \mathbb{R}^3$. Then, given a scalar field $\phi(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$, we have

$$\iiint_V \phi \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_V \phi \,\mathrm{d}V \equiv \iiint_R \phi\big(\boldsymbol{x}(\xi,\eta,\zeta)\big) J(\xi,\eta,\zeta) \,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta, \tag{1.80a}$$

$$\iiint_V \boldsymbol{u} \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_V \boldsymbol{u} \,\mathrm{d}V \equiv \iiint_R \boldsymbol{u}\big(\boldsymbol{x}(\xi,\eta,\zeta)\big) J(\xi,\eta,\zeta) \,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta, \tag{1.80b}$$

where J is the *Jacobian* of the transformation from (ξ,η,ζ) to (x,y,z), that is

$$J(\xi,\eta,\zeta) \equiv \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} := \begin{vmatrix} \partial x/\partial\xi & \partial x/\partial\eta & \partial x/\partial\zeta \\ \partial y/\partial\xi & \partial y/\partial\eta & \partial y/\partial\zeta \\ \partial z/\partial\xi & \partial z/\partial\eta & \partial z/\partial\zeta \end{vmatrix}. \tag{1.81}$$

1.6.4 Integral theorems

These are all multidimensional generalisations of the Fundamental Theorem of Calculus. For any volume V with boundary ∂V we have the **Divergence Theorem**

$$\iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{u} \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iint_{\partial V} \boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}S, \tag{1.82}$$

and a corollary of this is the identity

$$\iiint_V \boldsymbol{\nabla}\phi \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iint_{\partial V} \phi\boldsymbol{n} \,\mathrm{d}S. \tag{1.83}$$

For any surface S spanning a simple closed curve C, **Stokes' Theorem** states that

$$\iint_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} \,\mathrm{d}S \equiv \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}, \tag{1.84}$$

where the orientation of the normal $\boldsymbol{n}$ is chosen such that C rotates around it in a right-handed sense.