2 Two-dimensional incompressible irrotational flow

2.1 Velocity potential and streamfunction

We now focus on purely two-dimensional flows, in which the velocity takes the form

$$\boldsymbol{u}(x,y,t) = u(x,y,t)\boldsymbol{i} + v(x,y,t)\boldsymbol{j}. \tag{2.1}$$

With the velocity given by (2.1), the vorticity takes the form

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k}. \tag{2.2}$$

We assume throughout that the flow is *irrotational*, *i.e.* that $\boldsymbol{\nabla} \times \boldsymbol{u} \equiv \boldsymbol{0}$ and hence

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{2.3}$$

We have already shown in Section 1 that this condition implies the existence of a *velocity potential* ϕ such that $\boldsymbol{u} \equiv \boldsymbol{\nabla}\phi$, that is

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}. \tag{2.4}$$

We also recall the definition of ϕ as

$$\phi(x,y,t) = \phi_0(t) + \int_{\boldsymbol{0}}^{\boldsymbol{x}} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{x} = \phi_0(t) + \int_{\boldsymbol{0}}^{\boldsymbol{x}} (u\,\mathrm{d}x + v\,\mathrm{d}y)\,, \tag{2.5}$$

where the scalar function $\phi_0(t)$ is arbitrary, and the value of $\phi(x,y,t)$ is independent of the integration path chosen to join the origin $\boldsymbol{0}$ to the point $\boldsymbol{x} = (x,y)$. This fact is even easier to establish when we restrict our attention to two dimensions. If we consider two alternative paths, whose union forms a simple closed contour C in the (x,y)-plane, Green's Theorem implies that

$$\oint_C (u\,\mathrm{d}x + v\,\mathrm{d}y) \equiv \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y, \tag{2.6}$$

where S is the region enclosed by C, and the right-hand side of (2.6) is zero by virtue of (2.3).

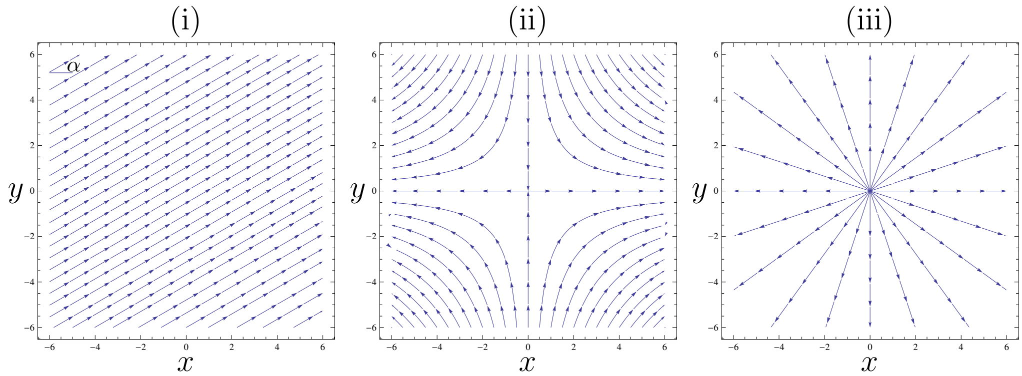

Figure 2.1: Streamline plots for (i) uniform flow; (ii) stagnation point flow; (iii) line source.

We also assume that the flow is *incompressible*, so that $\nabla \cdot \boldsymbol{u} = 0$, which in two dimensions reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{2.7}$$

By substituting (2.4) into (2.7), we find that ϕ satisfies Laplace's equation, that is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0. \tag{2.8}$$

As noted in Section 1, Laplace's equation is linear, and therefore much easier to solve than the nonlinear Euler equations. We will see that many flows may be constructed using well-known simple solutions of Laplace's equation. Once we have solved for ϕ, the velocity components are easily recovered from (2.4), and the pressure may be found using Bernoulli's Theorem. We recall from Section 1 that this reads

$$\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \chi \text{ is constant in steady irrotational flow,} \tag{2.9}$$

or

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \chi = F(t) \text{ in unsteady irrotational flow.} \tag{2.10}$$

Example 2.1 ***Uniform flow***
A linear function of x and y trivially satisfies Laplace's equation. The velocity potential

$$\phi = Ux\cos\alpha + Uy\sin\alpha, \tag{2.11}$$

where U and α are constants, represents the velocity field

$$(u, v) = U(\cos\alpha, \sin\alpha). \tag{2.12}$$

This corresponds to uniform flow*, with speed U, in a direction making an angle α with the x-axis, as shown in* Figure 2.1(i)*.*

Example 2.2 ***Stagnation point flow***
The velocity potential

$$\phi = \frac{1}{2}\left(x^2 - y^2\right) \tag{2.13}$$

clearly satisfies Laplace's equation and corresponds to the stagnation point flow

$$(u, v) = (x, -y) \tag{2.14}$$

considered previously in Section 1*. We recall that the streamlines are hyperbolae, as illustrated in* Figure 2.1(ii)*.*

Example 2.3 ***Line source***
We recall that the two-dimensional Laplace's equation may be written as

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0, \tag{2.15}$$

in terms of plane polar coordinates (r, θ)*. If we seek a radially-symmetric solution, with* $\phi = \phi(r)$*, we find that*

$$\phi = \frac{Q}{2\pi}\log r, \tag{2.16}$$

where Q *is a constant of integration, and the additional arbitrary constant may be set to zero without loss of generality. The corresponding velocity field is given by*

$$\boldsymbol{u} = \frac{\mathrm{d}\phi}{\mathrm{d}r}\boldsymbol{e}_r = \frac{Q}{2\pi r}\boldsymbol{e}_r, \tag{2.17}$$

where $\boldsymbol{e}_r$ *is the unit-vector in the* r*-direction. The velocity (2.17) may be rewritten in terms of Cartesian variables as*

$$\boldsymbol{u} = (u, v) = \frac{Q}{2\pi\left(x^2 + y^2\right)}(x, y). \tag{2.18}$$

We see that the fluid flows radially outward, at a speed that increases without bound as we approach the origin. The streamlines are straight rays pointing outward from the origin, as shown in Figure 2.1(iii)*. This flow is called a* line source*: a "source" because fluid is being squirted out of the origin, and a "line" because the point* $(0, 0)$ *in the* (x, y)*-plane really corresponds to a line (the* z*-axis) in three-dimensional space.*

The constant Q *is called the* source strength*. To see why, let us calculate the rate at which fluid crosses a contour* C *containing the origin, namely*

$$\oint_C \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s,$$

where $\boldsymbol{n}$ *is the outward-pointing unit normal to* C*. The integral is particularly straightforward if we choose* C *to be the circle* $r = a$*, for some constant radius* a*, in which case* $\boldsymbol{n} \equiv \boldsymbol{e}_r$ *and hence*

$$\oint_C \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_0^{2\pi} \frac{Q}{2\pi a}\boldsymbol{e}_r\cdot\boldsymbol{e}_r a\,\mathrm{d}\theta = Q. \tag{2.19}$$

It is easily shown that any other simple closed curve C *containing the origin gives the same value of the flux. Therefore* Q *is the rate at which fluid is produced by the source.*

If Q *is negative, then fluid is being consumed rather than produced, and the origin is called a* line sink*, of strength* $|Q|$*.*

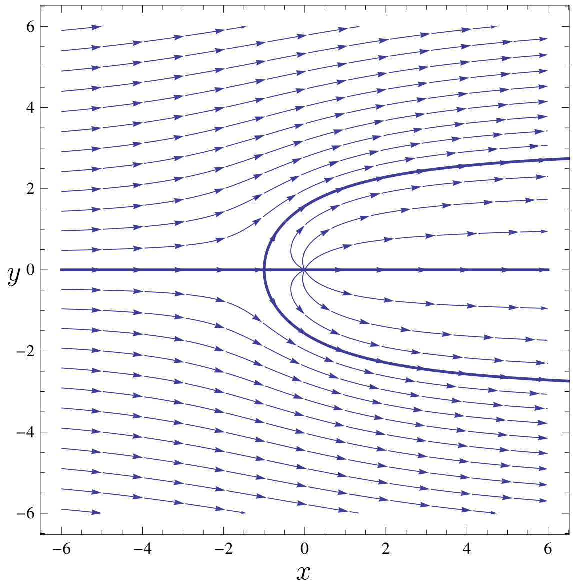

Figure 2.2: Streamlines for the flow produced by a line source of strength Q in a uniform flow at speed U in the x-direction. (In this plot, $Q = 2\pi U$.)

One of the great advantages of dealing with a *linear* partial differential equation is that the result of combining two flows may be found by simply adding the corresponding velocity potentials. This is emphatically *not* the case for the nonlinear Euler equations in general.

Example 2.4 ***Line source in a uniform flow***
Consider the flow produced by placing a line source of strength Q at the origin in a uniform flow of speed U in the x-direction. The velocity potentials due to the source and the uniform flow are Ux and $(Q/2\pi)\log r$ respectively, and the potential of the combined flow is thus

$$\phi = Ux + \frac{Q}{2\pi}\log r. \tag{2.20}$$

The corresponding velocity components are

$$u = U + \frac{Qx}{2\pi\,(x^2+y^2)}, \qquad v = \frac{Qy}{2\pi\,(x^2+y^2)}, \tag{2.21}$$

and the streamlines are shown in Figure 2.2.

It is easily seen that there is just one stagnation point (i.e. point where the velocity is zero) in the flow, at $(x, y) = (-Q/2\pi U, 0)$. The separatrix crossing the x-axis through this point separates the fluid emanating from the source from the fluid carried by the uniform flow.

By analogy with (2.3) and (2.4), we could instead start from (2.7) and deduce the existence of a *streamfunction* $\psi(x, y, t)$ satisfying

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{2.22}$$

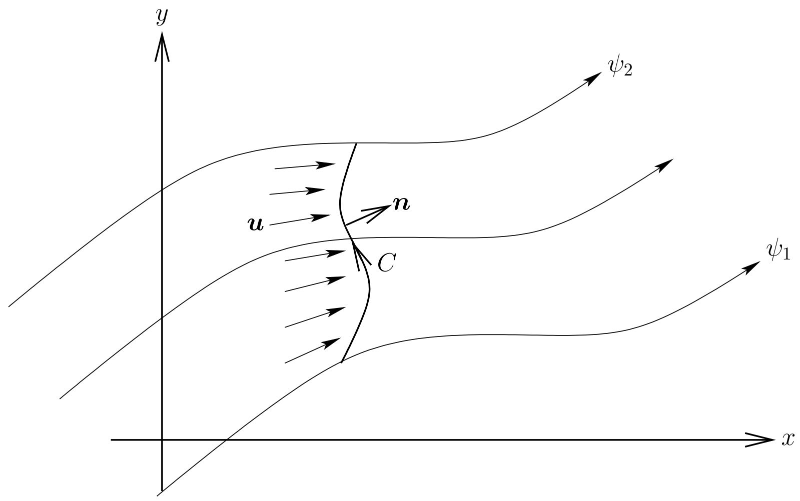

Figure 2.3: Schematic of a curve C joining two streamlines on which $\psi = \psi_1$ and $\psi = \psi_2$.

A handy shorthand for (2.22) is

$$\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi \boldsymbol{k}). \tag{2.23}$$

By comparison with (2.5), we infer that ψ may be defined as

$$\psi(x, y, t) = \psi_0(t) + \int_{\boldsymbol{0}}^{\boldsymbol{x}} (u \, \mathrm{d}y - v \, \mathrm{d}x), \tag{2.24}$$

where the function $\psi_0(t)$ is arbitrary: just as for ϕ, any function of t may be absorbed into ψ without influencing the velocity components given by (2.22). Again, the integral in (2.24) is independent of the path taken from the origin to the point $\boldsymbol{x}$, and the proof follows quickly from Green's Theorem.

The representation (2.22) implies that ψ satisfies

$$\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) \cdot \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) \equiv 0, \tag{2.25}$$

and it follows that *the streamfunction is constant along streamlines.* The contours of ψ therefore give us a handy tool for plotting the streamlines of a given flow.

Now let us consider two neighbouring streamlines, on which ψ takes the constant values $\psi = \psi_1$ and $\psi = \psi_2$, say. The *flux*, *i.e.* the net flow of fluid, between the two streamlines, is given by the integral

$$\text{flux} = \int_C \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}s, \tag{2.26}$$

where C is a smooth path joining the two streamlines, with unit normal $\boldsymbol{n}$, as shown schematically in Figure 2.3. Substituting for the velocity components from (2.4) and identifying $\boldsymbol{n}\,\mathrm{d}s$ with $(\mathrm{d}y, -\mathrm{d}x)$, we can rewrite (2.26) as

$$\text{flux} = \int_C \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)\cdot(\mathrm{d}y, \mathrm{d}x) = \int_C \left(\frac{\partial\psi}{\partial x}\mathrm{d}x + \frac{\partial\psi}{\partial y}\mathrm{d}y\right) = \big[\psi\big]_C = \psi_2 - \psi_1, \quad (2.27)$$

where the notation $[\psi]_C$ respresents the change in the value of ψ from one end of C to the other. Therefore,

the change in the value of ψ from one streamline to another is equal to the net flux of fluid between them. (2.28)

By substituting (2.22) into (2.3), we find that ψ, like ϕ, satisfies the two-dimensional Laplace's equation:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi = 0. \quad (2.29)$$

Again, we can pose familiar solutions of Laplace's equation for ψ and read off the corresponding velocity fields from (2.22). We begin by showing how the flows considered in Examples 2.1–2.4 may be described using the streamfunction.

Example 2.5 ***Uniform flow***
The streamfunction corresponding to the uniform flow (2.12) *is easily seen to be*

$$\psi = Uy\cos\alpha - Ux\sin\alpha, \quad (2.30)$$

up to an arbitrary constant. Hence the streamlines are given by $\psi = \text{constant}$, *that is,*

$$y = x\tan\alpha + \text{constant}, \quad (2.31)$$

in agreement with Figure 2.1(i).

Example 2.6 ***Stagnation point flow***
The streamfunction corresponding to the stagnation point flow (2.14), *satisfies the differential equations*

$$\frac{\partial\psi}{\partial x} = y, \qquad \frac{\partial\psi}{\partial y} = x. \quad (2.32)$$

We easily deduce that

$$\psi = xy \quad (2.33)$$

up to an arbitrary constant. The streamlines plotted in Figure 2.1(ii) *are the contours of* ψ, *namely the hyperbolae* $xy = \text{constant}$.

The representation (2.23) is helpful if we wish to use the streamfunction in plane polar coordinates (r, θ). Using the formulae given in §2.5.2, we find that

$$\boldsymbol{\nabla}\times(\psi\boldsymbol{k}) \equiv \frac{1}{r}\frac{\partial\psi}{\partial\theta}\boldsymbol{e}_r - \frac{\partial\psi}{\partial r}\boldsymbol{e}_\theta, \quad (2.34)$$

and the polar velocity components u_r and u_θ are therefore related to ψ by

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{\partial\psi}{\partial r}. \quad (2.35)$$

Example 2.7 ***Line source***
By substituting the velocity field (2.17) *due to a line source into equation* (2.35)*, we find that the corresponding streamfunction must satisfy*

$$\frac{\partial\psi}{\partial r} = 0, \qquad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{Q}{2\pi r}. \tag{2.36}$$

Hence, up to the addition of an arbitrary constant,

$$\psi = \frac{Q}{2\pi}\theta. \tag{2.37}$$

The contours of ψ are the rays θ = constant, in agreement with the streamlines plotted in Figure 2.1(iii)*.*

Example 2.7 demonstrates that the streamfunction due to a point source is *multivalued.* We should not be too surprised by this, since our uniqueness proof relied on Green's Theorem, which does not apply to a velocity field like (2.18), which is not differentiable (or even bounded) at the origin. Furthermore, the jump of Q in ψ as θ increases from 0 to 2π corresponds to the *flux* between these two streamlines, according to (2.28). This agrees with the result that Q is the rate at which the source produces fluid.

Example 2.8 ***Line source in a uniform flow***
The streamfunction corresponding to the flow (2.21) *may be found by combining the streamfunctions* (2.30) *and* (2.37) *found above, to obtain*

$$\psi = Uy + \frac{Q}{2\pi}\theta. \tag{2.38}$$

The streamlines plotted in Figure 2.2 *are the contours of this function, which may be written in the form*

$$r = \left(\frac{Q}{2\pi U}\right)\frac{C-\theta}{\sin\theta}, \tag{2.39}$$

where different streamlines correspond to different choices of the constant C.

As our final example, we examine the implications of a radially-symmetric streamfunction, by analogy with (2.3).

Example 2.9 ***Line vortex***
If the streamfunction depends only upon the radial coordinate r, then, as in equation (2.16)*, it must take the form*

$$\psi = \frac{-\Gamma}{2\pi}\log r, \tag{2.40}$$

up to the addition of an arbitrary constant. The minus sign in front of the integration constant Γ is included for later convenience. We see that this will give circular streamlines r = constant, as shown in Figure 2.4*. By using (2.35), we find that the velocity field is given by*

$$\boldsymbol{u} = -\frac{\mathrm{d}\psi}{\mathrm{d}r}\boldsymbol{e}_\theta = \frac{\Gamma}{2\pi r}\boldsymbol{e}_\theta. \tag{2.41}$$

Hence the fluid rotates around the origin, with a speed that increases without bound as $r \to 0$. This flow is called a line vortex*, the line in question being the z-axis in three-dimensional space.*

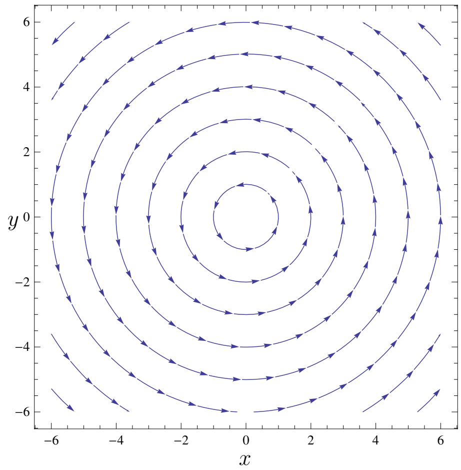

Figure 2.4: Streamlines due to a vortex at the origin.

The constant Γ *is called the* vortex strength. *We recall that the strength of a source measures the rate at which it produces fluid. In contrast, we will see that the strength of a vortex corresponds to the* circulation *around a contour* C *containing the origin, namely*

$$\oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \equiv \oint_C (u \, \mathrm{d}x + v \, \mathrm{d}y).$$

We make the integral as simple as possible by choosing C *to be the circle* $r = a$*, for some constant radius* a*, in which case* $(\mathrm{d}x, \mathrm{d}y) \equiv \boldsymbol{e}_\theta a \, \mathrm{d}\theta$ *and hence*

$$\oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \int_0^{2\pi} \frac{\Gamma}{2\pi a} \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta \, a \, \mathrm{d}\theta = \Gamma. \tag{2.42}$$

It is easily shown that the circulation around any other simple closed curve C *containing the origin is also equal to* Γ.

If Γ *is positive, the vortex rotates in an anticlockwise sense, as in* Figure 2.4*; a negative value of the circulation corresponds to clockwise rotation.*

We recall that the circulation is supposed to be identically zero in an irrotational flow. However, this result relies on Green's Theorem, which does not apply to a singular velocity field such as (2.41). A vortex may be thought of as a *source of vorticity* in an otherwise irrotational flow.

2.2 Complex potential

The velocity potential and streamfunction give us the two alternative representations (2.4) and (2.22) of the velocity. By comparing these expressions, we see that ϕ and ψ must satisfy the equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.43}$$

We recognise these as the *Cauchy–Riemann equations*, which relate the real and imaginary parts of a holomorphic function. Provided ϕ and ψ are continuously differentiable, we can deduce from (2.43) that

$$\phi(x, y) + \mathrm{i}\psi(x, y) \equiv w(z), \tag{2.44}$$

where w is a holomorphic function of $z = x + \mathrm{i}y$, known as the *complex potential.* We can use many of the powerful techniques of complex analysis to determine and manipulate the complex potential, and this allows us to solve for ϕ and ψ simultaneously.

We recall that the holomorphic function $w(z)$ has a well-defined derivative which is independent of the direction of differentiation in the complex plane, that is

$$\frac{\mathrm{d}w}{\mathrm{d}z} \equiv \frac{\partial}{\partial x}(\phi + \mathrm{i}\psi) \equiv \frac{1}{\mathrm{i}}\frac{\partial}{\partial y}(\phi + \mathrm{i}\psi). \tag{2.45}$$

Hence we find that the velocity components (u, v) may be recovered from w using the formula

$$\frac{\mathrm{d}w}{\mathrm{d}z} \equiv u - \mathrm{i}v. \tag{2.46}$$

We illustrate the use of $w(z)$ by listing the complex potentials corresponding to each of the examples introduced in §2.1.

Example 2.10 ***Uniform flow***
The complex potential

$$w(z) = U\mathrm{e}^{-\mathrm{i}\alpha}z \tag{2.47}$$

corresponds to uniform flow at speed U in a direction making an angle α with the x-axis. We can easily see this by calculating $\mathrm{d}w/\mathrm{d}z$ *and substituting into* (2.46).

Example 2.11 ***Stagnation point flow***
The velocity field $(u, v) = (x, -y)$ corresponds to the complex potential

$$w(z) = \frac{z^2}{2}. \tag{2.48}$$

We can easily see that the real and imaginary parts agree with equations (2.13) *and* (2.33).

Example 2.12 ***Line source***
A line source of strength Q at the origin is represented by the complex potential

$$w(z) = \frac{Q}{2\pi}\log z. \tag{2.49}$$

We note that this is a multifunction*, with a branch point at the origin. This is no surprise when we recall that a line source has a multivalued streamfunction.*

It is easy to generalise (2.49) to a source at an arbitrary point, say (a, b), in the (x, y)-plane. The required complex potential is

$$w(z) = \frac{Q}{2\pi}\log(z - c), \tag{2.50}$$

where $c = a + \mathrm{i}b$ is the complex number corresponding to the point (a, b) in the complex plane.

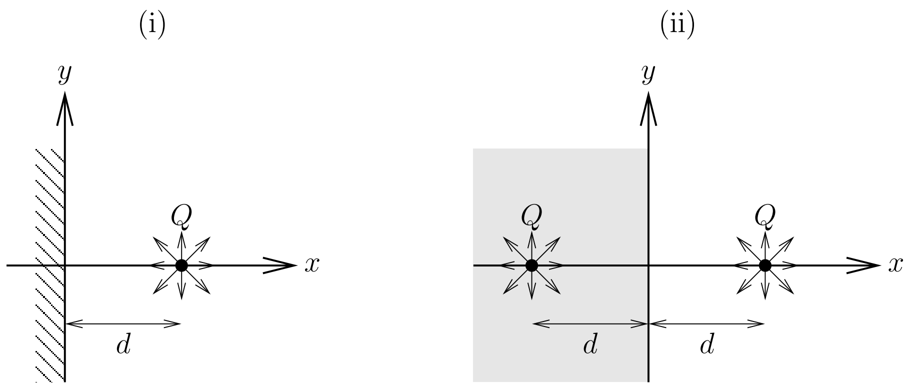

Figure 2.5: (i) Schematic of a source in the half-plane $x > 0$ bounded by a wall at $x = 0$. (ii) The same flow mimicked by inserting an image source.

Example 2.13 ***Line vortex***
A line vortex of strength Γ *at the origin is represented by the complex potential*

$$w(z) = \frac{-\mathrm{i}\Gamma}{2\pi} \log z. \tag{2.51}$$

This again is a multifunction, and we can infer from (2.51) *that the velocity potential is the multivalued function*

$$\phi = \frac{\Gamma}{2\pi}\theta. \tag{2.52}$$

The jump in ϕ *that occurs upon a circuit of the origin corresponds to the circulation* Γ.

A vortex at an arbitrary point (a, b) is represented by the complex potential

$$w(z) = \frac{-\mathrm{i}\Gamma}{2\pi} \log(z - c), \tag{2.53}$$

where $c = a + \mathrm{i}b$.

These basic flows can all be combined by simply superimposing the corresponding complex potentials.

2.3 Method of images

2.3.1 Plane boundaries

Now we consider how some of the flows that we have considered so far are modified in the presence of boundaries. As a first example, suppose fluid occupies the half-space $x > 0$ and there is a rigid impermeable wall along the y-axis $x = 0$. Now we wish to find the flow caused by a source of strength Q placed in the fluid at the point $(d, 0)$, where $d > 0$. We have to impose the boundary condition of zero normal velocity through the wall, that is

$$u = 0 \quad \text{at } x = 0. \tag{2.54}$$

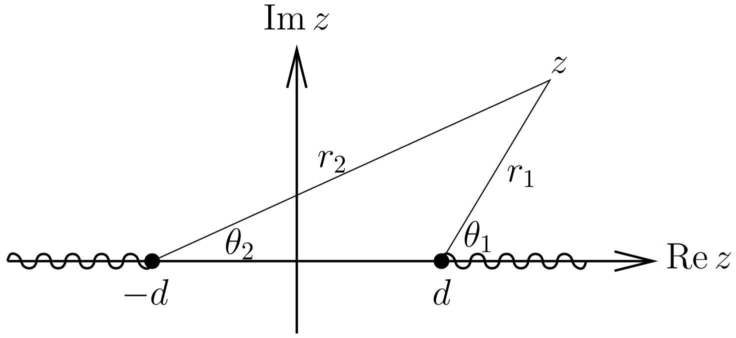

Figure 2.6: The lengths r_1, r_2, angles θ_1, θ_2 and branch cuts in the definition of the complex potential (2.59).

Alternatively, we could insist that $x = 0$ be a streamline for the flow, that is

$$\operatorname{Im} w(z) = \psi = \text{constant} \quad \text{at } x = 0. \tag{2.55}$$

The two boundary conditions (2.54) and (2.55) are equivalent by virtue of (2.22).

The idea of the *method of images* is to insert a fictitious "image" source at the point $(-d, 0)$ behind the wall. By symmetry we would expect the flows caused by the two sources to cancel each other out on the symmetry line $x = 0$, and thereby satisfy the condition of zero normal velocity there. Of course, we will have introduced a new singularity, but it is outside the physical region $x > 0$ in which we are interested.

By superimposing the real and image sources, we obtain the velocity potential

$$w = \frac{Q}{2\pi} \log(z - d) + \frac{Q}{2\pi} \log(z + d), \tag{2.56}$$

and the corresponding velocity components are given by

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{Qz}{\pi \left(z^2 - d^2\right)}. \tag{2.57}$$

In particular, on the x-axis we have $z = \mathrm{i}y$ and hence

$$u - \mathrm{i}v\big|_{x=0} = \frac{-Q\mathrm{i}y}{\pi \left(y^2 + d^2\right)}, \tag{2.58}$$

so the boundary condition (2.54) of zero normal velocity is indeed satisfied.

To extract the velocity potential and streamfunction from the velocity potential (2.56), we have to confront the fact that $w(z)$ is a multifunction. We can expand each log in (2.56) to obtain

$$w(z) = \phi + \mathrm{i}\psi = \frac{Q}{2\pi} \big(\log r_1 + \log r_2\big) + \frac{\mathrm{i}Q}{2\pi} \left(\theta_1 + \theta_2\right), \tag{2.59}$$

where the lengths r_1, r_2 and angles θ_1, θ_2 are defined as shown in Figure 2.6. There is a branch cut extending to infinity from each of the branch points $z = \pm d$, and it

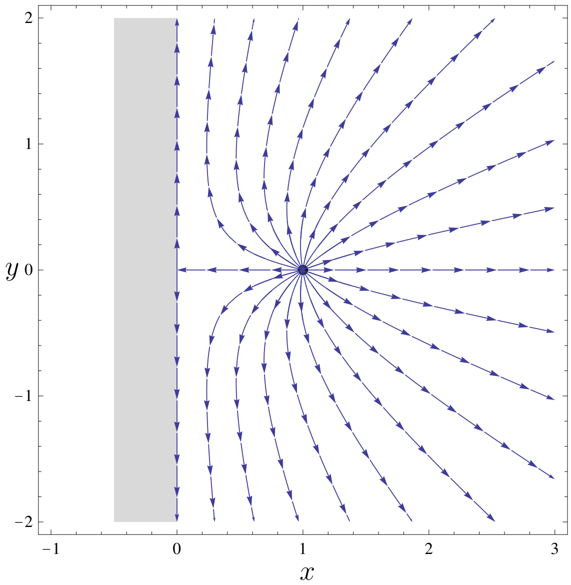

Figure 2.7: Streamlines for the flow produced by a source at the point $(1, 0)$ with an impermeable wall at $x = 0$.

is convenient to take them to lie along the x-axis as shown, so that $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in (-\pi, \pi]$. We easily see that $\theta_1 + \theta_2 \equiv \pi$ on the y-axis, and hence the boundary condition (2.55) is satisfied by the complex potential (2.56).

We can read off from (2.59) the streamfunction for the flow and deduce that the streamlines are given by $\theta_1 + \theta_2 =$ constant. This may be rearranged to show that the streamlines are hyperbolae satisfying the equation

$$x^2 - 2\beta xy - y^2 = d^2, \tag{2.60}$$

where $\beta = \cot(2\pi\psi/Q)$ is constant. The resulting curves are plotted in Figure 2.7, with $d = 1$. Notice that the competition between the real and image sources causes a stagnation point at the origin.

This idea is easily extended to describe the flow due to a vortex in a half-space. The only difference is that, to achieve the necessary cancellation, the strength of the image vortex must be equal and opposite to that of the original vortex, as shown schematically in Figure 2.8. Thus, the complex potential due to a vortex of strength Γ at the point $(d, 0)$ in the half-space $x > 0$ with an impermeable boundary at $x = 0$ is

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z - d) + \frac{\mathrm{i}\Gamma}{2\pi}\log(z + d). \tag{2.61}$$

We can verify that this satisfies the required boundary condition at $x = 0$ by calculating the velocity field, using

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{-\mathrm{i}\Gamma d}{\pi\left(z^2 - d^2\right)}, \tag{2.62}$$

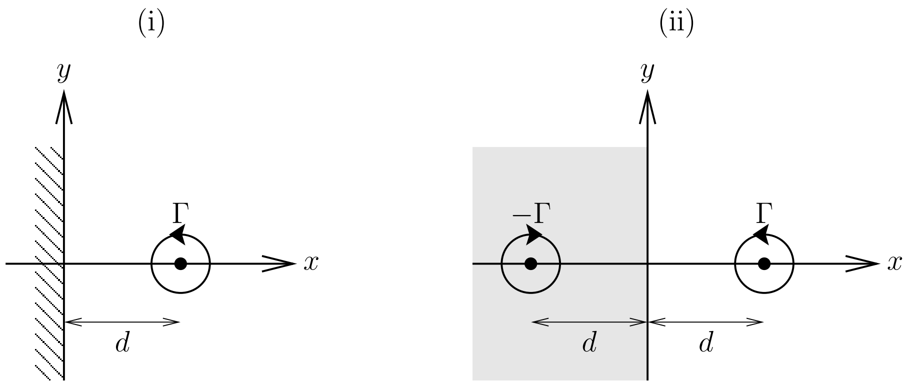

Figure 2.8: (i) Schematic of a vortex in the half-plane $x > 0$ bounded by a wall at $x = 0$. (ii) The same flow mimicked by inserting an image vortex.

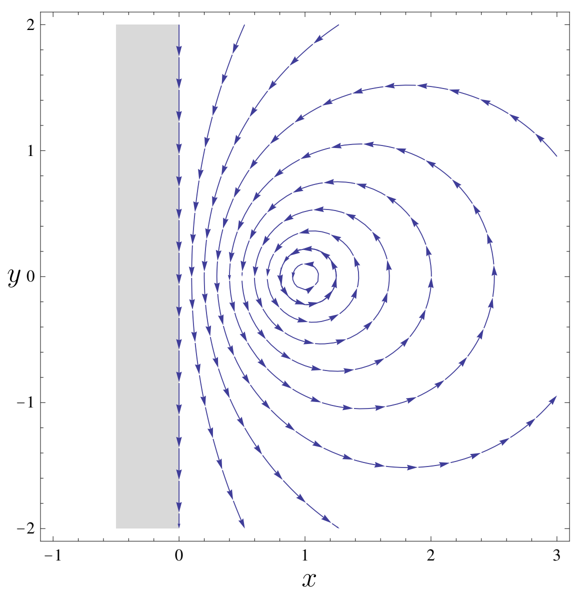

Figure 2.9: Streamlines for the flow produced by a vortex at the point $(1, 0)$ with an impermeable wall at $x = 0$.

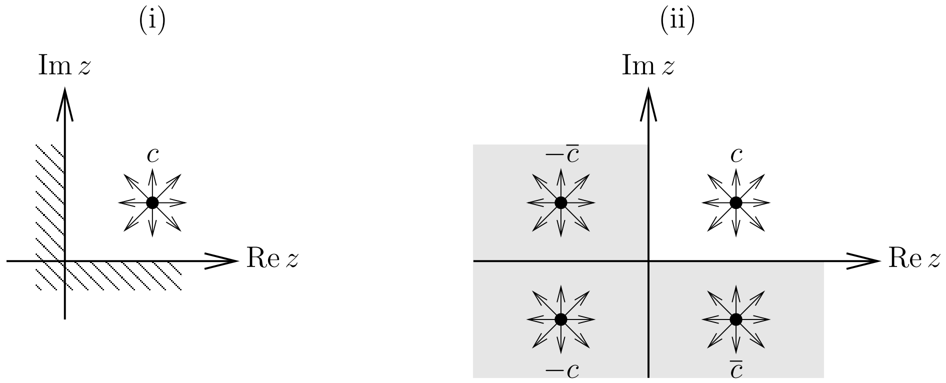

Figure 2.10: (i) Schematic of a source in the positive quadrant bounded by walls along the real and imaginary axes. (ii) The same flow mimicked by inserting three image sources.

which is clearly pure imaginary when $z = \mathrm{i}y$.

The streamfunction for this flow is found from (2.61) to be given by

$$\psi = \frac{\Gamma}{2\pi} \log\left(\frac{r_2}{r_1}\right), \tag{2.63}$$

where r_1 and r_2 are again defined as in Figure 2.6. Hence the streamlines are curves on which $r_2/r_1 = \text{constant}$, and it is easily shown that these are circles centred on the x-axis, as shown in Figure 2.9.

The same ideas are easily extended to describe flow in a quadrant. In Figure 2.10(i), we show a source at the point representing the complex number $c = a+\mathrm{i}b$ (with $a, b > 0$) in the Argand diagram, with impermeable boundaries along the real and imaginary axes. The condition of zero flow across the real axis may be satisfied by inserting an image source at $z = a - \mathrm{i}b = \overline{c}$, where a bar is used to denote the complex conjugate. Now, to impose the equivalent boundary condition on the imaginary axis, we need to insert an image corresponding to each of the sources now present in $\operatorname{Re} z > 0$, at $z = -a+\mathrm{i}b = -\overline{c}$ and at $z = -a - \mathrm{i}b = -c$ respectively. We therefore end up with three image sources altogther, as shown in Figure 2.10(ii), and the resulting flow, with the required symmetry in both the real and imaginary axes, is represented by the complex potential

$$w = \frac{Q}{2\pi}\log(z-c) + \frac{Q}{2\pi}\log(z+c) + \frac{Q}{2\pi}\log(z-\overline{c}) + \frac{Q}{2\pi}\log(z+\overline{c}). \tag{2.64}$$

An analogous approach works for a vortex in a quadrant, provided the signs of the images are adjusted as shown in Figure 2.11. The resulting complex potential is

$$w = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z-c) - \frac{\mathrm{i}\Gamma}{2\pi}\log(z+c) + \frac{\mathrm{i}\Gamma}{2\pi}\log(z-\overline{c}) + \frac{\mathrm{i}\Gamma}{2\pi}\log(z+\overline{c}). \tag{2.65}$$

In principle, the method of images may be extended to several other domains with straight boundaries. However, an infinite number of images will often be required to achieve symmetry in every boundary.

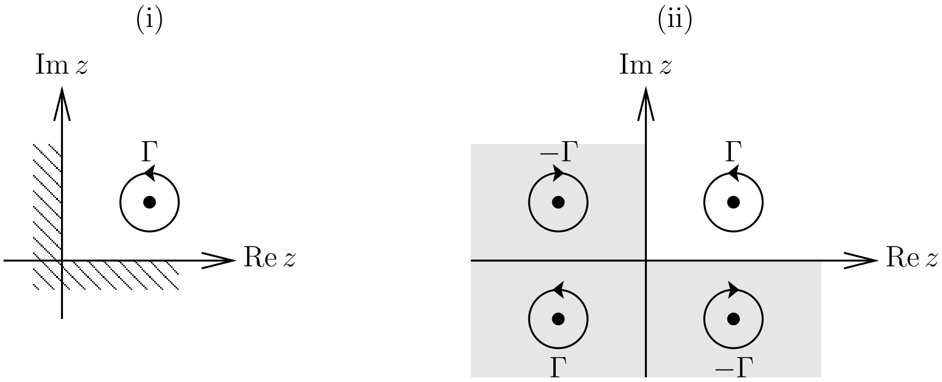

Figure 2.11: (i) Schematic of a vortex in the positive quadrant bounded by walls along the real and imaginary axes. (ii) The same flow mimicked by inserting three image vortices.

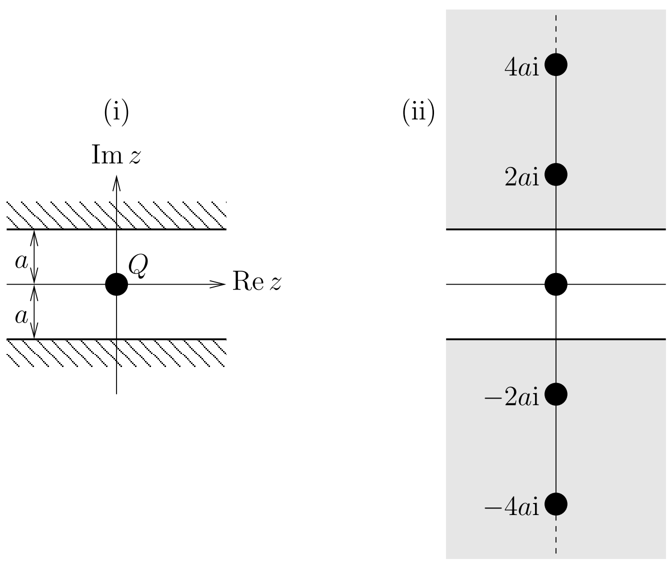

Figure 2.12: (i) Schematic of a source in the strip bounded by walls along the lines $\operatorname{Im} z = \pm a$. (ii) The same flow mimicked by inserting an infinite system of image sources.

Example 2.14 ***A source in a strip***
Fluid occupies the region $-a < y < a$ between two rigid boundaries at $y = \pm a$. We will use the method of images to find the flow caused by a source placed in the fluid at the origin $(x, y) = (0, 0)$, as shown in Figure 2.11(i).

At first glance, we can take care of the walls by inserting image sources at $z = \pm 2a\mathrm{i}$. However, then we realise that the image at (say) $z = 2a\mathrm{i}$ needs its own image in the wall $\mathrm{Im}\, z = -a\mathrm{i}$, *and we are therefore forced to include another image at $z = -a a\mathrm{i}$. By continuing this argument, we find that an infinite system of image sources along the imaginary axis is needed, at the points $z = 2na\mathrm{i}$ for all integers n, as shown in* Figure 2.11(ii).

The resulting velocity potential is apparently

$$w(z) = \frac{Q}{2\pi} \lim_{N\to\infty} \sum_{n=-N}^{N} \log(z - 2na\mathrm{i}), \tag{2.66}$$

although it is far from clear that this series converges! However, we can use it to calculate the velocity components, namely

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{Q}{2\pi} \left\{ \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2a^2} \right\}. \tag{2.67}$$

This series does *converge, and may be evaluated using the identity*

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + b^2} \equiv -\frac{1}{2b^2} + \frac{\pi \coth(b\pi)}{2b}. \tag{2.68}$$

We therefore find that

$$u - \mathrm{i}v = \frac{Q}{4a} \coth\left(\frac{\pi z}{2a}\right). \tag{2.69}$$

On the upper wall, where $z = \mathrm{i}a + x$, we therefore find that

$$u - \mathrm{i}v = \frac{Q}{4a} \tanh\left(\frac{\pi x}{2a}\right), \tag{2.70}$$

so the boundary condition $v = 0$ on $y = a$ is indeed satisfied.

Example 2.14 illustrates that in more complicated domains, although the method of images may work in principle, it is often unweildy in practice. As we will see, a far neater approach in such cases is to use *conformal mapping.* However, another example where the image system may be found relatively easily is where the boundary is a *circle.*

2.3.2 The Circle Theorem

Theorem 2.1 ***Milne-Thomson's Circle Theorem***
Suppose a velocity potential $w(z) = f(z)$ is given such that any singularities in $f(z)$ occur in $|z| > a$. Then the potential

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\overline{z}}\right)} \tag{2.71}$$

(where the bar again denotes complex conjugate) has

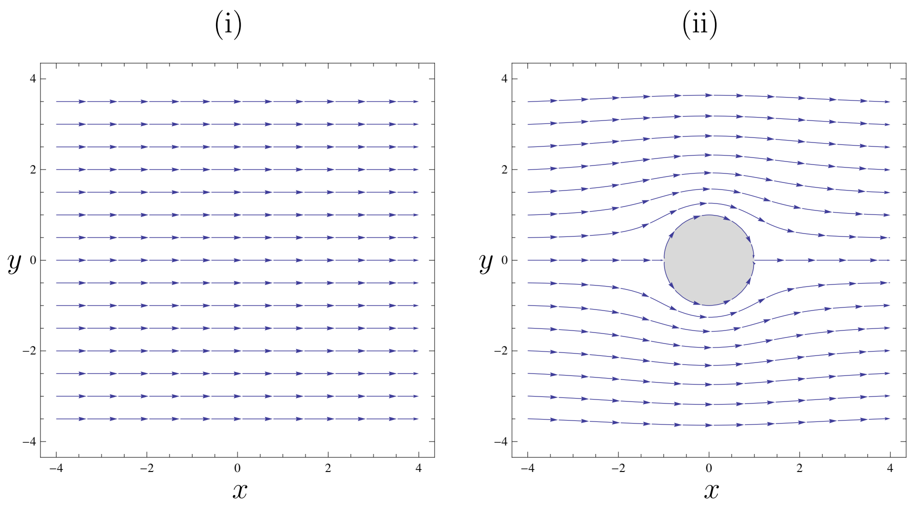

Figure 2.13: (i) Uniform flow in the x-direction. (ii) Uniform flow past a circular cylinder.

1. *the same singularities as* $f(z)$ *in* $|z| > a$;
2. *the circle* $|z| = a$ *as a streamline.*

Proof To prove part 1, we note that, if $|z| > a$, then $\left|a^2/\overline{z}\right| < a$. Since $f(z)$ is assumed to have no singularities in $|z| \leqslant a$, it follows that the second term in (2.71) has no singularities in $|z| > a$ and the result follows.

For part 2, note that, when $|z| = a$, we have $\bar{z} = a^2/z$ and hence

$$w(z)\Big|_{|z|=a} = f(z) + \overline{f\,(z)} = 2\,\mathrm{Re}\big(f(z)\big).$$

The right-hand side is evidently real, and it follows that $\mathrm{Im}\,w(z) = \psi = 0$ on $|z| = a$, and the circle $|z| = a$ is therefore a streamline. ∎

This theorem allows us to find the resulting flow when a circular cylinder is placed in a background flow with complex potential $f(z)$.

Example 2.15 ***Uniform flow past a circular cylinder***
Our starting point is uniform flow at speed U *in the* x*-direction, as shown in* Figure 2.13(i)*, with complex potential* $w(z) = Uz$. *If we insert an impermeable circular obstacle, at* $|z| = a$ *say, then the flow will be disturbed as illustrated in* Figure 2.13(ii). *Our task is to calculate the disturbed flow.*

Note that $f(z) = Uz$ *has a pole at infinity but is holomorphic in the disc* $|z| \leqslant a$ *and thus satisfies the hypotheses of the Circle Theorem. We can therefore obtain the solution by substituting* $f(z) = Uz$ *into equation* (2.71)*:*

$$w(z) = Uz + \overline{\frac{Ua^2}{\overline{z}}} \equiv Uz + \frac{Ua^2}{z}. \tag{2.72}$$

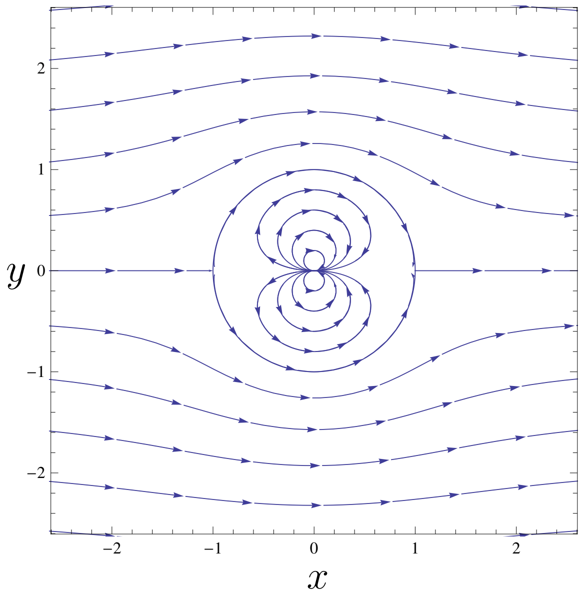

Figure 2.14: Streamlines caused by a doublet in a uniform flow.

The streamfunction is just the imaginary part of this function, namely

$$\psi = Uy\left(1 - \frac{a^2}{x^2 + y^2}\right), \tag{2.73}$$

and we see that the circle $x^2 + y^2 = a^2$ *is indeed a streamline, with* $\psi = 0$*. The resulting flow (with* $a = 1$*) is shown in* Figure 2.13(ii)*.*

The Circle Theorem may be viewed as an extension of the method of images to circular boundaries. In Example 2.15, we see that application of the Circle Theorem results in a singularity being inserted inside the cylinder. The flow due to the "image" singularity then repels the uniform flow, such that the circle $|z| = a$ ends up being a streamline. In this case, the required singularity turns out to be a so-called *doublet*, with complex potential $w(z) = Ua^2/z$. The singularity at the origin is inside the obstacle and thus does not adversely affect the external flow. The full streamline pattern, including the doublet inside the cylinder, is shown in Figure 2.14.

The Circle Theorem allows us to construct a velocity potential that shares the same singularities with the given external flow $f(z)$ and has the circle $|z| = a$ as a streamline. However, it does not guarantee that the resulting potential is unique. Consider the flow due to a *vortex*, as illustrated in Figure 2.4. This is a velocity field that has $|z| = a$ as a streamline, has no singularities in $|z| > a$ and decays to zero as $|z| \to \infty$. Hence, we could superimpose the flow due to a vortex at the origin onto any flow outside a circular cylinder, without introducing any further singularities in the fluid or disturbing the streamline at $|z| = a$. The most general flow produced by a inserting a circular

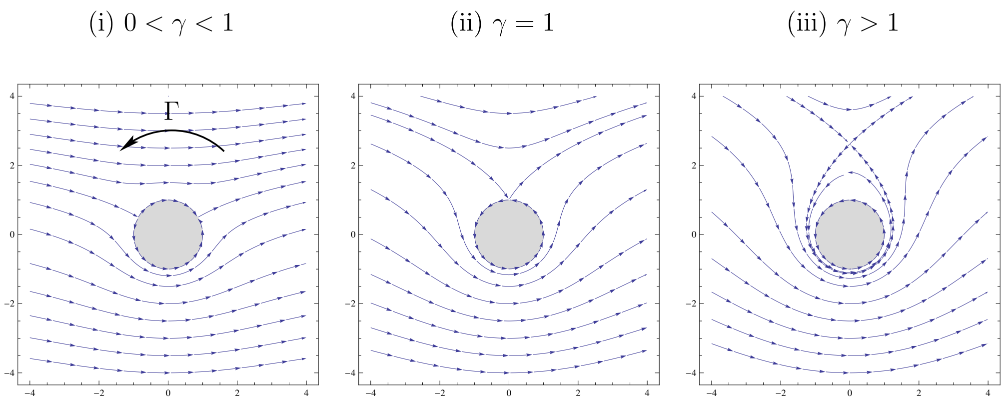

Figure 2.15: Streamlines for uniform flow at speed U flow past a cylinder pf radius a with with circulation Γ. (i) $0 < \gamma < 1$; (ii) $\gamma = 1$; (iii) $\gamma > 1$, where $\gamma = \Gamma/4\pi Ua$.

cylinder into an external flow $w(z) = f(z)$ is thus

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\overline{z}}\right)} - \frac{\mathrm{i}\Gamma}{2\pi}\log z, \tag{2.74}$$

where Γ is arbitrary, and represents the *circulation* about the cylinder.

Example 2.16 ***Uniform flow past a circular cylinder with circulation***
If we incorporate circulation in the complex potential (2.72) *corresponding to uniform flow past a circular cylinder, we obtain*

$$w(z) = Uz + \frac{Ua^2}{z} - \frac{\mathrm{i}\Gamma}{2\pi}\log z. \tag{2.75}$$

We can read off the streamfunction from the imaginary part of this function, namely

$$\psi = Uy\left(1 - \frac{a^2}{x^2+y^2}\right) - \frac{\Gamma}{4\pi}\log\left(x^2+y^2\right). \tag{2.76}$$

Hence the circle $x^2 + y^2 = a^2$ is still a streamline, with $\psi = -(\Gamma/2\pi)\log a$.

Any stagnation points in the flow satisfy

$$0 = u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = U - \frac{Ua^2}{z^2} - \frac{\mathrm{i}\Gamma}{2\pi z}, \tag{2.77}$$

which can be rearranged to the quadratic equation

$$z^2 - 2\mathrm{i}\gamma az - a^2 = 0, \tag{2.78}$$

where for convenience we introduce the shorthand

$$\gamma = \frac{\Gamma}{4\pi Ua}. \tag{2.79}$$

The roots of equation (2.78) *are given by*

$$\frac{z}{a} = \mathrm{i}\gamma \pm \sqrt{1-\gamma^2}. \tag{2.80}$$

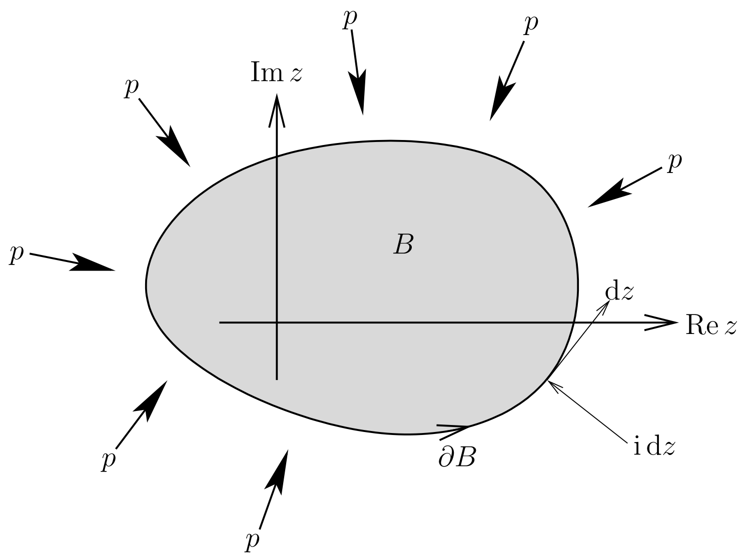

Figure 2.16: Schematic of an obstacle B, with boundary ∂B, subject to a pressure p.

When $\gamma = 0$, there is no circulation and the flow is as illustrated in Figure 2.13(ii)*, with stagnation points at $z = \pm a$. As γ increases, the anticlockwise circulation causes the stagnation points to move upwards around the cylinder, as shown in* Figure 2.15(i)*. When γ reaches the value 1, the two stagnation points coalesce at the top of the cylinder $z = \mathrm{i}a$, as shown in* Figure 2.15(ii)*. If $\gamma > 1$, then one stagnation point moves into the flow, as shown in* Figure 2.15(iii)*; the other one is inside the cylinder. Now there is a region of fluid, bounded by the separatrix through the stagnation point, that simply rotates around the cylinder, and is completely cut off from the external flow.*

2.4 Blasius' Theorem

Suppose we know the velocity potential $w(z)$ corresponding to flow past a given obstacle B, and we wish to determine the *force* exerted on the obstacle by the fluid flowing past. We will show how to achieve this for an arbitrarily-shaped obstacle. Although so far we have learnt how to determine $w(z)$ only when B is circular, we will find that the Circle Theorem may be generalised to other shapes by using conformal mapping.

Lemma 2.2 *Suppose fluid flows steadily past an obstacle B with simple closed boundary ∂B. If gravity is neglected, the net force (F_x, F_y) exerted on B by the fluid (per unit length out of the plane) is given by*

$$F_x + \mathrm{i}F_y = -\frac{\mathrm{i}\rho}{2}\oint_{\partial B}\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \mathrm{d}z. \tag{2.81}$$

Proof We recall that the pressure is assumed to act in the inward normal direction, so the net force experienced by the body B is given by

$$(F_x, F_y) = \oint_{\partial B} -p\boldsymbol{n}\,\mathrm{d}s = \oint_{\partial B} p(-\mathrm{d}y, \mathrm{d}x). \tag{2.82}$$

We translate the components (F_x, F_y) of $\boldsymbol{F}$ into a complex scalar by writing

$$F_x + \mathrm{i}F_y = \oint_{\partial B} p(-\mathrm{d}y + \mathrm{i}\mathrm{d}x) = \oint_{\partial B} p\mathrm{i}\,\mathrm{d}z, \tag{2.83}$$

where $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}y$. The directions of $\mathrm{d}z$ and $\mathrm{i}\,\mathrm{d}z$ are illustrated in Figure 2.16.

To evaluate the pressure from the complex potential, we turn to Bernoulli's Theorem. Restricting our attention to steady flow and neglecting gravity, we substitute the velocity components from (2.46) into (2.9) to obtain

$$p = P - \frac{\rho}{2}\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2, \tag{2.84}$$

where P is a constant reference pressure. Equation (2.81) follows from substitution of (2.84) into (2.83); the constant term P in the pressure integrates to zero by Cauchy's Theorem. ∎

Although equation (2.81) gives us formulae for the force components, it is very inconvenient to use it as it stands. The modulus signs mean that the integrand in (2.81) will in general be holomorphic nowhere. Hence, none of the standard tools of complex analysis is any use in evaluating the right-hand side of (2.81). Fortunately, there is a neat trick that eliminates the modulus signs in (2.81) and thus transforms the integrand into a meromorphic function.

Theorem 2.3 ***Blasius' Theorem***
Suppose fluid flows steadily past an obstacle B with simple closed boundary ∂B. If gravity is neglected, the net force (F_x, F_y) exerted on B by the fluid is given by

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2}\oint_{\partial B}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{2.85}$$

Proof By taking the complex conjugate of (2.81), we obtain

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2}\oint_{\partial B}\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \mathrm{d}\overline{z}, \tag{2.86}$$

where $\mathrm{d}\overline{z} = \mathrm{d}x - \mathrm{i}\mathrm{d}y$. By factorising the integrand in the form

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \equiv \frac{\mathrm{d}w}{\mathrm{d}z}\,\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{z}}, \tag{2.87}$$

we can rewrite (2.86) as

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2}\oint_{\partial B}\frac{\mathrm{d}w}{\mathrm{d}z}\,\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{z}}\,\mathrm{d}\overline{z} \equiv \frac{\mathrm{i}\rho}{2}\oint_{\partial B}\frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}\overline{w}. \tag{2.88}$$

Now we recall that ∂B is supposed to be a streamline for the flow. Hence $\psi = \operatorname{Im} w$ is constant on ∂B and it follows that $\mathrm{d}\overline{w} \equiv \mathrm{d}w$ on ∂B. We therefore obtain

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{\partial B} \frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}w, \tag{2.89}$$

and equation (2.85) follows. ∎

(For an alternative proof of Blasius' Theorem, see Acheson, section 4.10.)

Example 2.17 ***Uniform flow past a circular cylinder***
We illustrate the use of Blasius' Theorem for the flow considered in Example 2.15. *Substituting the velocity potential* (2.72), *into equation* (2.85), *we find the force components*

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{|z|=a} \left(U - \frac{Ua^2}{z^2}\right)^2 \mathrm{d}z. \tag{2.90}$$

This integral is easily evaluated using Cauchy's Residue Theorem. *There is just one pole, of order* 4, *at the origin. However, the Laurent expansion of the integrand, namely*

$$\frac{U^2a^4}{z^4} - \frac{2Ua^2}{z^2} + U^2$$

has no term of order $1/z$, *and the residue of the pole is therefore zero.*

Example 2.17 shows that *the net force exerted on a circular cylinder by a uniform flow is zero!* This counter-intuitive result is known as *D'Alembert's Paradox*. In practice, we would certainly expect a cylinder subject to a uniform flow to experience a nonzero force. The discrepancy between the theoretical prediction and experimental reality is now known to be due to our neglect of viscous effects.

Example 2.18 ***Uniform flow past a circular cylinder with circulation***
Next we apply Blasius' Theorem to the flow considered in Example 2.16, *with circulation included. Substituting the velocity potential* (2.75), *into equation* (2.85), *we find the force components*

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{|z|=a} \left(U - \frac{Ua^2}{z^2} - \frac{\mathrm{i}\Gamma}{2\pi z}\right)^2 \mathrm{d}z. \tag{2.91}$$

Again, there is just one pole, of order 4, at the origin. When we expand out the integrand, we easily find that the coefficient of $1/z$ *is* $-\mathrm{i}U\Gamma/\pi$, *and we therefore deduce from Cauchy's Residue Theorem that*

$$F_x - \mathrm{i}F_y = \mathrm{i}\rho U\Gamma. \tag{2.92}$$

Hence the so-called drag force F_x, *parallel to the flow, is zero, while the* lift force, *transverse to the flow, is given by*

$$F_y = -\rho U\Gamma. \tag{2.93}$$

We can interpret the lift formula (2.93) as follows. If Γ is positive (as in Figure 2.15), then the circulation opposes the uniform flow above the cylinder and reinforces it below. Hence the average speed along the bottom of the cylinder will be higher than that along the top. Bernoulli's Theorem tells us that higher speed is associated with lower pressure, and the cylinder therefore experiences a net downwards force. On the other hand, if Γ is negative, then the circulation is in the clockwise sense and the cylinder experiences an upwards force.

It may be shown that equation (2.93) applies to uniform flow past *any* obstacle, whether or not it is circular.

Theorem 2.4 ***Kutta–Joukowski Lift Theorem***
Consider steady uniform flow at speed U past an obstacle B, where there is a circulation Γ about B but there are no singularities in the flow. Then the obstacle experiences a drag force *D parallel to the flow and* lift force *L perpendicular to the flow given by*

$$D = 0, \qquad L = -\rho U \Gamma. \tag{2.94}$$

Proof Since there are assumed to be no singularities in the flow, $\mathrm{d}w/\mathrm{d}z$ must be holomorphic outside B and must therefore have a Laurent expansion of the form

$$\frac{\mathrm{d}w}{\mathrm{d}z} \sim U\mathrm{e}^{-\mathrm{i}\alpha} + \frac{b_1}{z} + \mathrm{O}\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty. \tag{2.95}$$

The first term in (2.95) corresponds to the imposed uniform flow at infinity, which we allow to approach at an arbitrary inclination α to the x-axis.

We recall from Blasius' Theorem that the components of the force on B are given by (2.85). Since there are no singularities in the flow, we can use Cauchy's Theorem to deform the contour of integration from ∂B to a large circle C of radius R, as shown schematically in Figure 2.17(i):

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{\partial B} \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z = \frac{\mathrm{i}\rho}{2} \oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{2.96}$$

We parametrise C using $z = R\mathrm{e}^{\mathrm{i}\theta}$ and use (2.95) to obtain

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \int_0^{2\pi} \left\{U\mathrm{e}^{-\mathrm{i}\alpha} + \frac{b_1\mathrm{e}^{-\mathrm{i}\theta}}{R} + \mathrm{O}\left(\frac{1}{R^2}\right)\right\}^2 \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta, \tag{2.97}$$

and, letting $R \to \infty$, we find that

$$F_x - \mathrm{i}F_y = -2\pi\rho b_1 U\mathrm{e}^{-\mathrm{i}\alpha}. \tag{2.98}$$

The drag and lift forces are defined to be the components parallel and perpendicular to the flow, as shown in Figure 2.17(ii). Simple trigonometry tells as that they are related to F_x and F_y by

$$D + \mathrm{i}L = (F_x + \mathrm{i}F_y)\,\mathrm{e}^{-\mathrm{i}\alpha}, \tag{2.99}$$

and we deduce from (2.98) that

$$D - \mathrm{i}L = -2\pi\rho b_1 U. \tag{2.100}$$

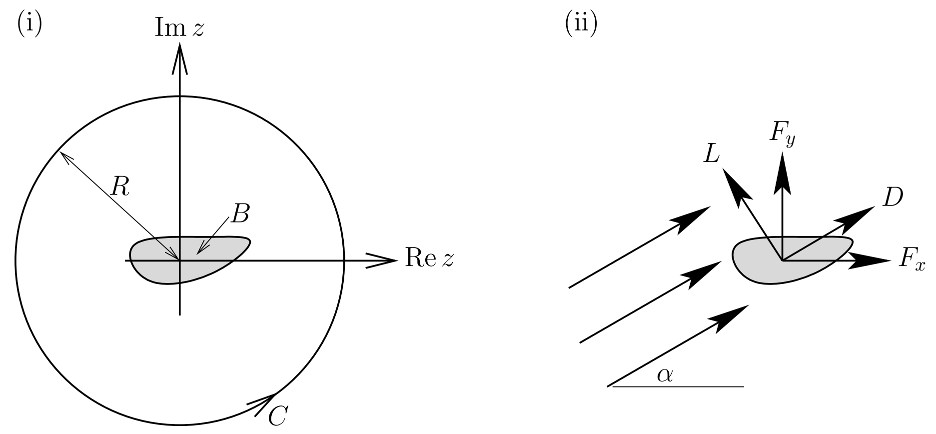

Figure 2.17: (i) Schematic showing deformation of the boundary ∂B of an obstacle B being deformed to a large circular contour of radius R. (ii) The relations between the drag and lift forces and the components F_x and F_y.

To evaluate the constant b_1, we note that

$$\oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}z = \int_0^{2\pi} \left\{ U\mathrm{e}^{-\mathrm{i}\alpha} + \frac{b_1 \mathrm{e}^{-\mathrm{i}\theta}}{R} + \mathrm{O}\left(\frac{1}{R^2}\right) \right\} \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = 2\pi\mathrm{i}b_1. \tag{2.101}$$

On the other hand, the real and imaginary parts of the left-hand side give us

$$\oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}z = \oint_C (u-\mathrm{i}v)\,(\mathrm{d}x+\mathrm{i}\mathrm{d}y) = \oint_C (u\,\mathrm{d}x+v\,\mathrm{d}y)+\mathrm{i}\oint_C (u\,\mathrm{d}y-v\,\mathrm{d}x) = \Gamma+\mathrm{i}Q, \tag{2.102}$$

where Γ is the *circulation* around C, while Q is the *flux* through C. (These are equal to the jumps in the values of ϕ and ψ respectively as the closed circuit C is traced.) The flux must be zero, as there are not supposed to be any sources or sinks in the flow, and we deduce from (2.101) and (2.102) that

$$b_1 = -\frac{\mathrm{i}\Gamma}{2\pi}. \tag{2.103}$$

Hence equation (2.100) becomes

$$D - \mathrm{i}L = \mathrm{i}\rho U\Gamma, \tag{2.104}$$

and the result (2.94) follows. ∎

2.5 Background material

2.5.1 Integral theorems

Green's Theorem Let the functions $P(x,y)$ and $Q(x,y)$ be continuously differentiable in the region D bounded by the simple closed curve ∂D (traversed in the anti-clockwise sense) with outward unit normal $\boldsymbol{n}$. Then the Divergence Theorem restricted to two dimensions tells us that

$$\iint_D \boldsymbol{\nabla} \cdot (P, Q) \, \mathrm{d}x\mathrm{d}y \equiv \oint_{\partial D} (P, Q) \cdot \boldsymbol{n} \, \mathrm{d}s, \tag{2.105}$$

which may be rewritten as

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}x\mathrm{d}y \equiv \oint_{\partial D} (P \, \mathrm{d}y - Q \, \mathrm{d}x). \tag{2.106}$$

Fundamental Theorem of Calculus for line integrals For any continuously differentiable scalar function $\phi(x, y)$, we have

$$\int_C \boldsymbol{\nabla}\phi \cdot \mathrm{d}\boldsymbol{x} = \int_C \left(\frac{\partial \phi}{\partial x} \mathrm{d}x + \frac{\partial \phi}{\partial y} \mathrm{d}y \right) = \phi(\boldsymbol{b}) - \phi(\boldsymbol{a}) \tag{2.107}$$

where C is a smooth path joining the two points with position vectors $\boldsymbol{x} = \boldsymbol{a}$ and $\boldsymbol{x} = \boldsymbol{b}$.

2.5.2 Plane polar coordinates

Definition (r, θ) are related to (x, y) by

$$x \equiv r \cos\theta, \qquad y \equiv r \sin\theta. \tag{2.108}$$

The corresponding unit basis vectors are

$$\boldsymbol{e}_r \equiv \boldsymbol{i} \cos\theta + \boldsymbol{j} \sin\theta, \qquad \boldsymbol{e}_\theta \equiv -\boldsymbol{i} \sin\theta + \boldsymbol{j} \cos\theta, \tag{2.109}$$

in terms of the usual Cartesian basis vectors $\{\boldsymbol{i}, \boldsymbol{j}\}$.

Grad, div and curl Given a two-dimensional scalar field $\phi(r, \theta)$ and a vector field $\boldsymbol{u}(r, \theta) = u_r(r, \theta)\boldsymbol{e}_r + u_\theta(r, \theta)\boldsymbol{e}_\theta$, we have

$$\boldsymbol{\nabla}\phi \equiv \frac{\partial \phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\boldsymbol{e}_\theta, \tag{2.110}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} \equiv \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \tag{2.111}$$

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \tag{2.112}$$

$$\boldsymbol{\nabla} \times \boldsymbol{u} \equiv \left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right) \boldsymbol{k}, \tag{2.113}$$

$$\boldsymbol{\nabla} \times (\phi \boldsymbol{k}) \equiv \frac{1}{r}\frac{\partial \phi}{\partial \theta}\boldsymbol{e}_r - \frac{\partial \phi}{\partial r}\boldsymbol{e}_\theta, \tag{2.114}$$

where $\boldsymbol{k} \equiv \boldsymbol{i} \times \boldsymbol{j} \equiv \boldsymbol{e}_r \times \boldsymbol{e}_\theta$ is a unit vector perpendicular to the (x, y)-plane.

2.5.3 Complex analysis

Standard notation Given a complex number $z = x + \mathrm{i}y$, we define:

- $\operatorname{Re} z = x$, the real part of z;
- $\operatorname{Im} z = y$, the imaginary part of z;
- $\overline{z} = x - \mathrm{i}y$, the complex conjugate of z;
- $|z| = \sqrt{x^2 + y^2}$, the modulus of z.

Note that the complex conjugate commutes with all standard mathematical operations, that is

$$\overline{z_1 + z_2} \equiv \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 z_2} \equiv \overline{z_1}\,\overline{z_2}, \qquad \overline{\mathrm{e}^z} \equiv \mathrm{e}^{\overline{z}}, \tag{2.115}$$

and so forth.

Derivatives and holomorphic functions A function $f(z)$ is *holomorphic* (or *analytic*) if it has a well-defined derivative

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \tag{2.116}$$

which is independent of the direction in which the complex variable h approaches zero.

Any holomorphic function has

- *derivatives of all orders*, and
- a *power series expansion*, with positive radius of convergence.

A complex-valued function may be decomposed into its real and imaginary parts, i.e. $f(z) \equiv \phi(x, y) + \mathrm{i}\psi(x, y)$. Then $f(z)$ is holomorphic if and only if the first partial derivatives of ϕ and ψ are continuous and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.117}$$

Laurent expansions If a function $f(z)$ is holomorphic on an *annulus*, say $\delta < |z - a| < R$, then it has a convergent *Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - a)^n. \tag{2.118}$$

The singularity at $z = a$ is

1. *removable* if $b_n = 0$ for all $n < 0$;
2. a *pole of order* N if $b_{-N} \neq 0$ but $b_n = 0$ for all $n < -N$;
3. an *essential singularity* if there does not exist an N such that $b_n = 0$ for all $n < -N$.

In case 2, the **residue** $\operatorname{Res}(f(z); a)$ of the pole is the coefficient b_{-1} of $(z - a)^{-1}$ in the Laurent expansion about $z = a$.

Multifunctions Functions such as $\log z$ and z^α for non-integer α cannot be defined both uniquely and continuously when z is complex. For example,

$$\log z = \log r + \mathrm{i}\theta, \tag{2.119}$$

where (r, θ) are the usual plane polar coordinates, also known as the *modulus* and *argument* of z. To define $\log z$ uniquely, we have to *choose a branch* of θ. For example, if we choose θ to lie in the range $\theta \in [0, 2\pi)$, then $\log z$ is uniquely defined, but discontinuous across the *branch cut* at $\theta = 0$, i.e. along the postive real axis.

By writing

$$z^\alpha \equiv r^\alpha \mathrm{e}^{\mathrm{i}\alpha\theta}, \tag{2.120}$$

we can also define z^α uniquely once we have chosen a branch for θ.

For more complicated multifunctions, see Priestley Chapter 9.

Contour integrals A closed curve C in the complex plane may be parametrised as $z = \gamma(\tau)$, where the real parameter τ lies in some interval $\tau \in [0, c]$ and $\gamma(0) = \gamma(c)$. Then we define the contour integral of $f(z)$ around C as

$$\oint_C f(z)\,\mathrm{d}z := \int_0^c f\big(\gamma(\tau)\big)\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{2.121}$$

Cauchy's Theorem If $f(z)$ is holomorphic inside and on the simple close curve C, then

$$\oint_C f(z)\,\mathrm{d}z \equiv 0. \tag{2.122}$$

Deformation Theorem Suppose that the contour $\tilde{C}$ lies inside the contour C, and that the function $f(z)$ is holomorphic on $\tilde{C}$, on C and on the region contained between them. Then the contour C may be deformed onto $\tilde{C}$ without changing the contour integral of $f(z)$:

$$\oint_C f(z)\,\mathrm{d}z \equiv \oint_{\tilde{C}} f(z)\,\mathrm{d}z. \tag{2.123}$$

Cauchy's Residue Theorem Suppose $f(z)$ is holomorphic on and inside a contour C, except for a finite number of poles at the points $a_1, \ldots, a_n$ inside C. Then the contour integral of $f(z)$ around C is equal to $2\pi\mathrm{i}$ times the sum of the residues of all the poles inside C, *i.e.*

$$\oint_C f(z)\,\mathrm{d}z \equiv 2\pi\mathrm{i}\sum_{k=1}^{n} \operatorname{Res}\left(f(z); a_k\right). \tag{2.124}$$