1 Similarity Solutions

1.1 Introduction

You are preparing a breakfast before you head out for your lectures for the day. You set up the frying pan ready to make some pancakes. As you carefully pour your oil into the pan you watch the spreading behaviour: the oil forms a shape that is very close to a circle, which expands outwards with time, slowing in its rate of expansion as it grows. The next day you're a little more tired and so when you deposit your oil into the pan it doesn't quite form a circular blob. But you notice that as it spreads out, the oil 'corrects' for your error and forms a circular shape once again. When the oil has spread across the pan you add your pancake batter. As you do so, you notice the spreading phenomenon is very similar, correcting for deviations from a circle in the initial blob of pancake mix you deposit and then spreading out and slowing as it does so.

Before you eat your pancake you decide to add some maple syrup onto the top. Once again the same pattern of events repeats itself. Your morning observations have uncovered a sense of universality in spreading. But can we quantify this more than simply through daily observations? Well, if you were to plot a graph of radius of the oil, pancake mix, or maple syrup, as a function of time, on log–log axes then you would see that, after a short transient the graph would settle to a straight line with a gradient of 1/8. In other words, the radius of the liquid, r, spreads in time like $r \propto t^{1/8}$, whether it's the oil, the pancake mix or the maple syrup and no matter how much of the liquid we initially deposit.†

The universality of our breakfast activities doesn't just apply to the spreading rate. If we were to take side-view photos to capture the height of each of the spreading liquids then the shape of the profile would be identical provided we stretched our photos vertically, so each liquid had the same peak height, and horizontally, so they had the same radius of the spreading front of the liquid.

This kind of behaviour extends much more generally, encompassing the spreading of liquids on tilted surfaces (e.g., rain running down a window pane), the spreading of liquid on a porous surface (e.g., spreading of honey on porous toast) and geological applications such as carbon sequestration, where we are trapping CO_2 underground.

In this first chapter we will explore the idea of this universality to see what it means mathematically and why so many seemingly different phenomena can all be described in the same way.

1.2 Spreading of a liquid on a vertical liquid-coated wall

Let us begin by considering the spreading of liquid on a vertical surface (see figure 1.1 for a schematic). Such a set-up could describe water running down a window pane. We assume

†To be strictly correct, this is only true for Newtonian fluids. If the fluid is non-Newtonian, such as pancake batter, then the behaviour will be a little different.

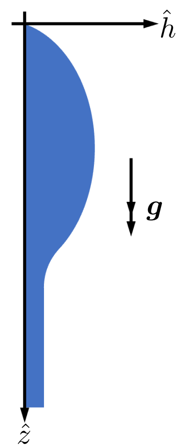

Figure 1.1: Schematic of liquid draining down a wall. The liquid profile is given by $\hat{h}(\hat{z}, \hat{t})$ at vertical position $\hat{z}$ and time $\hat{t}$.

here for simplicity that the entire surface of the wall is coated with liquid. If parts of the wall were dry then we would have contact lines, which correspond to places where we have air–liquid–solid junctions. We will study these in Section 1.4. If we let $\hat{z}$ denote the vertical position on the wall then the thickness of the liquid $\hat{h}$ as a function of vertical height $\hat{z}$ and time $\hat{t}$ is governed by

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{z}} = 0, \qquad \hat{Q} = \frac{\rho g \hat{h}^3}{3\mu}, \tag{1.1}$$

where ρ and μ denote respectively the density and viscosity of the liquid, g denotes acceleration due to gravity and $\hat{Q}$ is the flux of fluid (see Problem Sheet 1, question 1 for a derivation of this equation). Note that here, and elsewhere in these notes, we will use the convention that a hatted ($\hat{}$) variable represents a dimensional quantity; we will not, however, use hats for dimensional parameters (such as density, viscosity and acceleration due to gravity). We note that (1.1) may be written as a single equation:

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\rho g}{3\mu} \frac{\partial}{\partial \hat{z}} \left(\hat{h}^3 \right) = 0. \tag{1.2}$$

We will work with this form, but we note that all of the physical problems that we cover in this chapter may be expressed in the *conservative* form (1.1).

Equation (1.2) is a first-order hyperbolic equation so we require one initial condition $\hat{h}(\hat{z}, 0)$ and one boundary condition, for example $\hat{h}(0, \hat{t}) = 0$, to solve this. We can solve this problem analytically using the method of characteristics (see Problem Sheet 1, question 2). If we do this, we find that, irrespective of the initial condition, the liquid evolves towards a similar configuration (see figure 1.2).

Given our observation that two very different initial conditions eventually end up possessing the same time-dependent profile, let us analyse the system in a different way to try to extract and understand this property. Before we proceed any further though, we will first non-dimensionalize the system. This will allow us to work with the simplest form of the system. When non-dimensionalizing, we choose characteristic scales that reduce the problem to

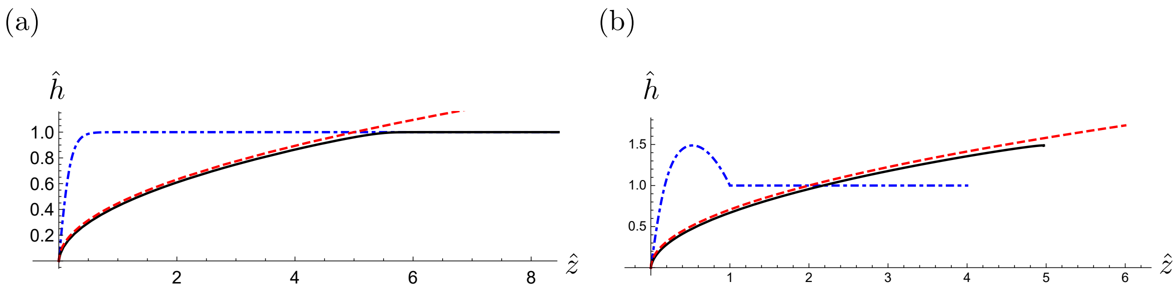

Figure 1.2: Analytic solution for the spreading of a liquid on a vertical substrate with an initial condition (a) $\hat{h}(\hat{z},0) = \tanh(5\hat{z})$ at $\hat{t} = 5$; (b) $\hat{h}(\hat{z},0) = \tanh(5\hat{z}) + 2\left(1 - \text{Heaviside}(\hat{z}-1)\right)\hat{z}(\hat{z}-1)$ at $\hat{t} = 2$. The red dashed lines show the similarity solution, towards which all profiles evolve, regardless of the initial condition. The blue dot-dashed line shows the initial condition.

its simplest form. In this case, we choose to non-dimensionalize by introducing the following scales:

$$\hat{z} = \hat{z}_0 z, \qquad \hat{t} = \hat{t}_0 t, \qquad \hat{h} = \hat{h}_0 h, \tag{1.3}$$

where $\hat{z}_0$, $\hat{t}_0$ and $\hat{h}_0$ are length, time and liquid thickness scales that we are free to choose; generally it is useful to choose these using typical scales in the problem. If we choose

$$\hat{t}_0 = \frac{\mu \hat{z}_0}{\rho g \hat{h}_0^2} \tag{1.4}$$

then the dimensionless equation is

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left(h^3\right) = 0. \tag{1.5}$$

We can use the initial condition to choose appropriate values for $\hat{h}_0$ and $\hat{z}_0$, for example the height of the liquid as $\hat{z} = \infty$. However, there is no natural length scale to allow us to choose $\hat{x}_0$, meaning that we are free to choose this however we like. Indeed, the lack of a natural length scale indicates that a similarity solution may be possible.

Now that we have a dimensionless system we are in a position to analyse this more easily. First, we shall seek a solution of the form

$$h(z,t) = f(\eta) \qquad \text{where} \quad \eta = \frac{z}{t^\alpha}, \tag{1.6}$$

where α is to be determined. On the face of it, it is not obvious why we should seek a solution of this form. We will look into why this is the case later on. However, if we do so and substitute this into (1.5) then we find that if we choose $\alpha = 1$ the problem is transformed from a partial differential equation into an *ordinary differential equation* in terms of the new variable η as

$$(f^2 - \eta)f' = 0. \tag{1.7}$$

From here we immediately obtain the solution

$$f = \eta^{1/2} = \left(\frac{z}{t}\right)^{1/2}, \tag{1.8}$$

or, in dimensional terms,

$$\hat{h} = \left(\frac{\mu}{\rho g}\right)^{1/2} \left(\frac{\hat{z}}{\hat{t}}\right)^{1/2}. \tag{1.9}$$

If we plot this function we find that it does an excellent job of replicating the analytic solution for the thickness profile (figure 1.2). However, we never used the initial condition to derive the similarity solution. As a result, the similarity solution cannot satisfy an arbitrary initial condition $\hat{h}(\hat{z}, 0)$. In fact, as $\hat{t} \to 0$ $\hat{h} \to \infty$. This highlights the fact that our solution does not accurately capture the early time behaviour. In fact, the system 'forgets' this initial information over time. Indeed, all initial conditions approach the same time-evolving state, but just take different amounts of time to reach that point.

The analysis we have performed is advantageous because it bypasses the need to determine an analytic solution of the entire system, which is often difficult, if not impossible. The similarity solution (and in this case the non-dimensionalization (1.4) alone) tells us the parametric dependence of the problem: there is a parametric grouping, $\rho g/\mu$, that indicates how the liquid spreads. This tells us how the experiment will change if, for example, we used a more viscous liquid or a more dense liquid and also tells us the typical time taken for a given liquid to drain under gravity.

Before we move on to another example, we will finish by examining the equation using an even simpler *scaling-law* approach. Here, we approximate derivatives $\partial y/\partial x \sim Y/X$, where Y and X correspond to the typical variations in y and x respectively. If we substitute this scaling ansatz into (1.5) and assume that the two terms in our equation are in balance then we get

$$\frac{H}{T} \sim \left(\frac{\rho g}{3\mu}\right) \frac{H^3}{Z}, \tag{1.10}$$

which may be rearranged to give

$$H \sim \left(\frac{3\mu}{\rho g}\right)^{1/2} \left(\frac{Z}{T}\right)^{1/2}. \tag{1.11}$$

This result tells us almost as much information as the similarity solution, namely, that the height scales like $(\hat{z}/\hat{t})^{1/2}$ and that the dependence on the parameters appears as a prefactor of the form $(\mu/\rho g)^{1/2}$. The only part that we are missing is the numerical prefactor (which, in this case happens to be $1/\sqrt{3}$).

To summarize what we have seen: a scaling law gives the long-time functional $((\hat{z}/\hat{t})^{1/2})$ and parametric $(\mu/\rho g)^{1/2}$ *dependence* of the solution; the similarity solution gives the *long-time solution*; the analytic solution gives the correct behaviour for *all time*.

1.3 The heat equation

We will now move onto the familiar example of the heat equation for temperature $\hat{T}$,

$$\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0, \qquad \hat{Q} = -D\frac{\partial \hat{T}}{\partial \hat{x}}, \tag{1.12}$$

where $\hat{x}$ denotes space, $\hat{t}$ denotes time and D denotes the (constant) diffusivity; $\hat{Q}$ is the heat flux. This may be written as the following single equation

$$\frac{\partial \hat{T}}{\partial \hat{t}} = D\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}, \tag{1.13}$$

We require one initial condition $\hat{T}(\hat{x}, 0)$ and two boundary conditions, say,

$$\hat{T}(\hat{x}, \hat{t}) \to \hat{T}_{-\infty} \qquad \text{as} \quad \hat{x} \to -\infty, \tag{1.14a}$$

$$\hat{T}(\hat{x}, \hat{t}) \to \hat{T}_{+\infty} \qquad \text{as} \quad \hat{x} \to +\infty, \tag{1.14b}$$

where $\hat{T}_{\pm\infty}$ are constants.

As in the previous example, before we proceed any further, we non-dimensionalize the system via:

$$\hat{x} = \hat{x}_0 x, \qquad \hat{t} = \frac{\hat{x}_0^2}{D}t, \qquad \hat{T} = \left(\hat{T}_{+\infty} - \hat{T}_{-\infty}\right)T + \hat{T}_{-\infty}. \tag{1.15}$$

Note that this non-dimensionalization is only suitable if $\hat{T}_{-\infty} \neq \hat{T}_{+\infty}$. We shall suppose this is the case for now, then study the case where $\hat{T}_{-\infty} = \hat{T}_{+\infty}$ afterwards. As in the previous example, there is no natural lengthscale, so $\hat{x}_0$ is arbitrary. With these scalings, the governing equation (1.13) becomes

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \tag{1.16}$$

while the boundary conditions (1.14) become

$$T(x, t) \to 0 \qquad \text{as} \quad x \to -\infty, \tag{1.17a}$$

$$T(x, t) \to 1 \qquad \text{as} \quad x \to +\infty. \tag{1.17b}$$

In this example, trying our simple scaling arguments, where we set $\partial y/\partial x \sim Y/X$, gives only that $X^2/T \sim 1$. This provides evidence that our similarity variable is $x/t^{1/2}$ but doesn't provide any further functional form for the solution T. We therefore proceed as in the previous example to seek a similarity solution of the form $T = f(\eta)$ where $\eta = x/t^{\alpha}$. Substituting this ansatz into (1.16) we find that we require $\alpha = 1/2$, which confirms our simple scaling argument prediction above. Choosing $\alpha = 1/2$, the governing equation becomes

$$f'' + \frac{\eta}{2}f' = 0, \tag{1.18}$$

where primes denote differentiation, while the boundary conditions become

$$f \to 0 \qquad \text{as} \quad \eta \to -\infty, \tag{1.19a}$$

$$f \to 1 \qquad \text{as} \quad \eta \to +\infty. \tag{1.19b}$$

The solution to (1.18) subject to (1.19) is

$$f = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\eta}{2}\right)\right), \tag{1.20}$$

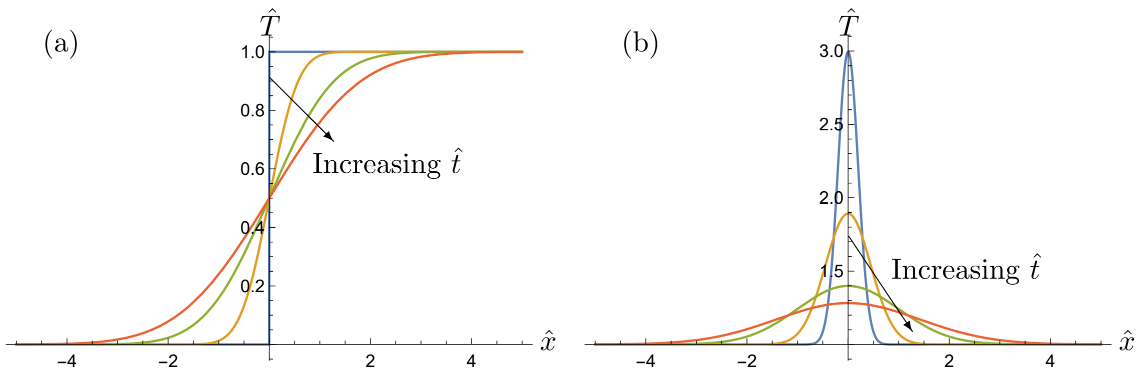

Figure 1.3: Similarity solution for the heat equation with (a) $\hat{T}_{-\infty} = 0$, $\hat{T}_{+\infty} = 1$ and $D = 1$ for times $t = 0, 0.1, 0.5, 1$; (b) $\hat{T}_{-\infty} = 0$, $\hat{T}_{+\infty} = 0$ and $D = 1$ for times $t = 0.02, 0.1, 0.5, 1$

where

$$\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} \mathrm{e}^{-s^2}\, \mathrm{d}s \tag{1.21}$$

is the error function. The solution in terms of dimensional variables is

$$\hat{T} = \frac{\left(\hat{T}_{+\infty} - \hat{T}_{-\infty}\right)}{2} \left(1 + \mathrm{erf}\left(\frac{\hat{x}}{2\sqrt{D\hat{t}}}\right)\right) + \hat{T}_{-\infty}, \tag{1.22}$$

We show this solution in figure 1.3(a).

As in the previous case, we did not use the initial condition to obtain this solution, and so this result only holds for long time, unless the initial condition of (1.22), *i.e.*, a step function at $\hat{x}_0 = 0$ to the left of which, $\hat{T} = \hat{T}_{-\infty}$ and to the right, $\hat{T} = \hat{T}_{+\infty}$, happens to be the initial condition that we impose. In that case, the similarity solution (1.22) is the analytic solution for all time. The fact that all solutions evolve to (1.22) irrespective of the initial configuration again shows that systems that possess similarity solutions lose the initial information for long time.

This case was fairly similar to the previous example. However, now let's make a small change and consider the aforementioned case where $\hat{T}_{-\infty} = \hat{T}_{+\infty}$. In this case we non-dimensionalize instead via

$$\hat{x} = \hat{x}_0 x, \qquad \hat{t} = \frac{\hat{x}_0^2}{D} t, \qquad \hat{T} = \hat{T}_{+\infty} T + \hat{T}_{+\infty}. \tag{1.23}$$

Our dimensionless governing equation remains the same (1.16), but our boundary conditions are now

$$T(x, t) \to 0 \qquad \text{as} \quad x \to \pm\infty. \tag{1.24}$$

While seemingly similar to the previous problem, our similarity solution (1.22) gives the trivial solution. However, if we integrate (1.16) over $-\infty < x < \infty$ and apply the boundary conditions (1.24) we obtain

$$\int_{-\infty}^{\infty} T(x, t)\, \mathrm{d}x = \int_{-\infty}^{\infty} T(x, 0)\, \mathrm{d}x = E, \quad \text{say,} \tag{1.25}$$

where E is the dimensionless energy in the system. This gives an additional constraint that was not required in the previous case (in that case the total initial energy in the domain was infinite). Substituting our previous similarity ansatz $T = f(\eta)$ with $\eta = x/t^{1/2}$ into this equation we get

$$t^{1/2} \int_{-\infty}^{\infty} f \,\mathrm{d}\eta = E, \tag{1.26}$$

which can only hold if $E = 0$ (*i.e.*, the trivial solution $T = 0$) since the left-hand side is a function of t but the right-hand side is a constant. But E is prescribed. This means that we need to seek a more general similarity ansatz. We try a solution of the form

$$T = t^{\beta} f(\eta) \qquad \text{where} \quad \eta = \frac{x}{t^{\alpha}}. \tag{1.27}$$

Substituting this ansatz into (1.16) and (1.25) we find that we must choose $\alpha = 1/2$ and $\beta = -1/2$ to make the resulting system depend only on η. In doing so, equations (1.16) and (1.25) become, respectively:

$$f'' + \frac{1}{2}\eta f' + \frac{1}{2} f = 0, \tag{1.28a}$$

$$\int_{-\infty}^{\infty} f \,\mathrm{d}\eta = E. \tag{1.28b}$$

We can solve (1.28) to obtain the solution

$$f = \frac{E}{2\sqrt{\pi}} \mathrm{e}^{-\eta^2/4}, \tag{1.29}$$

thus giving the full dimensional solution

$$\hat{T} = \frac{\hat{E}}{2\sqrt{\pi D \hat{t}}} \exp\left(-\frac{\hat{x}^2}{4D\hat{t}}\right) + \hat{T}_{+\infty}, \tag{1.30}$$

where $\hat{E}$ is the dimensional counterpart to E, defined by

$$\hat{E} = \int_{-\infty}^{\infty} \left(\hat{T}(\hat{x}, \hat{t}) - \hat{T}_{+\infty}\right) \mathrm{d}\hat{x}. \tag{1.31}$$

Again, we have not imposed an initial condition, which indicates that all initial profiles evolve to (1.30) in the long term. In this case, if the initial profile is proportional to a delta function centred at the origin, $T(x, 0) = E\delta(x)$ then the similarity solution and the analytic solution are equivalent. We show this solution in figure 1.3(b).

Thus we have shown in this section that the form of the similarity solution depends on both the governing equation and the boundary conditions. We note that in this case the ansatz was chosen somewhat arbitrarily. Before we proceed to the next example, we will consider a method for determining the form of the similarity solution. To do this, we pose the following rescalings for our independent variables:

$$x = a\tilde{x}, \qquad t = b\tilde{t}, \qquad T = c\tilde{T}. \tag{1.32}$$

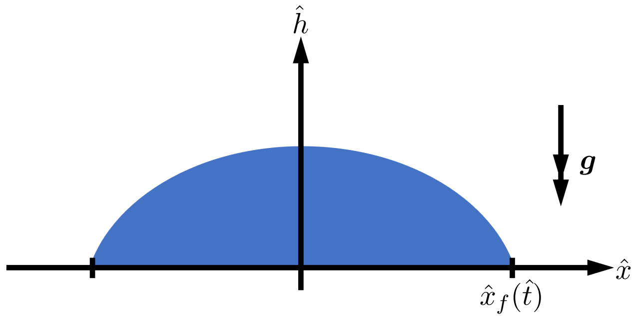

Figure 1.4: Schematic of a liquid droplet on a horizontal surface. The profile is $\hat{h}(\hat{x}, \hat{t})$ and the droplet front is located at $\hat{x}_f(\hat{t})$.

If we substitute these scalings into (1.16) and (1.25), we find that we recover the original equation and boundary condition in terms of the new primed variables if and only if we choose $a^2 = b$ and $ac = 1$. This means that

$$\frac{x}{t^{1/2}} = \frac{\tilde{x}}{\tilde{t}^{1/2}} \qquad \text{and} \qquad Tt^{1/2} = \tilde{T}\tilde{t}^{1/2}. \tag{1.33}$$

This indicates that the groupings $x/t^{1/2}$ and $Tt^{1/2}$ are *invariants* of the system, and this is the information we need to guide us into proposing the ansatz that the solution depends on $x/t^{1/2} = \eta$ and has a form $f(\eta) = t^{1/2}T(x, t)$.

1.4 Spreading of liquid on a horizontal plate

The next example we consider is the spreading of a finite blob of liquid under gravity on a horizontal plate (see figure 1.4 for a schematic). Such a phenomenon is termed a *gravity current*. The governing equation is (see [2] for details)

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0, \qquad \hat{Q} = -\frac{\Delta\rho g}{3\mu}\hat{h}^3\frac{\partial \hat{h}}{\partial \hat{x}}, \tag{1.34}$$

where $\hat{Q}$ is the fluid flux and $\Delta\rho$ is the density difference between the spreading liquid and the surrounding fluid. For water spreading in air, $\Delta\rho$ is approximately equal to the water density (since the density of air is much lower than that of water) but this equation could also be used to describe spreading of a viscous liquid such as oil at the bottom of the sea. All other variables are defined as before. This is a nonlinear diffusion equation with diffusion coefficient proportional to $\hat{h}^3$ (cf. equation (1.13) with $\hat{h}$ replaced with $\hat{T}$ and D replaced with $\Delta\rho g\hat{h}^3/3\mu$).

The system (1.34) may be written as a single equation,

$$\frac{\partial \hat{h}}{\partial \hat{t}} - \frac{\Delta\rho g}{3\mu}\frac{\partial}{\partial \hat{x}}\left(\hat{h}^3\frac{\partial \hat{h}}{\partial \hat{x}}\right) = 0, \tag{1.35}$$

We need one initial condition $\hat{h}(\hat{x}, 0)$ and two boundary conditions to close the system (cf. spreading on a vertical wall where we only needed one boundary condition from which the information propagates away). One boundary condition is

$$\hat{h}(\hat{x}_f(\hat{t}), \hat{t}) = 0, \tag{1.36}$$

where $\hat{x}_f(\hat{t})$ denotes the position of the front of the liquid drop at time $\hat{t}$. But $\hat{x}_f$ itself is unknown so we need another boundary condition. This comes in the form of volume conservation:

$$\hat{V}(\hat{t}) = \int_0^{\hat{x}_f(\hat{t})} \hat{h}(\hat{x},\hat{t})\,\mathrm{d}\hat{x}, \tag{1.37}$$

where $\hat{V}(\hat{t})$ is a prescribed amount of fluid. Here we have used symmetry to only define the problem for $\hat{x} > 0$ so $\hat{V}$ is half of the total amount of fluid on the plate. If we deposit a blob of liquid and simply watch it spread then $\hat{V}$ will be a constant; the time dependence in $\hat{V}$ allows for possibilities where we are injecting fluid. Note that for the spreading of a liquid on a vertical substrate considered in Section 1.2 there was an infinite amount of liquid, and so we did not have an associated volume conservation law. The spreading of an infinite amount of liquid on a vertical substrate considered in Section 1.2 is analogous to the first example considered in Section 1.3 while the spreading of a finite droplet is analogous to the second example considered in Section 1.3.

The final condition to close the problem is

$$\hat{h}^3\frac{\partial \hat{h}}{\partial \hat{x}} \to 0 \qquad \text{as } \hat{x} \to \hat{x}_f. \tag{1.38}$$

which corresponds to no fluid flux $\hat{Q}$ at the leading edge. Since we already know that $\hat{h} = 0$ at $\hat{x} = -\hat{x}_f$ on first glance this condition may not appear to provide any extra information. However, the condition places a constraint on how well behaved the solution is at the contact line, which, along with (1.36) and (1.37) forms a closed problem for (1.35).

Solving the system (1.35)–(1.38) numerically can be challenging since (1.37) is a global constraint rather than a boundary condition and (1.38) only enforces how well behaved the solution is at the front. However, we can transform each of these into boundary conditions, which are much easier to work with in a numerical scheme.

First, if we integrate (1.35) over $0 < \hat{x} < \hat{x}_f$ and apply (1.37) and (1.38) then this gives

$$-\frac{\Delta\rho g}{3\mu}\hat{h}^3\frac{\partial \hat{h}}{\partial \hat{x}} = \hat{V}'(\hat{t}) \qquad \text{on } \hat{x} = 0, \tag{1.39}$$

where a prime denotes differentiation. This replaces the global condition (1.37).

Second, we rescale into a region local to the moving front by introducing the coordinate change $(\hat{x},\hat{t}) \to (\hat{\xi},\hat{\tau})$ defined by

$$\hat{x} = \hat{x}_f(\hat{\tau}) + \epsilon\hat{\xi}, \qquad \hat{t} = \hat{\tau}, \tag{1.40}$$

and rescale the height in this region local to the front,

$$\hat{h}(\hat{x},\hat{t}) = \epsilon^{1/3}\hat{H}(\hat{\xi},\hat{\tau}), \tag{1.41}$$

where $\epsilon \ll 1$; the $\epsilon^{1/3}$ choice is made to achieve a leading-order balance at the front. In this new coordinate system, (1.35) becomes

$$\epsilon\frac{\partial \hat{H}}{\partial \hat{\tau}} - \frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{\tau}}\frac{\partial \hat{H}}{\partial \hat{\xi}} - \frac{\Delta\rho g}{3\mu}\frac{\partial}{\partial \hat{\xi}}\left(\hat{H}^3\frac{\partial \hat{H}}{\partial \hat{\xi}}\right) = 0. \tag{1.42}$$

Considering this equation at leading order in ϵ we find that we can integrate the resulting equation and apply the condition (1.36) to obtain the local behaviour at the front

$$\hat{H}^2 \frac{\partial \hat{H}}{\partial \hat{\xi}} = -\frac{3\mu}{\Delta\rho g}\frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{\tau}} \qquad \text{on} \quad \hat{x} = \hat{x}_f, \tag{1.43}$$

which in terms of the original variables gives

$$\hat{h}^2 \frac{\partial \hat{h}}{\partial \hat{x}} \sim -\frac{3\mu}{\Delta\rho g}\frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{t}} \qquad \text{as} \quad \hat{x} \to \hat{x}_f. \tag{1.44}$$

This condition informs us of the solution behaviour as we approach the contact point, which is more useful than (1.38) when solving this numerically as this allows us to initiate the numerical scheme accurately.

Thus the system (1.35) subject to (1.36), either (1.37) or (1.39), and either (1.38) or (1.44), plus an initial condition $\hat{h}(\hat{x}, 0)$ forms a closed problem.

We will restrict our attention to the case of a finite blob of liquid, so $\hat{V} =$constant. Analytic solutions are not possible, so we proceed directly to analysing the system using similarity solutions. First, we perform the simple scaling argument, setting derivatives $\partial y/\partial x = Y/X$. Substituting into (1.35) and (1.37) gives

$$\frac{H}{T} \sim \frac{\Delta\rho g}{3\mu}\frac{H^4}{X^2}, \qquad HX \sim \hat{V}, \tag{1.45}$$

which may be rearranged to obtain the scalings

$$\hat{x} \sim \left(\frac{\Delta\rho g \hat{V}^3}{3\mu}\right)^{1/5} \hat{t}^{1/5}, \qquad \hat{h} \sim \left(\frac{\Delta\rho g}{3\mu \hat{V}^2}\right)^{-1/5} \hat{t}^{-1/5}. \tag{1.46}$$

As in the case of vertical drainage, the scaling laws quickly reveal the parametric dependence of the problem. The scaling laws also tell us how we could perform experiments with different fluids and replicate the same results. For example, if we conducted an experiment with a liquid that has half the density and half the viscosity of another liquid then the results would be identical. As in the previous examples, the scaling arguments also give insight into the correct form of the similarity solution that we should seek. However, as before, while the scaling argument gives us a lot of information, it does not tell us the shape of the interface.

We now consider the full similarity solution. Before doing so, we non-dimensionalize the problem via the following scalings:

$$\hat{x} = \hat{x}_0 x, \qquad \hat{x}_f = \hat{x}_0 x_f, \qquad \hat{t} = \hat{t}_0 t, \qquad \hat{h} = \hat{h}_0 h, \tag{1.47}$$

where we choose

$$\hat{x}_0 = \hat{x}_f(0), \qquad \hat{t}_0 = \frac{3\mu \hat{x}_f(0)^5}{\Delta\rho g \hat{V}^3}, \qquad \hat{h}_0 = \frac{\hat{V}}{\hat{x}_f(0)}. \tag{1.48}$$

Here the timescale is chosen to remove all parameter dependence in the governing equation (1.34).

The governing equation and boundary conditions are then

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) = 0, \tag{1.49}$$

$$h = 0, \qquad \text{on} \quad x = x_f(t) \tag{1.50}$$

$$\int_0^{x_f} h \, \mathrm{d}x = 1, \tag{1.51}$$

$$h^2 \frac{\partial h}{\partial x} \sim -\frac{\mathrm{d}x_f}{\mathrm{d}t} \qquad \text{as} \quad x \to x_f, \tag{1.52}$$

We seek a solution of the form $h(x,t) = t^\beta f(\eta)$ where $\eta = x/t^\alpha$. Substituting into (1.49)–(1.52) we find that we must choose $\alpha = 1/5$ and $\beta = -1/5$, which is consistent with the prediction of the scaling analysis, equation (1.46). The system then becomes

$$\left(f^3 f'\right)' + \frac{1}{5}\left(f + \eta f'\right) = 0, \tag{1.53}$$

$$f = 0, \qquad \text{on} \quad \eta = \eta_f, \tag{1.54}$$

$$\int_0^{\eta_f} f \, \mathrm{d}\eta = 1, \tag{1.55}$$

$$f^2 f' \sim -\frac{1}{5}\eta_f \qquad \text{as} \quad \eta \to \eta_f, \tag{1.56}$$

where

$$\eta_f = \frac{x_f(t)}{t^{1/5}} \tag{1.57}$$

denotes the position of the front in similarity variables. Note that (1.57) enforces an assumption on the motion of the contact line.

We then make a second change of variables to decouple the position of the front, η_f, from the differential equation:

$$s = \frac{\eta}{\eta_f}, \qquad k(s) = \frac{f(\eta_f s)}{\eta_f^{2/3}}. \tag{1.58}$$

Then the system becomes

$$\left(k^3 k'\right)' + \frac{1}{5}\left(k + s k'\right) = 0, \tag{1.59}$$

$$k(1) = 0, \tag{1.60}$$

$$\eta_f = \left(\int_0^1 k(s) \, \mathrm{d}s\right)^{-3/5}, \tag{1.61}$$

$$k^2 k' = -1 \qquad \text{as} \quad s \to 1. \tag{1.62}$$

The solution to (1.59)–(1.62) is

$$k = \left(\frac{3}{10}\right)^{1/3} \left(1 - s^2\right)^{1/3} \tag{1.63}$$

with

$$\eta_f = \left[\left(\frac{3}{10}\right)^{1/3} \frac{\pi^{1/2}}{5} \frac{\Gamma(1/3)}{\Gamma(5/6)} \right]^{-3/5} \approx 1.411, \tag{1.64}$$

where Γ is the Gamma function defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} \mathrm{e}^{-x} \, \mathrm{d}x. \tag{1.65}$$

In dimensional terms, the solution is

$$\hat{h}(\hat{x}, \hat{t}) = \frac{\eta_f^{2/3}}{\hat{t}^{1/5}} \left(\frac{\Delta\rho g}{3\mu \hat{V}^2}\right)^{-1/5} k\left(\frac{\hat{x}}{\hat{x}_f}\right), \tag{1.66a}$$

$$\hat{x}_f(\hat{t}) = \eta_f \hat{t}^{1/5} \left(\frac{\Delta\rho g \hat{V}^3}{3\mu}\right)^{1/5}. \tag{1.66b}$$

Comparing this with (1.46) we see that the scaling analysis provided the time dependence and the parametric dependence but lacked the shape that this similarity solution (1.66) provides.

In every example we have considered so far, we have been able to determine the required solution by a scaling argument. Examples of this kind are called *similarity solutions of the first kind*. In the next section we will explore an example where our scaling analysis alone does not determine the solution. In these cases, we have a *similarity solution of the second kind*.

1.5 Spreading of a groundwater mound

We will continue with the theme of a spreading liquid on an impermeable horizontal plate, but where the liquid is within a porous medium. Such an example could describe the spreading of oil in a well in the earth. In this case, as the liquid drains it may leave behind a residue in the pores. Thus we have two regions (see figure 1.5):
(i) Region 1: A region in which the fluid is invading the surrounding porous medium, so $\partial\hat{h}/\partial\hat{t} > 0$ (see figure 1.5). Here, the height is governed by the porous-medium equation [1],

$$\phi\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0, \qquad \hat{Q} = -\frac{\Delta\rho g K}{\mu}\hat{h}\frac{\partial \hat{h}}{\partial \hat{x}}. \tag{1.67}$$

where K and ϕ are respectively the permeability and porosity of the porous medium.
(i) Region 2: A region in which the fluid is draining, so $\partial\hat{h}/\partial\hat{t} < 0$, leaving behind a region partially occupied by fluid. We suppose that the fraction of space occupied by fluid in the drained region is a constant, $0 \leq s \leq 1$ (the light blue region in figure 1.5). In this region the equation governing the height is [1]

$$(1-s)\phi\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0, \qquad \hat{Q} = -\frac{\Delta\rho g K}{\mu}\hat{h}\frac{\partial \hat{h}}{\partial \hat{x}}. \tag{1.68}$$

We denote the position of the moving front by $\hat{x}_f(\hat{t})$ and the position of the *joint* that separates the two regions by $\hat{x}_s(\hat{t})$. Thus, Regions 1 and 2 can also be respectively described by $\hat{x}_s \leq \hat{x} \leq \hat{x}_f$ and $0 \leq \hat{x} \leq \hat{x}_s$ (see figure 1.5).

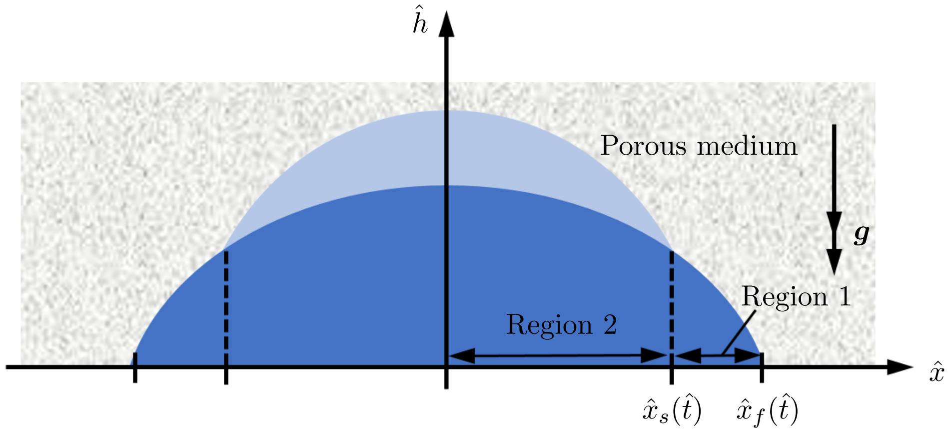

Figure 1.5: Schematic of a groundwater mound comprising a porous medium (shaded grey). Region 1 denotes the $\hat{x}$ domain where the liquid (blue) is invading the porous medium. Here there is a region of fully saturated medium (dark blue) surrounded by dry porous medium (grey). Region 2 denotes the $\hat{x}$ domain where the liquid is draining from the porous medium. This leaves behind a residue (light blue) while the dark blue region corresponds to the porous medium that is still fully saturated.

We have two second-order partial differential equations with two additional unknowns: $\hat{x}_s$ and $\hat{x}_f$. We therefore require initial conditions for the two regions and six boundary conditions (four for the equations and two for the two unknowns). We apply the following boundary conditions:

$$\hat{h}(\hat{x}_f, \hat{t}) = 0, \tag{1.69}$$

$$\frac{\partial \hat{h}}{\partial \hat{t}}(\hat{x}_s, \hat{t}) = 0, \tag{1.70}$$

which respectively define the front and the joint location and so determine $\hat{x}_s$ and $\hat{x}_f$. We then apply

$$\hat{h}\frac{\partial \hat{h}}{\partial \hat{x}} = 0 \qquad \text{at} \quad \hat{x} = 0, \tag{1.71}$$

$$\hat{h}\frac{\partial \hat{h}}{\partial \hat{x}} = 0 \qquad \text{at} \quad \hat{x} = \hat{x}_f, \tag{1.72}$$

which represent, respectively, no flux at the centre and at the front. We close the problem by enforcing continuity of height $\hat{h}$ and flux $\hat{h}\partial\hat{h}/\partial\hat{x}$ at the joint.

As for the case of a spreading droplet considered in Section 1.4, on first glance, condition (1.72) appears to give no additional information than (1.69). However, it does indicate the behaviour of the solution near the contact line, which is sufficient to close the problem. As in Section 1.4, we can also consider a local analysis near the contact line to transform (1.72) into a condition that allows us to obtain a numerical solution more easily. We achieve this by making a change of variables $(\hat{x}, \hat{t}) \to (\hat{\xi}, \hat{\tau})$ defined by

$$\hat{x} = \hat{x}_f(\hat{\tau}) + \epsilon\hat{\xi}, \qquad \hat{t} = \hat{\tau}, \tag{1.73}$$

and scale the height in this region local to the front,

$$\hat{h}(\hat{x},\hat{t}) = \epsilon \hat{H}(\hat{\xi},\hat{\tau}). \tag{1.74}$$

Substituting these scalings into (1.67) and considering the resulting equation at leading order, we find that we can integrate the result and apply (1.69) to give the local behaviour

$$\frac{\partial \hat{H}}{\partial \hat{\xi}} = -\frac{\mu\phi}{\Delta\rho g K}\frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{\tau}}, \tag{1.75}$$

which in terms of the original variables yields

$$\frac{\partial \hat{h}}{\partial \hat{x}}(\hat{x}_f,\hat{t}) = -\frac{\mu\phi}{\Delta\rho g K}\frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{t}}. \tag{1.76}$$

This condition replaces (1.72).

We employ the following non-dimensionalization:

$$\hat{x} = \hat{x}_0 x, \qquad \hat{x}_s = \hat{x}_0 x_s, \qquad \hat{x}_f = \hat{x}_0 x_f, \qquad \hat{t} = \hat{t}_0 t, \qquad \hat{h} = \hat{h}_0 h, \tag{1.77}$$

with $\hat{t}_0 = \mu\phi\hat{x}_0^2/\Delta\rho g K\hat{h}_0$. The resulting dimensionless system is then

$$\frac{\partial h}{\partial t} - \kappa\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) = 0, \qquad 0 \le x \le x_s(t), \tag{1.78}$$

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) = 0, \qquad x_s(t) \le x \le x_f(t), \tag{1.79}$$

where $\kappa = 1/(1-s)$, subject to the boundary conditions

$$h(x_f,t) = 0, \tag{1.80}$$

$$\frac{\partial h}{\partial t}(x_s,t) = 0, \tag{1.81}$$

$$\frac{\partial h}{\partial x}(0,t) = 0, \tag{1.82}$$

$$\frac{\partial h}{\partial x}(x_f,t) = -\frac{\mathrm{d}x_f}{\mathrm{d}t} \tag{1.83}$$

plus continuity in h and $h\partial h/\partial x$ at $x = x_s$.

As the liquid spreads and leaves a residue this means that we cannot apply a simple global conservation law as we did in the previous cases. So, we try a similarity solution of the form

$$h = t^\beta f(\eta) \qquad \text{where} \quad \eta = \frac{x}{t^\alpha}. \tag{1.84}$$

Substituting into (1.78) and (1.79) we find that the system can be written in similarity form if $\beta = 2\alpha - 1$:

$$\kappa(ff')' + \alpha\eta f' - (2\alpha-1)f = 0, \qquad 0 \le \eta \le \eta_s, \tag{1.85}$$

$$(ff')' + \alpha\eta f' - (2\alpha-1)f = 0, \qquad \eta_s \le \eta \le \eta_f, \tag{1.86}$$

where $\eta_f = x_f/t^\alpha$ and $\eta_s = x_s/t^\alpha$ are defined by

$$f(\eta_f) = 0, \tag{1.87}$$

$$(2\alpha - 1)f(\eta_s) - \alpha\eta_s f'(\eta_s) = 0, \tag{1.88}$$

which come from (1.80) and (1.81) respectively, while (1.82) and (1.83) become

$$f'(0) = 0, \tag{1.89}$$

$$f'(\eta_f) = -\alpha\eta_f, \tag{1.90}$$

as well as continuity in f and f' at η_s. At this point, unlike in all of our previous examples, we have no information to determine the value of α. We thus proceed further, keeping α arbitrary for the moment.

We make a change of variable by defining $z = \eta/\eta_f$ and $k(z) = f(\eta_f z)/\eta_f^2$. In terms of this new variable and function, the equations and boundary conditions are

$$\kappa\left(kk'\right)' + \alpha z k' - (2\alpha - 1)k = 0, \qquad 0 \le z \le z_s, \tag{1.91}$$

$$\left(kk'\right)' + \alpha z k' - (2\alpha - 1)k = 0, \qquad z_s \le z \le 1, \tag{1.92}$$

subject to

$$k(1) = 0, \tag{1.93}$$

$$(2\alpha - 1)k(z_s) - \alpha z k'(z_s) = 0, \tag{1.94}$$

$$k'(0) = 0, \tag{1.95}$$

$$k'(1) = -\alpha, \tag{1.96}$$

and continuity in k and k' at z_s, where $z_s = \eta_s/\eta_f$. Within this system we have eliminated the unknown front location. We now have one additional boundary condition and so this forms an eigenvalue problem: for a given κ there is a value of α for which there exists a solution. Thus, for this problem the natural similarity variables do not emerge from a scaling argument and instead these emerge as part of the solution. Such problems are called *similarity solutions of the second kind*.

Importantly, in this case the similarity solution only provides the shape of the interface h up to a constant, since k is scaled with the unknown η_f. To determine the appropriate scaling we must use conservation of mass. However, as mentioned earlier, there is no integral conservation law in this case since some of the liquid is retained in the pores. Thus, to determine the scaling coefficient we must compare our similarity solution with the full numerical solution at one point in time. As a result, the similarity solution in this case provides only part of the solution, namely the shape and there is one effective fitting parameter determined from the full numerical simulation.

1.6 Summary

We have seen in this chapter how a scaling law can provide a significant amount of information about the solution of a partial differential equation system. A similarity solution of the first kind provides the full behaviour for long time but cannot capture the early-time behaviour (unless the actual solution happens by chance to have the same early-time behaviour as the similarity solution). We obtain a similarity solution of the second kind when we cannot fully determine the form of the similarity solution from the governing equation and boundary condition. The functional dependence in this case emerges from the solution to the ordinary differential equation, which forms an eigenvalue problem. The similarity solution of the second kind provides the shape of the solution up to a scaling constant.

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