# Dynamics: Problem Sheet 8 (of 8)

1. (a) Consider a rigid body that is rotating about a general point $O$ that is fixed both in the body *and* fixed in an inertial frame. Starting from the point particle model of a rigid body, show that its kinetic energy is

$$T \equiv \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 = \frac{1}{2} \sum_{i,j=1}^{3} \mathcal{I}_{ij}^{(O)} \omega_i \omega_j ,$$

where $\mathcal{I}^{(O)}$ is the inertia tensor of the body about $O$, and $\boldsymbol{\omega}$ is its angular velocity. [*Hint*: You might find the vector identity $|\mathbf{a} \wedge \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ helpful.]

(b) Using the result in part (a), hence show that the kinetic energy of the heavy pendulum, considered at the end of section 8.3 of the lecture notes, is

$$T = \frac{1}{6} M l^2 \dot{\theta}^2 .$$

Here recall that $\theta$ is the angle the pendulum makes with the downward vertical, and the pendulum has length $l$ and mass $M$.

(c) Given that the potential energy is $V = MgZ_G$, where $Z_G$ is the height of the centre of mass of the pendulum, hence write down the total energy $E = T + V$. Show that conservation of $E$ is implied by the equation of motion (8.47) derived in the lecture notes.

2. A smooth straight wire rotates with constant angular speed $\omega$ about the vertical axis through a fixed point $O$ on the wire, and the angle between the wire and the upward vertical is constant and equal to $\alpha$, where $0 < \alpha < \pi/2$. A bead of mass $m$ is free to slide on the wire.

(a) Starting from the general form of Newton's second law in a rotating frame, show that

$$m \left( \frac{\mathrm{d}^2 \mathbf{r}}{\mathrm{d}t^2} \right)_{\mathcal{S}} = \mathbf{N} - mg\,\mathbf{k} - 2m\omega\,\mathbf{k} \wedge \left( \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right)_{\mathcal{S}} - m\omega^2\,\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{r}) ,$$

where $\mathbf{r}(t)$ is the position of the bead in the frame $\mathcal{S}$ that rotates with the wire, $\mathbf{k}$ is a unit vector pointing vertically, and $\mathbf{N}$ is the normal reaction of the wire.

(b) Hence show that $z(t)$, the height of the bead above $O$, satisfies the equation

$$\ddot{z} - (\omega^2 \sin^2 \alpha) z = -g \cos^2 \alpha .$$

Show that an equilibrium point for the bead exists, and determine its stability.

3. A bead $P$ of mass $m$ slides on a smooth circular wire of radius $a$ and centre $C$. The wire lies in a horizontal plane and is forced to rotate at a constant angular speed $\omega$ about the vertical axis through a fixed point $O$ in the plane of the wire. The distance from $O$ to $C$ is constant and equal to $b$.

Let $\theta$ be the angle that the line joining the centre $C$ to the bead makes with the diameter through $O$. (See the figure on the next page.)

(a) Show that the position of the bead may be written as

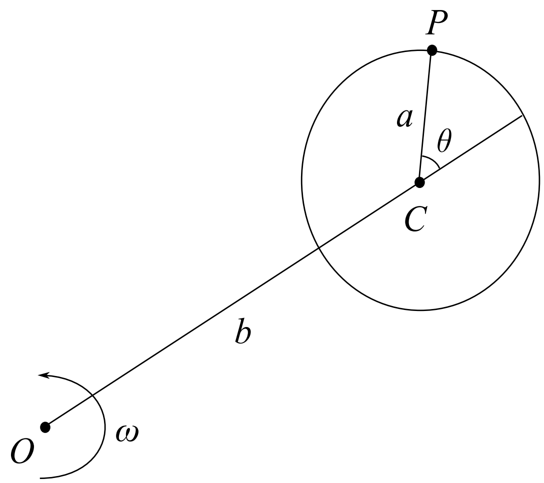
$$\mathbf{r} = (b + a \cos\theta)\,\mathbf{e}_1 + a \sin\theta\,\mathbf{e}_2 ,$$

where $\mathbf{e}_i$, $i = 1, 2, 3$, are an orthonormal basis for a frame that rotates with the wire.

(b) Using Newton's second law in this rotating frame, hence show that

$$\ddot{\theta} + \frac{b}{a}\omega^2 \sin\theta \quad = \quad 0 \ .$$

(c) Show that the bead can remain in equilibrium relative to the wire at two points. Decide whether these positions of equilibrium are stable or unstable.




---

4. *Optional*

Consider (again) dropping a particle from the top of a tower, of height $h$ above the ground, at time $t = 0$. Use a reference frame $\mathcal{S}$ fixed to the surface of the Earth with $\mathbf{e}_1$ is a unit vector pointing North, $\mathbf{e}_2$ is a unit vector pointing West, and $\mathbf{e}_3$ is a unit vector pointing upwards. The origin of this frame is at a latitude $\theta$ on the surface of the Earth and hence the angular velocity of the Earth is given by $\boldsymbol{\omega} = \omega\cos\theta\,\mathbf{e}_1 + \omega\sin\theta\,\mathbf{e}_3$, where $\omega = 2\pi/[1 \text{ day}]$.

(a) Including only the effects of a uniform gravitational field and the Coriolis force, show that Newton's second law in the frame $\mathcal{S}$ implies

$$\ddot{\mathbf{r}} \quad \simeq \quad -g\,\mathbf{e}_3 + 2gt\,\boldsymbol{\omega}\wedge\mathbf{e}_3 \ ,$$

where we have assumed that terms of order $\omega^2$ are negligible.

(b) Hence show that the trajectory of the particle is given by

$$\mathbf{r} \quad \simeq \quad \left(h - \frac{1}{2}gt^2\right)\mathbf{e}_3 - \frac{1}{3}\omega g t^3\,\cos\theta\,\mathbf{e}_2 \ ,$$

and that it lands a distance $d$ to the East of the tower, where

$$d \quad = \quad \frac{1}{3}\omega\sqrt{\frac{8h^3}{g}}\,\cos\theta \ .$$

Please send comments and corrections to `gaffney@maths.ox.ac.uk`.