B4.4 Fourier Analysis HT22

Lecture 6: The Fourier inversion formula on tempered distributions

1. The adjoint identity scheme in the tempered context
2. Multiplication with polynomials and convolution with Schwartz test functions
3. The Fourier transform on $\mathscr{S}'$
4. The Fourier inversion formula on $\mathscr{S}'$
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The material corresponds to pp. 25–27 in the lecture notes and should be covered in Week 3.

The adjoint identity scheme in the tempered context

We discussed this in Lecture 5 and recall that the procedure is as in B4.3. The only difference is that $\mathscr{D}(\Omega)$ is replaced by $\mathscr{S}(\mathbb{R}^n)$ (and correspondingly, we then require $\mathscr{S}$ continuity instead of $\mathscr{D}$ continuity).

Set-up: Given an *operation* T on $\mathscr{S}(\mathbb{R}^n)$, assumed to be a linear map

$$T\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n),$$

that we would like to extend to tempered distributions.

Assume $S\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a linear and $\mathscr{S}$ continuous map, and that we have the adjoint identity:

$$\int_{\mathbb{R}^n} T(\phi)\psi \, \mathrm{d}x = \int_{\mathbb{R}^n} \phi S(\psi) \, \mathrm{d}x$$

holds for all ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$.

The adjoint identity scheme in the tempered context

We can then define $\overline{T}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ for each $u \in \mathscr{S}'(\mathbb{R}^n)$ by the rule

$$\langle \overline{T}(u), \phi \rangle := \langle u, S(\phi) \rangle,\ \phi \in \mathscr{S}(\mathbb{R}^n).$$

Hereby $\overline{T}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is well-defined, linear and $\mathscr{S}'$ continuous

The adjoint identity ensures that the extension is consistent, $\overline{T}|_{\mathscr{S}(\mathbb{R}^n)} = T$ and so as in $\mathscr{D}$ context we shall in the sequel write T also for its extension $\overline{T}$.

We used it to define the Fourier transform of a tempered distribution, and also extended the notions of differentiation, translation, dilation and composition with orthogonal maps to tempered distributions.

Multiplication with polynomials

Let $p(x) \in \mathbb{C}[x]$. Clearly the operation $\phi \mapsto p(x)\phi$ $(= \phi p(x))$ defines a linear and $\mathscr{S}$ continuous map of $\mathscr{S}(\mathbb{R}^n)$ to itself. Furthermore we have the trivial adjoint identity:

$$\int_{\mathbb{R}^n} \big(p(x)\phi(x)\big)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \phi(x)\big(p(x)\psi(x)\big)\,\mathrm{d}x$$

for all ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$. Hence by the adjoint identity scheme we extend this operation to tempered distributions consistently if for $u \in \mathscr{S}'(\mathbb{R}^n)$ we define $p(x)u$ $(= up(x))$ by the rule

$$\langle p(x)u, \phi \rangle := \langle u, p(x)\phi \rangle \text{ for } \phi \in \mathscr{S}(\mathbb{R}^n)$$

Note that the operation is consistent beyond $\mathscr{S}$: when u is a tempered L^1 function, then also $p(x)u$ is a tempered L^1 function and we have $T_{p(x)u} = p(x)T_u$.

Note: In the tempered context we cannot multiply with general C^∞ functions. Polynomials is of course a very narrow class and we extend the definition to a larger class of C^∞ functions in a later lecture.

Convolution with Schwartz test functions

Let $\theta \in \mathscr{S}(\mathbb{R}^n)$. Then we have by use of Fubini's theorem that

$$\int_{\mathbb{R}^n} (\theta * \phi)\psi \, \mathrm{d}x = \int_{\mathbb{R}^n} \phi(\widetilde{\theta} * \psi) \, \mathrm{d}x$$

holds for all ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$. Furthermore, the maps

$$\phi \mapsto \theta * \phi \ \text{ and } \ \psi \mapsto \widetilde{\theta} * \psi$$

are linear and $\mathscr{S}$ continuous maps of $\mathscr{S}(\mathbb{R}^n)$ to itself. We can therefore apply the adjoint identity scheme to define $\theta * u$ for $u \in \mathscr{S}'(\mathbb{R}^n)$ by the rule

$$\left\langle \theta * u, \phi \right\rangle := \left\langle u, \widetilde{\theta} * \phi \right\rangle, \quad \phi \in \mathscr{S}(\mathbb{R}^n)$$

Because $\theta * \phi = \phi * \theta$ the adjoint identity scheme definition of $u * \theta$ will result in the same distribution: $\theta * u = u * \theta$.

We have consistency beyond $\mathscr{S}$: if $u \in \mathscr{E}'(\mathbb{R}^n)$ we defined in B4.3 the convolution $u * \psi$ $(= \psi * u)$ for any $\psi \in \mathrm{C}^\infty(\mathbb{R}^n)$ as $\langle u * \psi, \phi \rangle := \langle u, \widetilde{\psi} * \phi \rangle$ for $\phi \in \mathscr{D}(\mathbb{R}^n)$. If $\psi \in \mathscr{S}(\mathbb{R}^n)$, then as $\widetilde{\psi} * \phi \in \mathscr{S}(\mathbb{R}^n)$ we have consisteny with our definition on $\mathscr{S}'$.

The Fourier transform on $\mathscr{S}'$

We defined it using the adjoint identity scheme: for $u \in \mathscr{S}'(\mathbb{R}^n)$ its Fourier transform $\mathcal{F}u = \widehat{u}$ is

$$\langle \widehat{u}, \phi \rangle := \langle u, \widehat{\phi} \rangle, \quad \phi \in \mathscr{S}(\mathbb{R}^n).$$

Hereby $\mathcal{F} \colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is a linear and $\mathscr{S}'$ continuous map. It is not difficult to see that many of the rules for the Fourier transform on $\mathscr{S}$ remain valid in the wider context of $\mathscr{S}'$:

Proposition For $u \in \mathscr{S}'(\mathbb{R}^n)$ we have that

- Differentiation rules: $\widehat{\partial_j u} = \mathrm{i}\xi \widehat{u}$ and $\widehat{x_j u} = \mathrm{i}\partial_j \widehat{u} \quad (1 \leq j \leq n)$
- invariance under orthogonal maps, dilation and translation rules

all remain valid.

The proofs are easy and follow from our definition of the Fourier transform on $\mathscr{S}'$. It simply allows us to deduce properties of the Fourier transform on $\mathscr{S}'$ from the corresponding properties of the Fourier transform on $\mathscr{S}$.

The Fourier transform on $\mathscr{S}'$

If $f \colon \mathbb{R}^n \to \mathbb{C}$ is a tempered L^p function for some $p > 1$, then we have defined its Fourier transform as a tempered distribution. In this case we can also note that if we define $f_j := f\mathbf{1}_{B_j(0)}$, then $f_j \in \mathrm{L}^1(\mathbb{R}^n)$ and $f_j \to f$ in $\mathscr{S}'(\mathbb{R}^n)$ as $j \to \infty$, hence

$$\widehat{f} = \lim_{j\to\infty} \widehat{f_j}(\xi) = \lim_{j\to\infty} \int_{B_j(0)} f(x)\mathrm{e}^{-\mathrm{i}\xi\cdot x}\,\mathrm{d}x \ \text{ in } \ \mathscr{S}'(\mathbb{R}^n)$$

as $j \to \infty$ because the Fourier transform is $\mathscr{S}'$ continuous. While this observation makes the connection to more familiar ground, it is seldom useful for actually finding the Fourier transform of f. At the end of the lecture we give two examples of possible ways to calculate Fourier transforms in the $\mathscr{S}'$ context.

The Fourier inversion formula in $\mathscr{S}'$

Theorem The Fourier transform $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is a linear, $\mathscr{S}'$ continuous and bijective map with inverse

$$\mathcal{F}^{-1} = (2\pi)^{-n}\widetilde{\mathcal{F}}.$$

Remark The operations $\mathcal{F}$ and $\widetilde{(\cdot)}$ commute, so

$$\widetilde{\mathcal{F}}(u) = \widetilde{\mathcal{F}(u)} = \mathcal{F}\big(\widetilde{u}\big).$$

Proof. The proof is an illustration of how powerful good definitions can be, as we shall see that they allow us to obtain the result from that on $\mathscr{S}$!
It clearly suffices to show that

$$\left((2\pi)^{-n}\widetilde{\mathcal{F}}\right) \circ \mathcal{F} = \mathcal{F} \circ \left((2\pi)^{-n}\widetilde{\mathcal{F}}\right) = \mathrm{I},$$

the identity map on $\mathscr{S}'(\mathbb{R}^n)$.

The Fourier inversion formula in $\mathscr{S}'$

Let $u \in \mathscr{S}'(\mathbb{R}^n)$. Then for $\phi \in \mathscr{S}(\mathbb{R}^n)$ we calculate (using abbreviation FIF to mean *Fourier inversion formula in* $\mathscr{S}$):

$$\begin{aligned}
\left\langle \left[\left((2\pi)^{-n} \widetilde{\mathcal{F}} \right) \circ \mathcal{F} \right](u), \phi \right\rangle &= \left\langle (2\pi)^{-n} \widetilde{\mathcal{F}}(\mathcal{F}(u)), \phi \right\rangle \\
&\overset{defs}{=} \left\langle u, \mathcal{F}\left((2\pi)^{-n} \widetilde{\mathcal{F}}(\phi) \right) \right\rangle \\
&\overset{FIF}{=} \langle u, \phi \rangle \\
&\overset{FIF}{=} \left\langle u, \left((2\pi)^{-n} \widetilde{\mathcal{F}} \right) (\mathcal{F}(\phi)) \right\rangle \\
&\overset{defs}{=} \left\langle \mathcal{F}\left((2\pi)^{-n} \widetilde{\mathcal{F}}(u) \right), \phi \right\rangle \\
&= \left\langle \left[\mathcal{F} \circ \left((2\pi)^{-n} \widetilde{\mathcal{F}} \right) \right](u), \phi \right\rangle .
\end{aligned}$$

So defs and FIF do the job! $\square$

The Fourier-Gel'fand formula

The Dirac delta function at 0 on $\mathbb{R}^n$ can be expressed as

$$\delta_0 = \lim_{r \to \infty} (2\pi)^{-n} \int_{B_r(0)} \mathrm{e}^{-\mathrm{i}\xi \cdot x} \, \mathrm{d}x$$

with convergence in $\mathscr{S}'(\mathbb{R}^n)$.

This is an easy consequence of our earlier observation that $\widehat{\delta_0} = \mathbf{1}_{\mathbb{R}^n}$ (the constant function) and the Fourier inversion formula. Accordingly

$$\widehat{\mathbf{1}_{\mathbb{R}^n}} = \widehat{\widehat{\delta_0}} = (2\pi)^n \widetilde{\delta_0} = (2\pi)^n \delta_0,$$

where the last equality is because δ_0 is an *even* distribution (so one with $\widetilde{u} = u$). Now it is easy to check that

$$\mathbf{1}_{B_r(0)} \to \mathbf{1}_{\mathbb{R}^n} \text{ in } \mathscr{S}'(\mathbb{R}^n) \text{ as } r \to \infty$$

and therefore the conclusion by $\mathscr{S}'$ continuity of the Fourier transform.

The principle about smoothness versus decay at infinity revisited

Proposition Let $u \in \mathscr{S}'(\mathbb{R}^n)$ and assume that $\widehat{u} \in \mathrm{L}^1(\mathbb{R}^n)$ and that for some $m \in \mathbb{N}$ also $|\xi|^m \widehat{u}(\xi) \in \mathrm{L}^1(\mathbb{R}^n)$. Then $u \in \mathrm{C}^m(\mathbb{R}^n)$ and for each multi-index $\alpha \in \mathbb{N}_0^n$ of length at most m, $\partial^\alpha u \in \mathrm{C}_0(\mathbb{R}^n)$.
Note: The conclusion is as usual that the distribution u can be represented by a function with the listed properties.

Proof. Fix a multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, and note that

$$\left|\xi^\alpha\right| \leq 1 + |\xi|^m$$

holds for all $\xi \in \mathbb{R}^n$. Therefore we have that $\xi^\alpha \widehat{u}(\xi)$ is integrable, so by the differentiation rule, the Fourier inversion formula in $\mathscr{S}'$ and the Riemann-Lebesgue lemma we get

$$\partial^\alpha u(x) = (2\pi)^{-n} \mathcal{F}_{\xi \to -x}\left(\left(-\mathrm{i}\xi\right)^\alpha \widehat{u}(\xi)\right) \in \mathrm{C}_0(\mathbb{R}^n).$$

This concludes the proof. $\square$

Example 1 *Find the Fourier transform of* $\mathrm{pv}\left(\frac{1}{x}\right)$*.*

Recall from B4.3 that $\frac{\mathrm{d}}{\mathrm{d}x}\log|x| = \mathrm{pv}\left(\frac{1}{x}\right)$ holds in $\mathscr{D}'(\mathbb{R})$. Since $\log|x|/\left(1+|x|^2\right) \in \mathrm{L}^1(\mathbb{R})$ the function $\log|x|$ is a tempered L^1 function and so defines a tempered distribution. But then also its distributional derivative is tempered:

$$\mathrm{pv}\left(\tfrac{1}{x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\log|x| \in \mathscr{S}'(\mathbb{R}).$$

The Fourier transform is therefore defined as a tempered distribution. To find it we recall from B4.3 that also $x\mathrm{pv}\left(\frac{1}{x}\right) = 1$ in $\mathscr{D}'(\mathbb{R})$. By $\mathscr{S}$ density of $\mathscr{D}(\mathbb{R})$ in $\mathscr{S}(\mathbb{R})$ and since tempered distributions are $\mathscr{S}$ continuous we extend the identity to $\mathscr{S}(\mathbb{R})$. But then we can Fourier transform it using the differentiation rule and Fourier-Gel'fand:

$$2\pi\delta_0 = \widehat{\mathbf{1}_{\mathbb{R}}} = \widehat{x\mathrm{pv}\left(\tfrac{1}{x}\right)} = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{\mathrm{pv}\left(\tfrac{1}{x}\right)},$$

hence $-2\pi\mathrm{i}\delta_0 = \frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{\mathrm{pv}\left(\frac{1}{x}\right)}$, so

$$\widehat{\mathrm{pv}\left(\tfrac{1}{x}\right)} = -2\pi\mathrm{i}H + c \text{ for some constant } c \in \mathbb{C}.$$

Example 1 continued...

Because the Fourier transform is invariant under orthogonal maps, the Fourier transform in particular preserves parity: since $\mathrm{pv}\left(\frac{1}{x}\right)$ is odd, so is its Fourier transform. Thus

$$
\begin{aligned}
0 = \widehat{\mathrm{pv}\left(\tfrac{1}{x}\right)} + \widetilde{\widehat{\mathrm{pv}\left(\tfrac{1}{x}\right)}} &= -2\pi\mathrm{i}H + c - 2\pi\mathrm{i}\widetilde{H} + c \\
&= -2\pi\mathrm{i} + 2c,
\end{aligned}
$$

and so $c = \pi\mathrm{i}$. Therefore

$$\widehat{\mathrm{pv}\left(\tfrac{1}{x}\right)} = \pi\mathrm{i} - 2\pi\mathrm{i}H = -\pi\mathrm{i}\,\mathrm{sgn}(\xi),$$

where $\mathrm{sgn}(\xi) := \xi/|\xi|$ is the *signum function*.

Example 2 *Find the Fourier transform of Heaviside's function H.*

We clearly have $H \in \mathscr{S}'(\mathbb{R})$ and $H' = \delta_0$ in $\mathscr{S}'(\mathbb{R})$. By the differentiation rule $1 = \widehat{H'} = \mathrm{i}\,\xi \widehat{H}$, so $-\mathrm{i} = \xi \widehat{H}$. Extending a result from B4.3 this implies that

$$\widehat{H} = -\mathrm{i}\,\mathrm{pv}\left(\tfrac{1}{\xi}\right) + c\delta_0$$

holds for some constant $c \in \mathbb{C}$. Now $1 = H + \widetilde{H}$, so

$$2\pi\delta_0 = \widehat{H} + \widehat{\widetilde{H}} = 2c\delta_0,$$

whereby $c = \pi$. Consequently

$$\widehat{H} = -\mathrm{i}\,\mathrm{pv}\left(\tfrac{1}{\xi}\right) + \pi\delta_0.$$

Note that $H \in \mathrm{L}^{\infty}(\mathbb{R})$ and $\widehat{H}$ is a distribution of order 1.

Example 3 *Let $f \in \mathrm{L}^\infty(\mathbb{R})$. Show that the order of $\widehat{f}$ is at most 2.*

For $\phi \in \mathscr{S}(\mathbb{R})$ we calculate using the bound from the inclusion $\mathscr{S} \subset \mathrm{L}^1$ and the Fourier bounds:

$$\begin{aligned} \langle \widehat{f}, \phi \rangle &= \langle f, \widehat{\phi} \rangle = \int_{-\infty}^{\infty} f(x)\widehat{\phi}(x)\,\mathrm{d}x \leq \|f\|_\infty \|\widehat{\phi}\|_1 \\ &\leq c(1,1)\|f\|_\infty \overline{S}_{2,0}\big(\widehat{\phi}\big) \leq c(1,1)c\|f\|_\infty \overline{S}_{2,2}\big(\phi\big). \end{aligned}$$

This implies that the order of $\widehat{f}$ is at most 2.

Exercise Show that if f is a tempered L^∞ function on $\mathbb{R}$ satisfying for an $m \in \mathbb{N}$,

$$\mathrm{ess.sup} \frac{|f(x)|}{\big(1+|x|^2\big)^{\frac{m}{2}}} < \infty,$$

then $\widehat{f}$ is of order at most $m+2$.