B4.4 Fourier Analysis HT22

Lecture 11: The Paley-Wiener theorem for compactly supported distributions

1. The Fourier transform of a compactly supported distribution
2. The Fourier-Laplace transform
3. A qualitative uncertainty principle
4. Nonexistence of compactly supported fundamental solutions
5. The Paley-Wiener theorem for distributions
6. An application

The material corresponds to pp. 41–45 in the lecture notes and should be covered in Week 6.

The Fourier transform of a compactly supported distribution

For $u \in \mathscr{E}'(\mathbb{R}^n)$ its Fourier transform is defined as the distribution

$$\left\langle \widehat{u}, \phi \right\rangle := \left\langle u, \widehat{\phi} \right\rangle, \quad \phi \in \mathscr{S}(\mathbb{R}^n).$$

In lecture 9 we saw that $\widehat{u}$ is a moderate C^∞ function given by the formula

$$\widehat{u}(\xi) = \left\langle u, \mathrm{e}^{-\mathrm{i}\xi\cdot(\cdot)} \right\rangle, \quad \xi \in \mathbb{R}^n.$$

Here the right-hand side means that u acts on the C^∞ function $x \mapsto \mathrm{e}^{-\mathrm{i}\xi\cdot x}$. Recall from B4.3 that each compactly supported distribution u admits a unique extension to a linear functional defined on C^∞ functions and satisfying the boundedness property: for each compact neighbourhood K of $\mathrm{supp}(u)$ there exist $c = c_K \geq 0$, $m = m_K \in \mathbb{N}_0$ such that

$$\left|\langle u, \phi \rangle\right| \leq c \sum_{|\alpha| \leq m} \sup_K \left|\partial^\alpha \phi\right| \tag{1}$$

holds for all $\phi \in C^\infty(\mathbb{R}^n)$.

The Fourier transform of a compactly supported distribution

Evidently the function $x \mapsto \mathrm{e}^{-\mathrm{i}\zeta\cdot x}$ remains a C^∞ function for $\zeta \in \mathbb{C}^n$ and we may still consider

$$\widehat{u}(\zeta) = \langle u, \mathrm{e}^{-\mathrm{i}\zeta\cdot(\cdot)}\rangle$$

for such $\zeta \in \mathbb{C}^n$. This extension of the Fourier transform, denoted again by $\widehat{u}$, is also here called the *Fourier-Laplace transform* of u. We will show that it is an entire function. Write $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$ and consider the Fourier-Laplace transform of u as a function of the $2n$ real variables (ξ, η), $\widehat{u} = \widehat{u}(\xi, \eta) \colon \mathbb{R}^{2n} \to \mathbb{C}$.

Claim. $\widehat{u}(\xi, \eta)$ is a C^1 function of $(\xi, \eta) \in \mathbb{R}^{2n}$ and we may calculate its partial derivatives by differentiation behind the distribution sign.

To show it we consider $\widehat{u}$ as a function of one the variables, say ξ_j, while keeping the other variables fixed. For an increment $h \in \mathbb{R} \setminus \{0\}$ we have

$$\frac{\widehat{u}(\zeta + he_j) - \widehat{u}(\zeta)}{h} = \left\langle u, \frac{\mathrm{e}^{-\mathrm{i}(\zeta + he_j)\cdot(\cdot)} - \mathrm{e}^{-\mathrm{i}\zeta\cdot(\cdot)}}{h} \right\rangle$$

The Fourier transform of a compactly supported distribution

Here

$$\Delta_h(x) := \frac{\mathrm{e}^{-\mathrm{i}(\zeta + he_j)\cdot x} - \mathrm{e}^{-\mathrm{i}\zeta\cdot x}}{h} \to -\mathrm{i}x_j\mathrm{e}^{-\mathrm{i}\zeta\cdot x} \text{ as } h \to 0$$

locally uniformly in $x \in \mathbb{R}^n$. Likewise, we have for any multi-index $\alpha \in \mathbb{N}_0^n$ that

$$\partial_x^\alpha \Delta_h(x) \to -\mathrm{i}x_j\mathrm{e}^{-\mathrm{i}\zeta\cdot x}\big(-\mathrm{i}\zeta\big)^\alpha \text{ as } h \to 0$$

locally uniformly in $x \in \mathbb{R}^n$. In combination with the boundedness property (1) of u this yields

$$\frac{\widehat{u}(\zeta + he_j) - \widehat{u}(\zeta)}{h} \to \left\langle u, \frac{\partial}{\partial \xi_j}\mathrm{e}^{-\mathrm{i}\zeta\cdot(\cdot)}\right\rangle \text{ as } h \to 0.$$

Using the boundedness property (1) again we see that the partial derivative $\partial_{\xi_j}\widehat{u}(\zeta)$ is a continuous function of ζ. The same argument applies to the remaining $2n - 1$ real variables and so we have established the validity of claim.

The Fourier transform of a compactly supported distribution

We can now check that $\widehat{u}$ satisfies the Cauchy-Riemann equation with respect to each of the variables ζ_j, where $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$:

$$\frac{\partial}{\partial \overline{\zeta}_j} \widehat{u}(\zeta) = \left\langle u, \frac{\partial}{\partial \overline{\zeta}_j} \mathrm{e}^{-\mathrm{i}\zeta \cdot (\cdot)} \right\rangle = 0,$$

hence $\widehat{u}$ is a holomorphic function of ζ_j, and since $1 \leq j \leq n$ was arbitrary we have shown that $\widehat{u}$ is an entire function on $\mathbb{C}^n$. We have shown:

The Fourier transform of a compactly supported distribution $u \in \mathscr{E}'(\mathbb{R}^n)$ extends to $\mathbb{C}^n$ as an entire function called the Fourier-Laplace transform of u.

This result allows us to give a short proof of a qualitative *uncertainty principle*.

A qualitative uncertainty principle

Proposition If $u \in \mathscr{E}'(\mathbb{R}^n)$ and $\widehat{u} \in \mathscr{E}'(\mathbb{R}^n)$, then $u = 0$.

Proof. Because u has compact support, the Fourier-Laplace transform $\widehat{u}\colon \mathbb{C}^n \to \mathbb{C}$ is entire, and because the Fourier transform $\widehat{u}$ has compact support we can find $r > 0$ so $\mathrm{supp}(\widehat{u}) \subset [-r, r]^n$. Now fix $\xi_0 \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ and consider the entire function

$$\mathbb{C} \ni \zeta_n \mapsto \widehat{u}(\xi_0 + \zeta_n e_n)\,, \qquad \text{where } (e_j)_{j=1}^n \text{ is the standard basis for } \mathbb{C}^n.$$

This function vanishes when $\zeta_n \in \mathbb{R} \setminus [-r, r]$, and so by the identity theorem for holomorphic functions it must vanish identically:

$$\widehat{u}(\xi_0 + \zeta_n e_n) = 0$$

for all $\zeta_n \in \mathbb{C}$. Because $\xi_0 \in \mathbb{R}^{n-1} \times \{0\}$ was arbitrary we have shown that the Fourier transform $\widehat{u}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$. But then $u = 0$ by the Fourier inversion formula in $\mathscr{S}'$. □

Nonexistence of compactly supported fundamental solutions

Recall from B4.3 that a fundamental solution to a linear differential operator with constant coefficients

$$p(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$$

is any distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ satisfying $p(\partial)E = \delta_0$ in $\mathscr{D}'(\mathbb{R}^n)$.

A fundamental solution for a differential operator of order at least one is never compactly supported:

We proceed by contradiction and assume $E \in \mathscr{E}'(\mathbb{R}^n)$ and $p(\partial)E = \delta_0$. We can then Fourier transform the equation. Using the differentiation rule we get

$$p(\mathrm{i}\xi)\widehat{E} = 1 \text{ in } \mathscr{S}'(\mathbb{R}^n).$$

Nonexistence of compactly supported fundamental solutions

Here $\widehat{E}$ is a moderate C^∞ function on $\mathbb{R}^n$ that admits an extension to $\mathbb{C}^n$ as an entire function, so, by the identity theorem for holomorphic functions, we still have

$$p(\mathrm{i}\zeta)\widehat{E}(\zeta) = 1 \text{ on } \mathbb{C}^n.$$

Because $p(\mathrm{i}\zeta)$ is a polynomial of degree at least one, we can find $\zeta_0 \in \mathbb{C}^n$ and $1 \leq j \leq n$ such that $\mathbb{C} \ni \zeta_j \mapsto p(\mathrm{i}(\zeta_0 + \zeta_j e_j))$ is a polynomial of degree at least one. It therefore has a zero in $\mathbb{C}$ and because

$$p\big(\mathrm{i}(\zeta_0 + \zeta_j e_j)\big)\widehat{E}(\zeta_0 + \zeta_j e_j) = 1 \text{ for all } \zeta_j \in \mathbb{C}$$

it follows that the holomorphic function $\zeta_j \mapsto \widehat{E}(\zeta_0 + \zeta_j e_j)$ must have a singularity in $\mathbb{C}$. A contradiction proving the claim.

The Fourier transform of a compactly supported distribution

We return to the Fourier-Laplace transform $\widehat{u}$ of $u \in \mathscr{E}'(\mathbb{R}^n)$. As we did when considering the Fourier transform of a compactly supported test function we now want to derive a boundedness property for the Fourier-Laplace transform.

Take $R \geq 0$ so that u is supported in $\overline{B_R(0)}$. For the standard mollifier $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ on $\mathbb{R}^n$ we put

$$\chi = \chi_\varepsilon := \rho_\varepsilon * \mathbf{1}_{B_{R+\varepsilon}(0)}.$$

Then $\chi = 1$ on $B_R(0)$, it has support $\overline{B_{R+2\varepsilon}(0)}$ and we have

$$u = \chi u$$

Using the formula for the Fourier-Laplace transform we then get

$$\widehat{u}(\zeta) = \left\langle u, \chi \mathrm{e}^{-\mathrm{i}\zeta\cdot(\cdot)} \right\rangle$$

The Fourier transform of a compactly supported distribution

We estimate now the Fourier-Laplace transform using the boundedness property (1) corresponding to the compact set $K = \overline{B_{R+1}(0)}$:

$$\begin{aligned} \left|\widehat{u}(\zeta)\right| &= \left|\left\langle u, \chi \mathrm{e}^{-\mathrm{i}\zeta\cdot(\cdot)}\right\rangle\right| \\ &\leq c \sum_{|\alpha|\leq m} \sup_{B_{R+1}(0)} \left|\partial^\alpha\left(\chi \mathrm{e}^{-\mathrm{i}\zeta\cdot(\cdot)}\right)\right| \\ &\leq c \sum_{|\alpha|\leq m} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \sup_{x\in B_{R+1}(0)} \left|\mathrm{e}^{-\mathrm{i}\zeta\cdot x}\left(-\mathrm{i}\zeta\right)^{\alpha-\gamma}\partial^\gamma\chi(x)\right| \\ &\leq c \sum_{|\alpha|\leq m} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \sup_{x\in B_{R+1}(0)} \left|\partial^\gamma\chi(x)\right| \mathrm{e}^{(R+2\varepsilon)|\eta|}\left|\zeta^{\alpha-\gamma}\right| \end{aligned}$$

Here we have

$$\left|\partial^\gamma\chi(x)\right| = \varepsilon^{-|\gamma|}\left|\left((\partial^\gamma\rho)_\varepsilon * \mathbf{1}_{B_{R+\varepsilon}(0)}\right)(x)\right| \leq \varepsilon^{-|\gamma|}\|\partial^\gamma\rho\|_1$$

The Fourier transform of a compactly supported distribution

Therefore we get

$$\left|\widehat{u}(\zeta)\right| \leq c\mathrm{e}^{(R+2\varepsilon)|\eta|}\sum_{|\alpha|\leq m}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\varepsilon^{-|\gamma|}\|\partial^\gamma\rho\|_1|\zeta^{\alpha-\gamma}|$$

Put $C := \max_{|\gamma|\leq m}\|\partial^\gamma\rho\|_1$, whereby

$$\begin{aligned}\left|\widehat{u}(\zeta)\right| &\leq cC\mathrm{e}^{(R+2\varepsilon)|\eta|}\sum_{|\alpha|\leq m}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\varepsilon^{-|\gamma|}|\zeta^{\alpha-\gamma}| \\ &= cC\mathrm{e}^{(R+2\varepsilon)|\eta|}\sum_{|\alpha|\leq m}\prod_{j=1}^{n}\left(\varepsilon^{-1}+|\zeta_j|\right)^{\alpha_j} \\ &\leq cC\mathrm{e}^{(R+2\varepsilon)|\eta|}\sum_{|\alpha|\leq m}\left(\varepsilon^{-1}+(1+|\zeta|^2)^{\frac{1}{2}}\right)^{|\alpha|}\end{aligned}$$

Here we take corresponding to $\zeta\in\mathbb{C}^n$,

$$\varepsilon = \left(1+|\zeta|^2\right)^{-\frac{1}{2}}\in(0,1].$$

The Fourier transform of a compactly supported distribution

If

$$\varepsilon = \left(1+|\zeta|^2\right)^{-\frac{1}{2}} \in (0,1],$$

then

$$\mathrm{e}^{(R+2\varepsilon)|\eta|} \leq \mathrm{e}^{R|\eta|+2}$$

and for $|\alpha| \leq m$,

$$\left(\varepsilon^{-1} + (1+|\zeta|^2)^{\frac{1}{2}}\right)^{|\alpha|} \leq 2^m \left(1+|\zeta|^2\right)^{\frac{m}{2}}.$$

Hereby we arrive at

$$\left|\widehat{u}(\zeta)\right| \leq C_0 \left(1+|\zeta|^2\right)^{\frac{m}{2}} \mathrm{e}^{R|\eta|}$$

for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$, where $C_0 := cC\mathrm{e}^2 2^m \sum_{|\alpha|\leq m} 1$. This is the boundedness property for the Fourier-Laplace transform of a distribution supported in $\overline{B_R(0)}$.

The Paley-Wiener theorem for compactly supported distributions

(1) If $u \in \mathscr{E}'(\mathbb{R}^n)$ is of order $m \in \mathbb{N}_0$ and $\operatorname{supp}(u) \subseteq \overline{B_R(0)}$, then the Fourier-Laplace transform $\widehat{u}$ is an entire function on $\mathbb{C}^n$ given by

$$\widehat{u}(\zeta) = \left\langle u, \mathrm{e}^{-\mathrm{i}\zeta\cdot(\cdot)} \right\rangle, \, \zeta \in \mathbb{C}^n$$

and satisfying the boundedness condition

$$\left|\widehat{u}(\zeta)\right| \leq c\big(1 + |\zeta|^2\big)^{\frac{m}{2}} \mathrm{e}^{R|\eta|}$$

for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$, where $c \geq 0$ is a constant.

(2) If $\Phi \colon \mathbb{C}^n \to \mathbb{C}$ is an entire function satisfying for some constants $c \geq 0$, $m \in \mathbb{N}_0$ and $R \geq 0$ the boundedness condition

$$\left|\Phi(\zeta)\right| \leq c\big(1 + |\zeta|^2\big)^{\frac{m}{2}} \mathrm{e}^{R|\eta|}$$

for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$, then there exists a unique $u \in \mathscr{E}'(\mathbb{R}^n)$ whose Fourier-Laplace transform is Φ. Furthermore, u is supported in $\overline{B_R(0)}$ and has order at most $m + n + 1$.

The Paley-Wiener theorem–proof of (2)

We only give the proof for $n = 1$ [the proof in the case $n > 1$ is not examinable].

Define $\varphi := \Phi|_{\mathbb{R}}$. Then $\varphi \in \mathrm{C}^\infty(\mathbb{R})$ and because of the boundedness condition satisfied by Φ we have

$$\left|\varphi(\xi)\right| \leq c\left(1 + |\xi|^2\right)^{\frac{m}{2}}$$

for all $\xi \in \mathbb{R}$, so that φ is a tempered L^∞ function and so in particular $\varphi \in \mathscr{S}'(\mathbb{R})$. We can then by the Fourier inversion formula in $\mathscr{S}'$ define

$$u := \mathcal{F}^{-1}\varphi \in \mathscr{S}'(\mathbb{R}).$$

For the standard mollifier $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ on $\mathbb{R}$ we put $u_\varepsilon := \rho_\varepsilon * u$. Then u_ε is a moderate C^∞ function and by the convolution and dilation rules,

$$\widehat{u_\varepsilon} = \widehat{\rho_\varepsilon}\,\widehat{u} = d_\varepsilon\widehat{\rho}\widehat{u}.$$

The Paley-Wiener theorem–proof of (2)

Here $d_\varepsilon \widehat{\rho}(\xi) = \widehat{\rho}(\varepsilon \xi)$ and $\widehat{\rho}$ can by Paley-Wiener for test functions be extended to $\mathbb{C}$ as an entire function, hence so can $\widehat{u_\varepsilon}$:

$$\widehat{u_\varepsilon}(\zeta) = \widehat{\rho}(\varepsilon\zeta)\Phi(\zeta).$$

Furthermore, $\widehat{\rho}$ satisfies the boundedness condition given in Paley-Wiener for test functions: for any $k \in \mathbb{N}_0$ we find a constant $c_{k+m} \geq 0$ such that

$$\left|\widehat{\rho}(\varepsilon\zeta)\right| \leq c_{k+m}\left(1+|\varepsilon\zeta|^2\right)^{-\frac{k+m}{2}} \mathrm{e}^{\varepsilon|\eta|}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}$. Combine this with the boundedness condition we assume for Φ to get for $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}$:

$$\left|\widehat{u_\varepsilon}(\zeta)\right| \leq c_{k+m} c \frac{\left(1+|\zeta|^2\right)^{\frac{m}{2}}}{\left(1+|\varepsilon\zeta|^2\right)^{\frac{k+m}{2}}} \mathrm{e}^{(R+\varepsilon)|\eta|}$$

The Paley-Wiener theorem–proof of (2)

Here we have for $\varepsilon \in (0,1)$ and $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}$ that

$$\left(1+|\varepsilon\zeta|^2\right)^{\frac{k+m}{2}} = \varepsilon^{k+m}\left(\varepsilon^{-2}+|\zeta|^2\right)^{\frac{k+m}{2}} \geq \varepsilon^{k+m}\left(1+|\zeta|^2\right)^{\frac{k+m}{2}}$$

and therefore

$$\left|\widehat{u_\varepsilon}(\zeta)\right| \leq c_{k+m} c \varepsilon^{-k-m}\left(1+|\zeta|^2\right)^{-\frac{k}{2}} \mathrm{e}^{(R+\varepsilon)|\eta|}$$

Note that we have shown validity of this bound for each $k \in \mathbb{N}_0$. But then Paley-Wiener for test functions yields $\phi_\varepsilon \in \mathscr{D}(\mathbb{R})$ supported in $[-R-\varepsilon, R+\varepsilon]$ with Fourier-Laplace transform $\widehat{\phi_\varepsilon} = \widehat{u_\varepsilon}$. But then $u_\varepsilon = \phi_\varepsilon$ follows from the Fourier inversion formula in $\mathscr{S}'$, that is, we have shown that $u_\varepsilon \in \mathscr{D}(\mathbb{R})$ is supported in $[-R-\varepsilon, R+\varepsilon]$. Now we have clearly also that $u_\varepsilon \to u$ in $\mathscr{S}'(\mathbb{R})$ as $\varepsilon \searrow 0$ and it is not difficult to see that this implies that u is supported in $[-R, R]$ (check it as an exercise). □

Paley-Wiener bounds

We have encountered many different *boundedness conditions* during B4.3 and this course. It is useful to distinguish some of them with special names. Henceforth we will refer to the boundedness conditions in the Paley-Wiener theorems as *Paley-Wiener bounds*. More precisely:
Let $\Phi\colon \mathbb{C}^n \to \mathbb{C}$ be an entire function.

Φ satisfies a strong Paley-Wiener bound provided we can find $R \geq 0$ with the property that for each $m \in \mathbb{N}_0$ there exists $c_m \geq 0$ such that

$$\big|\Phi(\zeta)\big| \leq c_m\big(1+|\zeta|^2\big)^{-\frac{m}{2}}\mathrm{e}^{R|\eta|} \tag{2}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$.

Φ satisfies a weak Paley-Wiener bound provided we can find $R \geq 0$, $c \geq 0$ and $m \in \mathbb{N}_0$ such that

$$\big|\Phi(\zeta)\big| \leq c\big(1+|\zeta|^2\big)^{\frac{m}{2}}\mathrm{e}^{R|\eta|} \tag{3}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$.

Compactly supported distributions are Sobolev

Example Let $v \in \mathscr{E}'(\mathbb{R}^n)$. Then $v \in \mathrm{H}^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$.

Recall that $u \in \mathrm{H}^s(\mathbb{R}^n)$ provided

$$u \in \mathscr{S}'(\mathbb{R}^n) \text{ and } \left(1+|\xi|^2\right)^{\frac{s}{2}} \widehat{u} \in \mathsf{L}^2(\mathbb{R}^n).$$

Now by Paley-Wiener the Fourier-Laplace transform $\widehat{u}$ satisfies a weak Paley-Wiener bound, and more precisely we saw that if v is supported in $\overline{B_R(0)}$ and has order at most $m \in \mathbb{N}_0$ then for some constant $c \geq 0$ we have

$$\left|\widehat{v}(\zeta)\right| \leq c\left(1+|\zeta|^2\right)^{\frac{m}{2}} \mathrm{e}^{R|\eta|}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$. Thus if $\zeta = \xi \in \mathbb{R}^n$, then

$$\left(1+|\xi|^2\right)^{-m}\left|\widehat{v}(\xi)\right|^2 \leq c^2$$

It follows that $v \in \mathrm{H}^s(\mathbb{R}^n)$ for $s < -m - \frac{n}{2}$.