B4.4 Fourier Analysis HT22

Lecture 14: Periodic distributions and the Poisson summation formula

1. Examples and the periodisation of a test function
2. Periodic distributions are tempered
3. The Fourier transform of a periodic distribution
4. The Poisson summation formula

The material corresponds to pp. 48–53 in the lecture notes and should be covered in Week 7.

Periodic distributions

Definition Let $t > 0$. A distribution $u \in \mathscr{D}'(\mathbb{R})$ is t-periodic (or periodic with period t) if

$$\tau_t u = u.$$

Example 1 Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. Then u is t-periodic if $u(x+t) = u(x)$ almost everywhere. It follows from the fundamental lemma of the calculus of variations that u is t-periodic if and only if u is t-periodic as a distribution.

Example 2 Let $u \in \mathscr{D}'(\mathbb{R})$ and $t > 0$. Then u is t-periodic if and only if the dilated distribution

$$d_{\frac{t}{2\pi}} u$$

is 2π-periodic. Indeed, this is a consequence of the identity

$$\tau_{2\pi} d_{\frac{t}{2\pi}} u - d_{\frac{t}{2\pi}} u = d_{\frac{t}{2\pi}} \bigl(\tau_{2\pi} u - u \bigr).$$

Verify this as an exercise.

Periodic distributions

Intuitively a t-periodic distribution is fully determined if we know it on any interval of length t. This is clear for regular distributions. It is a little vague and unclear how this should be understood for general distributions. We assert that *if* (a, b) *is an interval of length* $b - a > t$*, then if we know that* $u \in \mathscr{D}'(\mathbb{R})$ *is t-periodic and know the values* $\langle u, \phi \rangle$ *for each* $\phi \in \mathscr{D}(a, b)$*, then we know* u.

Given any $\phi \in \mathscr{D}(\mathbb{R})$ with support contained in an interval $[c, d]$. Cover this compact interval with sets from the open cover

$$\left\{ (a + nt, b + nt) : \; n \in \mathbb{Z} \right\}$$

of $\mathbb{R}$. Use a smooth partition of unity for $[c, d]$ that is subordinated this cover (recall we constructed these in B4.3), say

$$[c, d] \subset \bigcup_{j=k}^{l} (a+jt, b+jt), \; \psi_j \in \mathscr{D}(a+jt, b+jt) \text{ and } \sum_{j=k}^{l} \psi_j = 1 \text{ on } [c, d].$$

Periodic distributions

Now because u is t-periodic we have

$$\langle u, \phi \rangle = \sum_{j=k}^{l} \langle u, \psi_j \phi \rangle = \sum_{j=k}^{l} \langle u, \tau_{-jt}(\psi_j \phi) \rangle$$

and since $\tau_{-jt}(\psi_j \phi) \in \mathscr{D}(a, b)$ for each j the value of u at ϕ is determined.

In the sequel we shall mainly consider 2π-periodic and 1-periodic distributions. As we have seen above this is not really restrictive as any period $t > 0$ can be obtained by dilation from, say, the 2π-periodic case.

Example 3 Use the Fourier bounds to show that

$$u = \sum_{n \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}nx}$$

is a tempered 2π-periodic distribution. [See details in the lecture notes]

The periodisation of a test function

Definition Let $\phi \in \mathscr{S}(\mathbb{R})$. Then *the periodisation of* ϕ is defined for each $x \in \mathbb{R}$ as

$$(P\phi)(x) = \sum_{n \in \mathbb{Z}} \phi(x + 2\pi n) \left(:= \lim_{\substack{k \to \infty \\ l \to \infty}} \sum_{n=-k}^{n=l} \phi(x + 2\pi n) \right).$$

We assert that $P\phi \colon \mathbb{R} \to \mathbb{C}$ is a 2π-periodic C^∞ function. Hereby $P \colon \mathscr{S}(\mathbb{R}) \to C^\infty_{\mathrm{per}}(\mathbb{R})$ is a linear map (valued in 2π-periodic C^∞ functions). First note that if $\phi \in \mathscr{D}(\mathbb{R})$, then the series becomes a finite sum and it is then clear that $P\phi \in C^\infty(\mathbb{R})$ and also that it is 2π-periodic. In the general case $\phi \in \mathscr{S}(\mathbb{R})$ the series defining $P\phi(x)$ is a genuine series and we must present a proof for our assertion:

Let $s \in \mathbb{N}_0$. For $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ we estimate

$$\left|\phi^{(s)}(x + 2\pi n)\right| = \frac{1 + (x + 2\pi n)^2}{1 + (x + 2\pi n)^2} \left|\phi^{(s)}(x + 2\pi n)\right| \leq \frac{2\overline{S}_{2,s}(\phi)}{1 + (x + 2\pi n)^2}$$

The periodisation of a test function

Consequently, given $r > 0$, we have for $x \in \mathbb{R}$, $k \in \mathbb{Z}$ satisfying $|x| \leq r < |k|$ that

$$\left|\phi^{(s)}(x + 2\pi k)\right| \leq \frac{2\overline{S}_{2,s}(\phi)}{1 + (2\pi|k| - r)^2}$$

and since

$$\sum_{n\in\mathbb{Z}, |n|>r} \frac{2\overline{S}_{2,s}(\phi)}{1 + (2\pi|n| - r)^2} < \infty$$

we infer from Weierstrass' M-test that the series $\sum_{n\in\mathbb{Z}} \phi^{(s)}(x + 2\pi n)$ converges uniformly in $x \in [-r, r]$. Because $s \in \mathbb{N}_0$, $r > 0$ were arbitrary we deduce that $P\phi$ is C^∞ and that the series, together with the term-by-term differentiated series, converge locally uniformly in $x \in \mathbb{R}$. Finally, it is clear that $P\phi$ is 2π-periodic.

The periodisation of a test function

Example 4 Show that if $\phi \in \mathscr{S}(\mathbb{R})$ and $\psi \in \mathscr{D}(\mathbb{R})$, then

$$\sum_{n=-k}^{n=l} \psi(x)\phi(x+2\pi n) \to \psi(x)\big(P\phi\big)(x) \ \text{ in } \ \mathscr{D}(\mathbb{R})$$

as $k,\ l \to \infty$.

Put $Z_{k,l}(x) = \sum_{n=-k}^{n=l} \phi(x+2\pi n)$. Then we have just shown that $Z_{k,l}^{(s)}(x) \to \big(P\phi\big)^{(s)}(x)$ locally uniformly in $x \in \mathbb{R}$ for each $s \in \mathbb{N}_0$ as k, $l \to \infty$. Because $\mathrm{supp}\big(\psi Z_{k,l}\big) \subseteq \mathrm{supp}\big(\psi\big)$ for all $k,\ l \in \mathbb{N}$ and by Leibniz' rule

$$\frac{\mathrm{d}^s}{\mathrm{d}x^s}\Big(\psi Z_{k,l}\Big) \to \frac{\mathrm{d}^s}{\mathrm{d}x^s}\Big(\psi P\phi\Big) \ \text{ uniformly}$$

as $k,\ l \to \infty$, the result follows.

Periodic distributions are tempered

Lemma Let $u \in \mathscr{D}'(\mathbb{R})$ be 2π-periodic. Then u is $\mathscr{S}$ continuous and hence extends to $\mathscr{S}(\mathbb{R})$ as a tempered distribution. (We also write u for this extension that necessarily must be unique.)

Proof. Put $\chi = \rho * \mathbf{1}_{(-1,2\pi+1]}$, where as usual ρ is the standard mollifier kernel on $\mathbb{R}$. Clearly, $\mathbf{1}_{(0,2\pi]} \leq \chi \leq \mathbf{1}_{(-2,2\pi+2]}$. The periodisation $P\chi$ is a 2π-periodic C^∞ function, and we must have $P\chi \geq 1$ everywhere, so that the function

$$\Psi = \frac{\chi}{P\chi}$$

is well-defined and $\Psi \in \mathscr{D}(\mathbb{R})$ and it has periodisation

$$P\Psi = 1 \text{ on } \mathbb{R}.$$

We use this to give a formula for the extension of u to $\mathscr{S}(\mathbb{R})$.

Periodic distributions are tempered–proof

For $\phi \in \mathscr{D}(\mathbb{R})$ we calculate:

$$
\begin{aligned}
\langle u, \phi \rangle &= \langle u, \phi P\Psi \rangle = \left\langle u, \sum_{n\in\mathbb{Z}} \phi \tau_{2\pi n} \Psi \right\rangle \\
&\overset{\text{Example 4}}{=} \sum_{n\in\mathbb{Z}} \left\langle u, \phi \tau_{2\pi n} \Psi \right\rangle \\
&= \sum_{n\in\mathbb{Z}} \left\langle u, \tau_{2\pi n} \left(\Psi \tau_{-2\pi n} \phi \right) \right\rangle \\
&= \sum_{n\in\mathbb{Z}} \left\langle u, \Psi \tau_{-2\pi n} \phi \right\rangle \\
&\overset{\text{Example 4}}{=} \left\langle u, \Psi \sum_{n\in\mathbb{Z}} \tau_{-2\pi n} \phi \right\rangle \\
&= \left\langle u, \Psi P\phi \right\rangle
\end{aligned}
$$

Periodic distributions are tempered–proof

Let $K = [-2, 2\pi + 2]$. By the boundedness property of u we can find constants $c = c_K \geq 0$, $m = m_K \in \mathbb{N}_0$ such that

$$\left|\langle u, \varphi\rangle\right| \leq c \sum_{s=0}^{m} \sup\left|\varphi^{(s)}\right|$$

holds for all $\varphi \in \mathscr{D}(K)$. Using the previous identity and this bound with $\varphi = \Psi P\phi$ we find:

$$\begin{aligned} \left|\langle u, \phi\rangle\right| &= \left|\langle u, \Psi P\phi\rangle\right| \\ &\leq c \sum_{s=0}^{m} \sup\left|\left(\Psi P\phi\right)\right| \\ &\overset{\text{Leibniz}}{\leq} cC(\Psi, m)\overline{S}_{0,m}(\phi), \end{aligned}$$

where $C(\Psi, m) = 3(m+1)2^m\overline{S}_{0,m}(\Psi)$. □

The Fourier transform of a periodic distribution

Assume $u \in \mathscr{D}'(\mathbb{R})$ is t-periodic. Then as we just saw, $u \in \mathscr{S}'(\mathbb{R})$ (abuse of notation...) and for some constants $c \geq 0$, $m \in \mathbb{N}_0$ we have

$$\bigl|\langle u, \phi \rangle\bigr| \leq c\overline{S}_{0,m}(\phi) \tag{1}$$

holds for all $\phi \in \mathscr{S}(\mathbb{R})$. Its Fourier transform $\widehat{u}$ is defined as a tempered distribution. Is there something special about it?

Obviously, we can Fourier transform the identity $\tau_t u = u$ using the translation rule to get

$$\mathrm{e}^{\mathrm{i}t\xi}\widehat{u} = \widehat{u}.$$

By the Fourier inversion formula any $v \in \mathscr{S}'(\mathbb{R})$ satisfying this equation, that is,

$$\mathrm{e}^{\mathrm{i}t\xi} v = v, \tag{2}$$

is the Fourier transform of a t-periodic distribution. Furthermore, from $\bigl(\mathrm{e}^{\mathrm{i}t\xi} - 1\bigr)v = 0$ we get that $\mathrm{supp}(v) \subseteq \frac{2\pi}{t}\mathbb{Z}$.

The Fourier transform of a periodic distribution

We used the following result:

Exercise If $v \in \mathscr{S}'(\mathbb{R})$ and $\Phi\colon \mathbb{R} \to \mathbb{C}$ is a moderate C^∞ function, then a *necessary* condition for

$$\Phi v = 0 \ \text{ in } \ \mathscr{S}'(\mathbb{R})$$

to hold is that $\operatorname{supp}(v) \subseteq \{x \in \mathbb{R} : \Phi(x) = 0\}$. Prove it. Prove also that the condition is *not* sufficient.

Now consider v restricted to the interval $(-\frac{2\pi}{t}, \frac{2\pi}{t})$. It is supported in $\{0\}$ and so by a result from B4.3 it follows that the restriction has the form

$$v\Big|_{(-\frac{2\pi}{t}, \frac{2\pi}{t})} = \sum_{s=0}^{m} a_s \delta_0^{(s)}$$

for some constants $a_s \in \mathbb{C}$ and $m \in \mathbb{N}_0$. Inspection shows that δ_0 satisfies (2). To see that the derivatives of δ_0 do not, and so that $m = 0$ above, we construct suitable test functions.

The Fourier transform of a periodic distribution

Fix $k \in \mathbb{N}$ and define

$$\varphi_k(x) = \frac{x^k}{k!}\big(\rho_\varepsilon * \mathbf{1}_{(-2\varepsilon,2\varepsilon)}\big)(x)$$

with $\varepsilon > 0$ so small that it is supported in $(-\frac{2\pi}{t}, \frac{2\pi}{t})$. We then have

$$\left\langle \left(\mathrm{e}^{\mathrm{i}t\xi} - 1\right) \sum_{s=0}^{m} a_s \delta_0^{(s)}, \varphi_{m-1} \right\rangle = m a_m (-1)^m \mathrm{i}t$$

and so (2) forces $a_m = 0$. Similarly, $a_{m-1} = a_{m-2} = \ldots = a_1 = 0$. We argue similarly at the other points of $\frac{2\pi}{t}\mathbb{Z}$ and so conclude that

$$v = \sum_{k\in\mathbb{Z}} c_k \delta_{\frac{2\pi}{t}k} \tag{3}$$

for some constants $c_k \in \mathbb{C}$. The doubly infinite sequence $\left(c_k\right)_{k\in\mathbb{Z}}$ cannot be arbitrary because the distribution (3) is tempered.

The Fourier transform of a periodic distribution

If we employ the Fourier inversion formula on the identity (3), assuming as we may that $v = \widehat{u}$, we find that

$$u = \sum_{k \in \mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}\frac{2\pi}{t}kx} \tag{4}$$

It follows that any t-periodic distribution admits an expansion of the form (4) for suitable coefficients $c_k \in \mathbb{C}$. In order to see what condition the doubly infinite sequence $\left(c_k\right)_{k \in \mathbb{Z}}$ must satisfy we return to the boundedness condition (1) for u and combine it with the Fourier bounds to get

$$\left| \sum_{k \in \mathbb{Z}} c_k \phi\Big(\frac{2\pi}{t}k\Big) \right| \leq C\overline{S}_{m+2,0}(\phi)$$

for $\phi \in \mathscr{S}(\mathbb{R})$, where $C \geq 0$ is a constant. We now construct suitable test functions to extract the information.

The Fourier transform of a periodic distribution

For each $j \in \mathbb{Z} \setminus \{0\}$ and $\varepsilon \in (0, \frac{\pi}{2t})$ define

$$\phi_j(x) = \frac{\overline{c_j}}{1+|c_j|}|j|^{-m-2}\big(\rho_\varepsilon * \mathbf{1}_{(\frac{2\pi}{t}j-2\varepsilon, \frac{2\pi}{t}j+2\varepsilon)}\big)(x).$$

Then ϕ_j and $\overline{S}_{m+2,0}(\phi_j) \leq \left(\frac{3\pi}{t}\right)^{m+2}$, hence

$$\left|\sum_{k\in\mathbb{Z}} c_k \phi_j\Big(\frac{2\pi}{t}k\Big)\right| = \frac{|c_j|^2}{1+|c_j|}|j|^{-m-2} \leq C\left(\frac{3\pi}{t}\right)^{m+2}$$

holds for all $j \neq 0$. The doubly infinite sequence $\left(c_k\right)_{k\in\mathbb{Z}}$ therefore satisfies

$$\left|c_k\right| \leq C\left(1+k^2\right)^{\frac{N}{2}} \tag{5}$$

for all $k \in \mathbb{Z}$, where $C \geq 0$, $N \in \mathbb{N}_0$ are constants. Such sequences are said to be of *moderate growth*. In turn, if a doubly infinite sequence $\left(c_k\right)_{k\in\mathbb{Z}}$ is of moderate growth, then (3) and (4) define tempered distributions.

The Fourier transform of a periodic distribution–summary

We have shown

Theorem The Fourier transform of a t-periodic distribution has the form

$$\sum_{k\in\mathbb{Z}} c_k \delta_{\frac{2\pi}{t}k},$$

where the doubly infinite sequence $\left(c_k\right)_{k\in\mathbb{Z}}$ has moderate growth. In turn, any such sum defines a tempered distribution that is the Fourier transform of a t-periodic distribution.

Corollary Any t-periodic distribution u admits an expansion

$$u = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}\frac{2\pi}{t}kx} \text{ in } \mathscr{S}'(\mathbb{R})$$

where the doubly infinite sequence $\left(c_k\right)_{k\in\mathbb{Z}}$ has moderate growth.

The Poisson summation formula

Theorem (2π-periodic version)

$$\sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kx} = 2\pi \sum_{k\in\mathbb{Z}} \delta_{2\pi k} \text{ in } \mathscr{S}'(\mathbb{R}) \tag{6}$$

The meaning of the formula is that for each $\phi \in \mathscr{S}(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} \widehat{\phi}(k) = 2\pi \sum_{k\in\mathbb{Z}} \phi(2\pi k).$$

If we apply the formula to the translate $\tau_x\phi$ we get

$$\big(P\phi\big)(x) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \widehat{\phi}(k)\mathrm{e}^{\mathrm{i}kx}.$$

At first this convergence is pointwise in $x \in \mathbb{R}$, but it is not difficult to show that it is uniform in $x \in \mathbb{R}$ and that the term-by-term differentiated series also all converge uniformly in $x \in \mathbb{R}$.

The Poisson summation formula

There are many variants of (6). For instance, for each $t > 0$, we get a t-periodic version by dilation:

Corollary (t-periodic version)

$$\sum_{k \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}\frac{2\pi}{t}kx} = t \sum_{k \in \mathbb{Z}} \delta_{kt} \text{ in } \mathscr{S}'(\mathbb{R})$$

As before the identity means that

$$\sum_{k \in \mathbb{Z}} \widehat{\phi}\left(\frac{2\pi}{t}k\right) = t \sum_{k \in \mathbb{Z}} \phi(kt) \tag{7}$$

holds for all $\phi \in \mathscr{S}(\mathbb{R})$. Again we can apply it to a translate of ϕ and this time one gets an expansion of the t-periodisation of ϕ.

For which ϕ beyond $\mathscr{S}(\mathbb{R})$ is (7) valid?

The Poisson summation formula–extension of scope

Exercise Let L, R be the distributions on the left-hand, right-hand side, respectively, of the t-periodic version of the Poisson summation formula. Show that there exists a constant $c = c(t) \geq 0$ such that

$$\left|\langle L, \phi\rangle\right| \leq c\overline{S}_{2,0}(\widehat{\phi})$$

and

$$\left|\langle R, \phi\rangle\right| \leq c\overline{S}_{2,0}(\phi)$$

hold for all $\phi \in \mathscr{S}(\mathbb{R})$.

Deduce that (7) remains valid for all continuous $\phi\colon \mathbb{R} \to \mathbb{C}$ with

$$\overline{S}_{2,0}(\phi) + \overline{S}_{2,0}(\widehat{\phi}) < \infty.$$

The Poisson summation formula–proof

Proof. Put

$$u = \sum_{k\in\mathbb{Z}} \delta_k$$

Then it is not difficult to see that $u \in \mathscr{S}'(\mathbb{R})$, that it is 1-periodic and

$$\widehat{u} = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}k\xi} = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}k\xi}.$$

By inspection, $\mathrm{e}^{\mathrm{i}\xi}\widehat{u} = \widehat{u}$ and $\tau_{2\pi}\widehat{u} = \widehat{u}$. We have seen that the first condition implies that

$$\widehat{u} = \sum_{k\in\mathbb{Z}} c_k \delta_{2\pi k}$$

for constants $c_k \in \mathbb{C}$. The second condition then implies that $c_k = c_0$ for all $k \in \mathbb{Z}$, thus

$$\widehat{u} = c_0 \sum_{k\in\mathbb{Z}} \delta_{2\pi k}.$$

The Poisson summation formula–proof

For $\phi \in \mathscr{S}(\mathbb{R})$ and $x \in (0, 2\pi]$ we apply $\widehat{u}$ to $\tau_x \phi \in \mathscr{S}(\mathbb{R})$:

$$\begin{aligned} \langle \widehat{u}, \tau_x \phi \rangle &= c_0 \sum_{k \in \mathbb{Z}} \phi(x + 2\pi k) \\ &= \sum_{k \in \mathbb{Z}} \widehat{\phi}(k) \mathrm{e}^{\mathrm{i}kx}. \end{aligned}$$

At first this identity holds pointwise in $x \in (0, 2\pi]$, but it is not difficult to see that it holds uniformly in $x \in (0, 2\pi]$. We can therefore integrate the identity by integrating the series term-by-term:

$$\int_0^{2\pi} c_0 \sum_{k \in \mathbb{Z}} \phi(x + 2\pi k) \, \mathrm{d}x = c_0 \sum_{k \in \mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} \phi(x) \, \mathrm{d}x = c_0 \int_{-\infty}^{\infty} \phi(x) \, \mathrm{d}x$$

equates

$$\int_0^{2\pi} \sum_{k \in \mathbb{Z}} \widehat{\phi}(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x = \sum_{k \in \mathbb{Z}} \int_0^{2\pi} \widehat{\phi}(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x = 2\pi \widehat{\phi}(0)$$

and so $c_0 = 2\pi$. □