Chapter 2

Dirac's Formalism and Continuum Normalisation

The subject of this section is, in some sense, a matter of notational formalism. We will (re-)introduce the *bra-ket* formalism of Dirac for representing states and observables in quantum mechanical systems. In reality, Dirac's formalism (in the broadest sense) is more substantive than just a change of notation. The novelty arises when discussing observables with a *continuous spectrum*. This can happen only in infinite-dimensional Hilbert spaces.

2.1 States, dual states, and matrix elements

The basic notational device introduced by Dirac is the *bra-ket*. Here we represent vectors in a Hilbert space as *kets*,

$$\psi \in \mathcal{H} \quad \longleftrightarrow \quad |\psi\rangle \ . \tag{2.1}$$

Since a Hilbert space is equipped with an inner product, we can also assign to a given state a dual vector

$$\begin{aligned} \varphi_\psi \ &: \ \mathcal{H} \ \to \ \mathbb{C} \,, \\ &: \ \chi \ \mapsto \ (\psi, \chi) \,. \end{aligned} \tag{2.2}$$

By sesquilinearity of the inner product on $\mathcal{H}$, this map is $\mathbb{C}$-conjugate-linear: $\varphi_{\lambda\psi} = \bar{\lambda}\varphi_\psi$ for $\lambda \in \mathbb{C}$. An important result in functional analysis is the following.

Theorem 2.1.1 (Riesz–Fréchet representation theorem)**.** Let $\mathcal{H}$ be a Hilbert space. For every continuous linear functional $\varphi \in \mathcal{H}^*$, there exists a unique $\psi \in \mathcal{H}$ such that $\varphi = \varphi_\psi$ (using the notations above).

We do not provide a proof here. In the finite-dimensional setting, it is not a difficult result, but in the infinite-dimensional case it is not as obvious. Indeed, the presence of the adjective *continuous* in the above is relevant precisely in the case of infinite-dimensional $\mathcal{H}$ (all linear functionals are continuous in a finite dimensional Hilbert space). This theorem establishes a canonical *bijective, antilinear isometry* between $\mathcal{H}$ and $\mathcal{H}^*$.

In Dirac notation, we denote elements of the (continuous) dual space $\mathcal{H}^*$ by *bras*:

$$\varphi \in \mathcal{H}^* \quad \longleftrightarrow \quad \langle\varphi| \ . \tag{2.3}$$

As a somewhat overloaded notational convention, we often use as the label for a *bra* the name of the state in $\mathcal{H}$ to which it corresponds under the Riesz–Fréchet isometry,

$$\varphi_\psi \quad \longleftrightarrow \quad \langle\psi| \ . \tag{2.4}$$

With these conventions in place, we denote the inner product between two states φ and ψ as a composite *bra-ket*, where the state and dual state are fused together in the visually natural manner,

$$(\varphi, \psi) \quad \longleftrightarrow \quad \langle\varphi|\psi\rangle \ . \tag{2.5}$$

An operator A can act on states/kets from the left, whereupon we will notationally allow it to be "absorbed" into the ket

$$A\,|\psi\rangle = |A\psi\rangle \ . \tag{2.6}$$

Similarly, operators act from the right upon bras, and are replaced by their adjoint when absorbed into the bra,

$$\langle\varphi|\,A = \langle A^*\varphi| \ . \tag{2.7}$$

Within an inner product, an operator can be moved about accordingly,

$$(\varphi, A\psi) = \langle\varphi|A\psi\rangle = \langle\varphi|A|\psi\rangle = \langle A^*\varphi|\psi\rangle = (A^*\varphi, \psi)\ . \tag{2.8}$$

The quantity $\langle\varphi|A|\psi\rangle$ will be referred to as the *matrix element* of A between φ and ψ.

2.2 Constructions with bra-kets

The bra-ket formalism is convenient for representing an array of natural constructions involving linear operators on Hilbert spaces. For example, given the state $|\alpha\rangle$ and the dual-state $\langle\beta|$, we can construct the *outer product*,

$$\begin{aligned} |\alpha\rangle\langle\beta| \ &: \ \mathcal{H} \ \to \ \mathcal{H}\ , \\ |\psi\rangle \ &\mapsto \ |\alpha\rangle\langle\beta|\psi\rangle = (\langle\beta|\psi\rangle)\,|\alpha\rangle\ . \end{aligned} \tag{2.9}$$

So we can concatenate *bras* and *kets* in the visually obvious manner and get meaningful operations. Outer products also behave well under taking adjoints,[15]

$$(|\alpha\rangle\langle\beta|)^* = |\beta\rangle\langle\alpha|\ . \tag{2.10}$$

Now let $\{|i\rangle\ , i \in I\}$ be an orthonormal basis for $\mathcal{H}$ (here I is some finite or countably infinite indexing set). Orthonormality means we have $\langle i|j\rangle = \langle j|i\rangle = \delta_{ij}$. We can then write an arbitrary vector in $\mathcal{H}$ uniquely as a (possibly infinite) linear combination of these basis vectors,

$$|\psi\rangle = \sum_{i\in I} c_i\,|i\rangle\ . \tag{2.11}$$

The components c_j for some $j \in I$ are extracted by acting with the *bra* corresponding to $|j\rangle$,

$$\langle j|\psi\rangle = \sum_{i\in I} c_i\,\langle j|i\rangle = \sum_{i\in I} c_i\delta_{ij} = c_j\ . \tag{2.12}$$

We see that we can realise the orthogonal projection Π_j onto the one-dimensional subspace spanned by the basis vector $|j\rangle$ using the outer product, $|j\rangle\langle j|$,[16]

$$|j\rangle\langle j|\psi\rangle = c_j\,|j\rangle\ . \tag{2.14}$$

More generally, for a linear subspace $\mathcal{H}' \subseteq \mathcal{H}$ with orthonormal basis $|i'\rangle\ , i' \in I'$, we can form the manifestly self-adjoint, orthogonal projection operator from $\mathcal{H}$ onto $\mathcal{H}'$:

$$\Pi_{\mathcal{H}'} = \sum_{i'\in I'} |i'\rangle\langle i'|\ . \tag{2.15}$$

In particular, for the case $\mathcal{H}' = \mathcal{H}$, we have an expression for the identity operator,

$$\Pi_{\mathcal{H}} \equiv 1_{\mathcal{H}} = \sum_{i\in I} |i\rangle\langle i|\ . \tag{2.16}$$

This expression is often referred to as a *resolution of the identity* or *completeness relation*. Given a linear operator $A : \mathcal{H} \to \mathcal{H}$, we can then resolve it in terms of its matrix elements with respect to the given basis,

$$A = 1_{\mathcal{H}} A 1_{\mathcal{H}} = \sum_{i,j\in I} |i\rangle\langle i|A|j\rangle\langle j| = \sum_{i,j\in I} A_{ij}\,|i\rangle\langle j|\ . \tag{2.17}$$

where

$$A_{ij} = \langle i|A|j\rangle\ . \tag{2.18}$$

[15] Verify this relation if it isn't obvious to you by inspection.

[16] For the a general state vector ψ, not necessarily normalised, we have the orthogonal projection operator,

$$\Pi_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\ . \tag{2.13}$$

Finally, for A an observable if the states $\{|i\rangle\}$ are an orthonormal basis of A eigenstates obeying $A|i\rangle = a_i|i\rangle$ then we have matrix elements $A_{ij} = a_i\delta_{ij}$ and (2.17) becomes the *spectral decomposition* of A,

$$A = \sum_i a_i |i\rangle \langle i| \ . \tag{2.19}$$

In the case where $\mathcal{H}$ is finite-dimensional, this is all pretty familiar. The outer product $|i\rangle\langle j|$ corresponds to the matrix that is all zeroes except for having a one in the i'th row at the j'th column, and (2.17) describes the building up the operator A entry by entry as a matrix, while (2.19) corresponds to the matrix expression for A in the basis where A is diagonalised, which is the usual spectral decomposition of an Hermitian matrix. The resolution of the identity is just the expression for the identity operator as the identity matrix.

In terms of bra-kets, we can represent the expectation value of an observable as follows. If our basis $\{|i\rangle\}$ diagonalises the observable A as above, then we have

$$\begin{aligned} \mathbb{E}_\psi(A) = \mathbb{E}_\psi(A1_{\mathcal{H}}) = &= \sum_{i\in I} \langle \psi|A|i\rangle \langle i|\psi\rangle \ , \\ &= \sum_{i\in I} a_i |\langle i|\psi\rangle|^2 \ , \end{aligned} \tag{2.20}$$

which matches the notion of expectation value for a random variable.

2.3 Continuous observables

We now come to the important issue of observables with *continuous spectrum*. In finite dimensional Hilbert spaces (and, it turns out, for something called a *compact operator* on an infinite-dimensional Hilbert space) the spectrum of any observable is discrete, being just the set of eigenvalues. For more general operators in infinite-dimensional Hilbert spaces we may potentially encounter a subtlety.

Just the definition of the spectrum of an operator is in fact more subtle in the infinite-dimensional case than just the eigenvalues. Indeed, we have the following:

Definition 2.3.1. The *spectrum* of a self-adjoint operator A on a Hilbert space $\mathcal{H}$ is the subset $\sigma(A) \subseteq \mathbb{R}$ such that for $\lambda \in \sigma(A)$, the shifted operator $A - \lambda 1_{\mathcal{H}}$ does not have a (bounded, everywhere defined) inverse.

We will not dwell on the bounded/everywhere-defined caveats, which are relevant for a fully rigorous treatment. In finite dimensions, the equivalence of non-invertibility and λ being an eigenvalue is automatic upon consideration of the characteristic polynomial.

In the infinite-dimensional case, more elaborate situations are possible, and in particular the spectrum can include a continuum. Indeed, we can think of the position operator X acting on $L^2(\mathbb{R})$. Considering our definition, a value $\lambda \in \sigma(X)$ if for any $g \in L^2(\mathbb{R})$ we can't find an $f \in L^2(\mathbb{R})$ that solves the problem,

$$(x - \lambda)f(x) = g(x) \ . \tag{2.21}$$

But clearly this is the case for any real λ (to be precise, we could take $g(x)$ to be an indicator function for a finite interval including $x = \lambda$). So for X the spectrum is the entire real line.

This is intuitively compatible with our Postulate III, since the possible observable values of the position operator should be roughly the entire real line. However now there is apparently some tension with our desire to assign a basis of eigenstates to the set of points in the spectrum of an observable. Dirac suggested in his original treatise on the subject to forge ahead and formally extend his bra-ket formalism to include kets associated even to elements of a continuous spectrum. This is indeed the approach method that is standard in the physics community. His proposal can in retrospect be understood as being essentially an application of the spectral theorem for self-adjoint operators in its most sophisticated form. We will introduce the method now in an operational sense.

2.3.1 Generalised position eigenstates

To get our discussion off the ground, let's continue with our discussion of the particle moving on the real line, so with Hilbert space $L^2(\mathbb{R})$.[17] The two fundamental observables in this setting are the position and momentum operators, and as we saw above, for the position operator X the spectrum is the entire real line.

Dirac instructs us to define an *generalised position eigenstate* $|\xi\rangle$ for this operator for each $\xi \in \mathbb{R}$,[18]

$$X\,|\xi\rangle = \xi\,|\xi\rangle \ . \tag{2.22}$$

Were we to use a wave function $\psi_\xi(x)$ to represent such a state, it would have to satisfy the unlikely-looking identity

$$x\psi_\xi(x) = \xi\psi_\xi(x) \ . \tag{2.23}$$

For this to hold, it must be that $\psi_\xi(x) = 0$ for $x \neq \xi$, and indeed if this were an element of $L^2(\mathbb{R})$ that would mean it was the zero function, so certainly this can't correspond to a non-zero element of the Hilbert space.

Nevertheless, we formally introduce such an object. Since this generalised state is meant to represent a situation where the particle is *definitely at* $x = \xi$, it is reasonable to demand

$$\langle\xi|\psi\rangle = \psi(\xi) \ , \qquad \langle\psi|\xi\rangle = \overline{\psi(\xi)} \ . \tag{2.24}$$

This is actually an important idea: *the value of the wave function at a point $x = \xi$ is the overlap of the state in question with the generalised position eigenstate $|\xi\rangle$*. Expressing this in terms of wave functions, we have

$$\int_{-\infty}^{\infty} \mathrm{d}x\,\overline{\psi_\xi(x)}\psi(x) = \psi(\xi) \ , \tag{2.25}$$

We recognise this to be precisely the *sifting property* of (confusingly named) *Dirac δ-function*. Rather than a function, this is a *distribution*, meaning it is a linear functional on functions. You have met the Dirac δ-function previously in **M4 Multivariable Calculus**, and maybe also in **ASO Integral Transforms**. Indeed, we will identify

$$|\xi\rangle \ \longleftrightarrow \ \psi_\xi(x) = \delta(x-\xi) \ . \tag{2.26}$$

Note that while these generalised position eigenstates are not normalisable in the usual sense of $L^2(\mathbb{R}$, they obey a *continuum normalisation condition*,[19]

$$\langle\xi|\xi'\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\,\delta(x-\xi)\delta(x-\xi') = \delta(\xi-\xi') \ . \tag{2.27}$$

This is a fairly natural generalisation of the usual normalisation condition where we have a Kronecker δ, but with the Dirac δ instead.

Happily, it turns out that we can for the most part use these generalised position eigenstates in the same ways we would use ordinary basis states as discussed previously, with various sums converted into integrals as appropriate. Justification for this rests upon some deep pieces of functional analysis that we are sweeping under the rug,[20] but as we mentioned

[17] A similar discussion here could take place for the particle moving on an interval $[0,1] \subset \mathbb{R}$, with Hilbert space $L^2([0,1])$. The free particle on the entire real line is even a bit more subtle.

[18] Here we begin to adopt a fairly standard notational choice: in the context of discussing a particular observable (in this case X), we denote states whose eigenvalue is some number (in this case $\xi \in \mathbb{R}$) by a *ket* whose label is *that same eigenvalue* (in this case $|\xi\rangle$). There is some danger of getting confused if not sufficiently diligent with this notational system, so be careful!

[19] Such (generalised) states are sometimes referred to as being δ-function normalisable states.

[20] There are several realisations of these generalised eigenstates within a more rigorous framework. In one version of the spectral theorem for self-adjoint operators on infinite-dimensional Hilbert spaces, one constructs the Hilbert space of interest as a *direct integral* of smaller Hilbert spaces, and these generalised states can be understood as elements of the (Hilbert-space) integrand of that direct integral. Alternatively, Hilbert spaces arising in quantum mechanics can be equipped with additional structure known as a *Gel'fand triple*. In this case the generalised states are elements of a larger space of distributions that form a part of that structure. You don't need to know any of this for the present course, but it is a beautiful subject!

above, the quantum mechanical formalism (due to Dirac) actually predated the rigorous justification. In particular, we have a resolution of the identity in terms of these position eigenstates,

$$1_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \mathrm{d}\xi \; |\xi\rangle \langle\xi| \; . \tag{2.28}$$

Acting on genuine states (corresponding to authentic, normalisable wave functions), we have

$$\begin{aligned} 1_{L^2(\mathbb{R})} |\psi\rangle &= \int_{-\infty}^{\infty} \mathrm{d}\xi \; |\xi\rangle \langle\xi|\psi\rangle \; , \\ &= \int_{-\infty}^{\infty} \mathrm{d}\xi \; \psi(\xi) \, |\xi\rangle \; , \end{aligned} \tag{2.29}$$

The final expression gives the continuum analogue of the decomposition of a general state in an orthonormal basis.

Generalising this resolution of the identity, if we integrate the outer product $|\xi\rangle\langle\xi|$ over any measurable subset $E \subset \mathbb{R}$, we obtain the self-adjoint projection operator corresponding to multiplication by the indicator function 1_E discussed in Chapter 1.3,[21]

$$\Pi_E = \int_E \mathrm{d}\xi \; |\xi\rangle \langle\xi| \; , \qquad \langle x| \Pi_E |\psi\rangle = 1_E(x)\psi(x) \; . \tag{2.30}$$

Indeed, we note that for a finite measure subset E this gives an actual projection operator on $\mathcal{H}$, while the outer product itself is not well-defined as a map on the Hilbert space. Since these projections are supposed to arise when considering measurements, this state of affairs is often understood as corresponding to the physical impossibility of measuring position with infinite precision; one could only ever check that a particle is within some error bar of a particular position.

2.3.2 Generalised momentum eigenstates

There is a similar story with the momentum operator $P = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}$. We introduce (generalised) momentum eigenstates $|p\rangle$,

$$P|p\rangle = p|p\rangle \; , \qquad p \in \mathbb{R} \, , \tag{2.31}$$

and if we denote the actual wave function associated to this state as $\psi_p(x)$, then we can easily solve the corresponding differential equation, at least formally,

$$-i\hbar\, \psi_p'(x) = p\psi_p(x) \quad \Longrightarrow \quad \psi_p(x) = \mathcal{N} e^{\frac{ipx}{\hbar}} \; , \tag{2.32}$$

where $\mathcal{N}$ is some normalisation factor. The problem is now clear and and feels familiar: these wave functions are not square-normalisable at all (on the entire real line), so this is not giving us an element of $L^2(\mathbb{R})$. Rather this is a kind of generalised eigenstate, which we can interpret as a distribution.

Using our previous understanding of the relationship between wave functions and generalised position eigenstates, we deduce the overlap equation

$$\langle x|p\rangle = \psi_p(x) = \mathcal{N} e^{\frac{ipx}{\hbar}} \; . \tag{2.33}$$

We can then derive the continuum normalisation condition for the momentum eigenstates,

$$\langle p|p'\rangle = |\mathcal{N}|^2 \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{ipx}{\hbar}} e^{\frac{ip'x}{\hbar}} = 2\pi\hbar\mathcal{N}^2 \int_{-\infty}^{\infty} \mathrm{d}s \, e^{2\pi i(p-p')s} = 2\pi\hbar\mathcal{N}^2 \delta(p-p') \, , \tag{2.34}$$

[21] In one rigorous treatment of these constructions, it is this assignment of a self-adjoint projection operator to (measurable) subsets of $\mathbb{R}$ that is rigorously defined and guaranteed to exist by the spectral theorem; such an assignment is called a *projection valued measure*.

where in the last equation we have used the integral representation for the delta function. It is then natural to adopt the normalisation conventions $\mathcal{N} = (2\pi\hbar)^{-1/2}$ giving canonical continuum normalisation to the generalised momentum eigenstates. We have an analogous resolution of the identity in terms of momentum states,

$$1_{\mathcal{H}} = \int_{-\infty}^{\infty} \mathrm{d}p \; |p\rangle \langle p| \; . \tag{2.35}$$

This formalism of position and momentum (generalised) bases for $L^2(\mathbb{R})$ gives us a nice new perspective on the quantum mechanics of a particle. To a given state vector $|\psi\rangle$, we can associated either its expression in *position space*,

$$\psi(x) = \langle x|\psi\rangle \; , \qquad |\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x \, \psi(x) \, |x\rangle \; , \tag{2.36}$$

or its expression in *momentum space*,

$$\widehat{\psi}(p) = \langle p|\psi\rangle \; , \qquad |\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}p \, \widehat{\psi}(p) \, |p\rangle \; . \tag{2.37}$$

So there are actually (at least) two wave functions associated to the state ψ of a particle on the real line. These, it turns out, are related by the Fourier transform,

$$\psi(x) \;=\; \langle x|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}p \; \langle x|p\rangle \, \langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}p \, e^{\frac{ipx}{\hbar}} \, \widehat{\psi}(p) \; . \tag{2.38}$$

$$\widehat{\psi}(p) \;=\; \langle p|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x \; \langle p|x\rangle \, \langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{ipx}{\hbar}} \, \psi(x) \; . \tag{2.39}$$

Indeed, the Fourier transform is a unitary map from $L^2(\mathbb{R})$ to itself (this is the Plancherel theorem), so the change from position to momentum representation is just *a change of basis* for our Hilbert space.

Remark 2.3.2. This formalism for generalised position eigenstates generalises immediately to the case of a particle moving in, say, $d = 2$ or $d = 3$ dimensions. There for $\mathbf{x} = (x_1, \ldots, x_d)$ we have the generalised eigenstates

$$X_i \, |\mathbf{x}\rangle = x_i \, |\mathbf{x}\rangle \; , \tag{2.40}$$

obeying the continuum normalisation condition,

$$\langle \mathbf{x}|\mathbf{x}'\rangle = \delta^d(\mathbf{x} - \mathbf{x}') \; , \tag{2.41}$$

and the corresponding resolution of the identity,

$$1_{L^2(\mathbb{R}^d)} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_d \; |\mathbf{x}\rangle \langle \mathbf{x}| \; . \tag{2.42}$$

Similarly, we have generalised momentum eigenstates corresponding to non-normalisable plane-wave wave functions,

$$|\mathbf{p}\rangle \quad \longrightarrow \quad \psi_{\mathbf{p}}(\mathbf{x}) = \frac{1}{(2\pi\hbar)^{d/2}} e^{\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}} \; , \tag{2.43}$$

obeying the same continuum normalisation condition and admitting the same type of resolution of the identity. d-dimensional wave functions in position space and momentum space are related now by the d-dimensional Fourier transform.

2.4 Application: free particle propagator

A nice application of the machinery we have developed here is in defining an important object in studying quantum mechanical dynamics: the *propagator*. Intuitively, this is the quantity that tells you the quantum mechanical *amplitude* (square root of probability density) for a particle that starts at a given position to be detected at some other position at some definite time in the future. In terms of generalised position eigenstates, this is the quantity

$$U(x_1,t_1;x_0,t_0) := \langle x_1|U(t_1;t_0)|x_0\rangle \ , \tag{2.44}$$

where $U(t_1;t_0)$ is the unitary time evolution operator introduced previously. If one has the propagator under good control, then the time evolution of general quantum states can be described using the following double integral

$$\begin{aligned}\langle\psi_1|U(t_1,t_0)|\psi_0\rangle &= \langle\psi_1|\left(\int_{-\infty}^{\infty}\mathrm{d}x_1\ |x_1\rangle\langle x_1|\right)U(t_1;t_0)\left(\int_{-\infty}^{\infty}\mathrm{d}x_0\ |x_0\rangle\langle x_0|\right)|\psi_0\rangle \ ,\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x_1\,\mathrm{d}x_0\ \overline{\psi_1(x_1)}U(x_1,t_1;x_0,t_0)\psi_0(x_0) \ .\end{aligned} \tag{2.45}$$

In general, the propagator is not so easy to compute. Here we will do it for the case of the free particle in one dimension. The Hamiltonian is $H = P^2/2m$ and the (generalised) energy eigenstates are precisely the (generalised) momentum eigenstates:

$$H|p\rangle = E_p|p\rangle = \frac{p^2}{2m}|p\rangle \ . \tag{2.46}$$

As we know well, time evolution for these states then proceeds via phase multiplication,

$$U(t_1,t_0)|p\rangle = \exp\left(\frac{-iE_p(t_1-t_0)}{\hbar}\right)|p\rangle = \exp\left(\frac{-ip^2(t_1-t_0)}{2m\hbar}\right)|p\rangle \ . \tag{2.47}$$

This means that the *momentum-space propagator* is very simple for the free particle,

$$\langle p_1|U(t_1;t_0)|p_0\rangle =: \widehat{U}(p_1,t_1;p_0,t_0) = \delta(p_1-p_0)\exp\left(-\frac{ip_0^2(t_1-t_0)}{2m\hbar}\right) \ . \tag{2.48}$$

The position-space propagator is then obtained by a double Fourier transform,[22]

$$U(x_1,t_1;x_0,t_0) = \langle x_1|\left(\int_{-\infty}^{\infty}\mathrm{d}p_0\,|p_0\rangle\langle p_0|\right)U(t_1;t_0)\left(\int_{-\infty}^{\infty}\mathrm{d}p_1\ |p_1\rangle\langle p_1|\right)|x_0\rangle \ , \tag{2.49}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}p_0\,\mathrm{d}p_1\ \langle x_1|p_1\rangle\,\widehat{U}(p_1,t_1;p_0,t_0)\,\langle p_0|x_0\rangle \ , \tag{2.50}$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}p_0\,\mathrm{d}p_1\,\delta(p_1-p_0)\exp\left(\frac{ip_1x_1-ip_0x_0}{\hbar}-\frac{ip_0^2(t_1-t_0)}{2m\hbar}\right) \ , \tag{2.51}$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}p\ \exp\left(\frac{ip(x_1-x_0)}{\hbar}-\frac{ip^2(t_1-t_0)}{2m\hbar}\right) \ , \tag{2.52}$$

$$= \left(\frac{m}{2\pi i\hbar(t_1-t_0)}\right)^{\frac{1}{2}}\exp\left(-\frac{m(x_1-x_0)^2}{2i\hbar(t_1-t_0)}\right) \ . \tag{2.53}$$

[22] The attentive reader will notice that upon setting $t\to -i\tau$, the final result for the propagator becomes the Green's function for the one-dimensional heat equation, with the thermal conductivity given by $\hbar/2m$. Indeed, the same "imaginary time" replacement applied to the time-dependent Schrödinger equation for this system yields the heat equation with said thermal conductivity.

The last integral is somewhat subtle, but can be computed using results for *Fresnel integrals*.[23]

It is interesting to observe that instantly when $t_1 > t_0$, the propagator is nonzero for arbitrarily large $x_1 - x_0$. This reflects the infinite uncertainty in momentum that is associated with the completely localised position eigenstate at time t_0. However, the phase in the exponential is also very large for large $x_1 - x_0$ and small $t_1 - t_0$, so when we average over positions (as we should if we start with a normalisable wave function) then there will be cancellations and the wave function will remain somewhat localised near its original support.

[23]The general formula is

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(\frac{i}{2}ax^2 + ibx\right) = \left(\frac{2\pi i}{a}\right)^{\frac{1}{2}} \exp\left(-\frac{ib^2}{2a}\right) .$$