Appendix A

Hilbert Space Minutiae

In the interest of making these notes fairly self-contained, we provide here an in depth account of the definition and important properties of Hilbert spaces. The fine details of the infinite-dimensional case go beyond the official syllabus for the course.

A.1 Definitions

Definition A.1.1. A *sesquilinear form* on a complex vector space V is a map $(\cdot,\cdot) : V \times V \to \mathbb{C}$ obeying

$$\begin{aligned}(\alpha\varphi_1 + \beta\varphi_2, \psi) &= \bar{\alpha}(\varphi_1, \psi) + \bar{\beta}(\varphi_2, \psi)\ .\\ (\varphi, \alpha\psi_1 + \beta\psi_2) &= \alpha(\varphi, \psi_1) + \beta(\varphi, \psi_2)\ .\end{aligned} \tag{A.1}$$

so it is linear in the second argument and conjugate-linear/$\mathbb{C}$-antilinear in the first argument.[54]

Definition A.1.2. An *Hermitian form* on a complex vector space V is a sesquilinear form that obeys

$$(\varphi, \psi) = \overline{(\psi, \varphi)}\ . \tag{A.2}$$

This means that for an Hermitian form, $(\psi, \psi) \in \mathbb{R}$ for any $\psi \in V$.

Definition A.1.3. An *Hermitian inner product* on a complex vector space V is a positive definite Hermitian form, so

$$(\varphi, \varphi) \geqslant 0\ , \qquad (\varphi, \varphi) = 0 \iff \varphi = 0\ . \tag{A.3}$$

As usual, this lets us define a norm on V,

$$\|\varphi\| := \sqrt{(\varphi, \varphi)}\ . \tag{A.4}$$

Definition A.1.4. A *complex Hilbert space* is a complex vector space with an Hermitian inner product such that all Cauchy sequences converge (so it is *complete*). This means that for a sequence of vectors $\varphi_1, \varphi_2, \ldots,$ such that $\forall \varepsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that

$$\|\varphi_n - \varphi_m\| < \varepsilon\ , \qquad m, n > N\ , \tag{A.5}$$

there is an element $\varphi \in \mathcal{H}$ such that $\{\varphi_n\} \to \varphi$.

***Definition A.1.5.** A *complex Hilbert space* is called *separable* if it admits a countable orthonormal basis.

Clearly every finite-dimensional Hilbert space is separable, so this is a technical condition relevant for the infinite-dimensional case. In this infinite dimensional case, an orthonormal basis is meant in the sense of *infinite* linear combinations,[55] so any vector ψ can be written in terms of the basis vectors $\{\psi_n\}$ as

$$\psi = \sum_{n=1}^{\infty} a_n \psi_n\ , \qquad a_n := (\psi_n, \psi)\ , \tag{A.6}$$

where

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty\ . \tag{A.7}$$

In standard quantum mechanical constructions (and indeed, in many branches of mathematics) one encounters only separable Hilbert spaces, so often one omits the modifier "separable" entirely. This will be our practice.

[54] As in the main text, here we adopt "physics conventions" in which the inner product is conjugate-linear in the *first* argument.
[55] What is sometimes called a *Schauder* basis.

A.2 Illustrative Examples

Example A.2.1 (Incomplete pre-Hilbert space)**.** The requirement of completeness is only relevant for infinite-dimensional Hilbert spaces (for finite-dimensional cases completeness is automatic). For readers who are interested, For an example, consider the space $C^([0,1],\mathbb{C})$. of continuous functions on the interval $[-1,1]$. This is an Hermitian inner product space, with

$$(f,g) = \int_{\mathbb{R}} \overline{f(x)} g(x)\,\mathrm{d}x\,. \tag{A.8}$$

However, this space is incomplete, as can be seen from the family of functions

$$f_n(x) = \begin{cases} 0 & -1 \leqslant x \leqslant -\frac{1}{n}\,, \\ \frac{xn+1}{2} & -\frac{1}{n} \leqslant x \leqslant +\frac{1}{n}\,, \\ 1 & +\frac{1}{n} \leqslant x \leqslant 1\,. \end{cases} \tag{A.9}$$

This set of functions can be checked to form a Cauchy sequence, but the limit is the discontinuous function that is zero for $x < 0$ and one for $x > 0$. Thus the space of continuous functions with the given inner product is not a Hilbert space; it is sometimes called a *pre-Hilbert space*. The completion of this space, which is the relevant Hilbert space for considering quantum mechanics on the interval, is the Lebesgue space $L^2([-1,1])$ of (equivalence classes of) complex-valued, square-integrable functions on $[-1,1]$, with two functions considered equivalent if they take the same value almost everywhere.

Example A.2.2 (Non-separable Hilbert space)**.** Though they won't show up in this course (or much of anywhere in quantum theory), I think it is useful to at least see a definition of a Hilbert space that is *not* separable, because seeing is believing. We define the space $\ell^2(\mathbb{R})$ to be the set of functions $f : \mathbb{R} \to \mathbb{C}$ such that $f(x) \neq 0$ for countably many x, and

$$\sum_{x\in\mathbb{R}} |f(x)|^2 < \infty\,. \tag{A.10}$$

The inner product is given by

$$(f,g) = \sum_{x\in\mathbb{R}} \overline{f(x)} g(x)\,. \tag{A.11}$$

This admits an uncountable, orthonormal basis which are the functions $\{f_x\}$ with $s \in \mathbb{R}$, where

$$f_s(x) = \begin{cases} 1 & x = s\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{A.12}$$

This clearly has a much different flavour from the sorts of Hilbert spaces we meet when discussing elementary particles or spin systems.

A.3 Operators on Hilbert Space

Here we collect some definitions and examples related to the subtleties of operator theory for infinite-dimensional Hilbert spaces. This is only provided for students curious about the details of dealing with infinite-dimensional subtleties; You will not need to deal with this material in the course for the problem sheets or the exam.

Definition A.3.1. An unbounded operator A on a Hilbert space $\mathcal{H}$ is a linear map $A : D(A) \to \mathcal{H}$ from a linear subspace $D(A) \subseteq \mathcal{H}$ to $\mathcal{H}$. It is conventional to require that $D(A)$ is dense in $\mathcal{H}$.

The operators we study in the quantum mechanics of $L^2(\mathbb{R})$ tend to be unbounded operators. For example, the momentum operator P naturally is defined for $D(P)$ the subspace of once-differentiable functions with square-integrable derivatives. The position operator X is defined for functions whose growth at infinity is sufficiently mild that they are still square-integrable after multiplying by x.

Definition A.3.2. The adjoint operator of an unbounded operator $(A, D(A))$ on a Hilbert space $\mathcal{H}$ is another unbounded operator $(A^*, D(A^*))$ on $\mathcal{H}$ obeying

$$(\varphi, A\psi) = (A^*\varphi, \psi) , \qquad \forall\psi \in D(A) , \ \forall\varphi \in D(A^*) . \tag{A.13}$$

The domain $D(A^*)$ is defined to be the linear subspace of $\mathcal{H}$ for which $\varphi \to (\varphi, A\psi)$ is continuous for any $\psi \in D(A)$. By the Riesz–Fréchet isomorphism, $A^*\varphi$ is uniquely defined for a $\varphi \in D(A^*)$ by virtue of the aforementioned continuous linear functional.

In this unbounded setting, a self adjoint operator is, importantly, an operator A for which not only $A = A^*$, but $D(A) = D(A^*)$.

Example A.3.3 (Unbounded operators and adjoints)**.** A very instructive example of this subtlety comes in the case of the particle in a box, as reviewed in the prologue of these lecture notes. Consider the Hamiltonian operator $H = P^2/2m$. This is in fact an unbounded operator, and so needs to be equipped with a domain; the domain used to define the particle in the box is the set of wave functions that vanish at the end points of the box and who are twice differentiable, with the result being square integrable. With this choice of $D(H)$, we find that $D(H^*) = D(H)$, so the Hamiltonian is truly self-adjoint and the spectral theorem must hold (as we have taken for granted in the past).

On the other hand, consider the momentum operator P for the particle in the box. If we take the same domain $D(P)$ as for H, then (because P is a first order differential operator), $D(P^*)$ is strictly larger than $D(P)$, and includes all differentiable functions with square-integrable derivative. Thus we cannot apply the spectral theorem in this case, and indeed we do not have a basis of P-eigenfunctions on the interval that obey the boundary conditions!