Fixed Point Methods for Nonlinear PDEs

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Introduction

The goal of this part of the course is to introduce some basic methods to establish existence of solutions to nonlinear equations in infinite-dimensional spaces, such as nonlinear partial differential equations and variational inequalities. In the first part we introduce and prove the major fixed point theorems by Picard, Brouwer and Schauder. In the second part we apply them to solve some nonlinear ordinary and partial differential equations. In the third chapter we then use the fixed point theorems to prove an abstract result on the existence of solutions to variational inequalities. This result applies for example to equations given by monotone operators, which often appear in connection with Euler-Lagrange equations of convex functionals, but can also arise in a nonvariational context. In the final chapter we apply the result on variational inequalities to quasilinear second order partial differential equations and obstacle problems. Throughout this lecture we will restrict ourselves to elliptic partial differential equations, but the methods can be extended without too much additional effort to parabolic equations.

These lecture notes are based on Yves Capdeboscq's and Melanie Rupflin's lecture notes.

Recommended Literature

H.-W. Alt, *Nonlinear Functional Analysis*, Lecture course given at the University of Bonn, 1990.

L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics 19, AMS, 1998.

O. Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Springer Paris, 1994.

H. Le Dret, *Équations aux dérivées partielles elliptiques*, Lecture course given at the University Pierre et Marie Curie (Paris VI), 2010.

Further Reading

M. Ruzicka, *Nichtlineare Funktionalanalysis, Eine Einführung*, Springer Berlin, 2004.

M.S. Berger, *Nonlinearity and Functional Analysis*, Academic Press, 1977.

K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.
E. Zeidler, *Nonlinear functional analysis and its applications I, Fixed Point theorems*, Springer New York, 1986
E. Zeidler, *Nonlinear functional analysis and its applications II A+B, Monotone operators*, Springer New York, 1990
L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute Lecture Notes, AMS, 2001.
R.E. Showalter, *Monotone operators in Banach spaces and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, AMS, 1997.

Before embarking on the theory we start out with some typical examples.

Examples.

(1) *Nonlinear ODE*

We look for a function $y\colon [a,b] \subset \mathbb{R} \to \mathbb{R}^N$, (or $y\colon [a,b] \subset \mathbb{R} \to X$, X Banach space) such that for some $t_0 \in (a,b)$

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

This is equivalent to finding a solution of the *integral equation*

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s.$$

This is again equivalent to finding a *zero of the map* F defined via

$$(Fy)(t) := y_0 - y(t) + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s$$

or a *fixed point* of the map T defined via $T := F + Id$, that is

$$(Ty)(t) := y_0 + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s\,.$$

(2) *Nonlinear PDE*

(a) *Semilinear elliptic equations*, e.g.

Look for a function $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ that solves

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega \\ u &= u_0 && \text{on } \partial\Omega\,, \end{aligned}$$

where $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is a – typically nonlinear – function. Equivalently look for a fixed point of

$$Tu := (-\Delta_{u_0})^{-1}(f(u)).$$

The subscript u_0 is intended to remind the reader of the boundary condition.

(b) *Quasilinear elliptic equations*, e.g. *p-Laplacian.*

Given a continuous function $f\colon \mathbb{R} \to \mathbb{R}$, we look for a function $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that [1]

$$\begin{aligned} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= f(u) && \text{in } \Omega \\ u &= u_0 && \text{on } \partial\Omega\,. \end{aligned}$$

(c) *Stationary Navier–Stokes equation.*

For $\Omega \subset \mathbb{R}^3$ find $u\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ (the velocity field) and $p\colon \Omega \to \mathbb{R}$ (the pressure) such that

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u &= -\nabla p + f && \text{in } \Omega \\ \mathrm{div} u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad .$$

Here $\nu > 0$ is the viscosity, f is an outer force (e.g. gravity) and $((u\cdot\nabla)u)_j = \sum_{i=1}^3 u_i\partial_i u_j$ is a convective term describing the transport of fluid particles with the flow.

(3) *Variational problems*

(i) **First example.** We want to find the shortest curve between two points $(0,a)$ and $(1,b)$ in $\mathbb{R}^2$. In a graph formulation this corresponds to finding a function $u\colon [0,1] \to \mathbb{R}$, $u(0) = a$, $u(1) = b$ which minimises the *length functional*

$$L(v) := \int_0^1 \sqrt{1 + |v'(x)|^2}\,\mathrm{d}x$$

under all curves with the same boundary conditions, that is, under all $v \in M := \{v \in C^1([0,1]) \mid v(0) = a, v(1) = b\}$. If u is a minimiser of L on M, then it satisfies $L(u) \leqslant L(v)$ for all $v \in M$.

We are going to derive the corresponding Euler-Lagrange equations. Let $u \in M$ and $\phi \in M_0 := \{\phi \in C^1([0,1]) \mid \phi(0) = 0, \phi(1) = 0\}$. Then $u + \varepsilon\phi \in M$ for all $\varepsilon \in \mathbb{R}$.

[1] We use the notation $\nabla u = (\partial_1 u, \ldots, \partial_n u)$, where $\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$, for the vector of partial derivatives of u. If $u\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by the components $u_1, \cdots, u_m$, then ∇u denotes the matrix $(\partial_j u_i), i = 1, \cdots m, j = 1, \cdots n$. Another notation is Du.

Define $\mathscr{I}(\varepsilon) := L(u + \varepsilon\phi)$. If u is a minimiser of L then $\mathscr{I}'(0) = 0$.
We now compute $\mathscr{I}'(0)$:

$$\mathscr{I}'(0) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathscr{I}(\varepsilon) = \int_0^1 \frac{u'(x)\phi'(x)}{\sqrt{1 + |u'(x)|^2}}\,\mathrm{d}x.$$

In an abstract form we can write the conclusion $\mathscr{I}'(0) = 0$ as

$$\langle A(u), \phi\rangle = 0 \quad \text{for all } \phi \in M,$$

where we consider A to be the operator that assigns to each $u \in M$ the linear map $A(u)$ on M_0 that is defined by

$$\langle A(u), \phi\rangle = \int_0^1 \frac{u'(x)\phi'(x)}{\sqrt{1 + |u'(x)|^2}}\,\mathrm{d}x.$$

Hence we are looking — in a weak form — for a zero of the operator A in the set M.
If we know that $u \in C^2([0,1])$ we can integrate by parts to obtain

$$0 = \int_0^1 -\left(\frac{u'(x)}{\sqrt{1 + |u'(x)|^2}}\right)' \phi(x)\,\mathrm{d}x \quad \text{for all } \phi \in M_0$$

which gives the corresponding *Euler–Lagrange equation* to L

$$-\left(\frac{u'}{\sqrt{1 + |u'|^2}}\right)' = 0 \text{ in } (0,1), \quad u(0) = a,\ u(1) = b.$$

Remarks.

(a) Easy to solve: integrate and square to obtain that $u' \equiv \text{const}$. Hence u describes the line segment connecting $(0, a)$ and $(1, b)$.

(b) In general not clear that solution of Euler–Lagrange equation is also minimiser. Here it is, since L is convex.

(c) We can do the same for surfaces that are described as graphs of functions $v\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. Set

$$I(v) = \mathit{Area}(\mathit{graph}(v)) = \int_\Omega \sqrt{1 + |\nabla v|^2}\,\mathrm{d}x, \quad v = v_0 \text{ on } \partial\Omega.$$

We obtain

$$\mathscr{I}'(0) = \int_\Omega \frac{\nabla u \cdot \nabla \phi}{\sqrt{1 + |\nabla u|^2}} \, \mathrm{d}x$$

and the Euler–Lagrange equation:

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0.$$

For $n > 1$ this is in general not easy to solve!

(ii) Now we consider the corresponding *obstacle problem*: Find a curve of shortest length which connects $(0, a)$ and $(1, b)$ and lies above an obstacle given by $h \in C^1([0, 1])$.

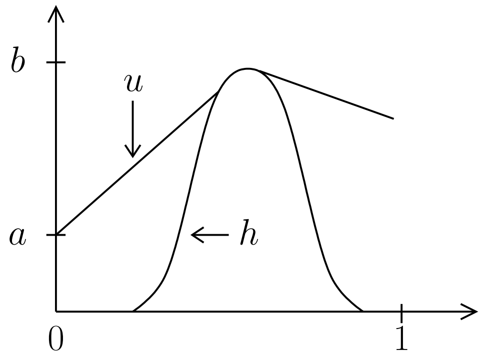


Figure 1: An obstacle problem

Thus we want to find minimiser u of $I(v)$ on

$$M := \{v \in C^1([0, 1]) \mid v(0) = a, v(1) = b, v(x) \geqslant h(x) \ \forall x \in [0, 1]\}.$$

To make sure that M is non-empty we require that $h(0) \leqslant a$ and $h(1) \leqslant b$. M is a convex set. Hence with $u, v \in M$ we also have $u + \varepsilon(v - u) \in M$ for all $\varepsilon \in [0, 1]$. If u is minimiser then $I(u) \leqslant I(u + \varepsilon(v - u))$ for all $\varepsilon \in [0, 1]$ and $v \in M$. Hence $\mathscr{I}'(0) \geqslant 0$ which gives the *variational inequality* $\langle A(u), v - u\rangle \geqslant 0$ for all $v \in M$ with A as above.

Remark. The above examples are given without specifying properties of the solutions, e.g. continuity or differentiability. In fact, one of the main tasks to apply Fixed Point Theorems and related results is to identify a suitable subset of an appropriate function space in which we are looking for a solution.

Chapter 1

Fixed Point Theorems

This section will discuss three fixed point theorems: the Contraction Mapping Theorem, Brouwer's Theorem and Schauder's Theorem.

Definition 1. Let (X, d) be a metric space and $T\colon M \subset X \to X$ be a map. A solution of $Tx = x$ is called a *fixed point* of T.

We will see several fixed point theorems with different assumptions on the space X and the map T respectively.

Example. Find a zero of a map $F\colon M \subset X \to X$. This problem can be formulated in different ways as a fixed-point problem, e.g.

(i) $Tx\colon = x - F(x)$

(ii) $Tx\colon = x - wF(x)$, $w \in \mathbb{R}$ (linear relaxation)

(iii) $Tx\colon = x - \big(DF(x)\big)^{-1}F(x)$ (Newton's method)

1.1 Banach's FPT and applications to ODEs

1.1.1 Banach's Fixed point theorem (= CMT)

Strategy. Find fixed point of map T as limit of the iteration defined by $x_{n+1} = Tx_n$.

Definition 2. Let (X, d) be a metric space and $T\colon M \subseteq X \to X$.
T is a contraction if there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leqslant kd(x, y) \quad \forall x, y \in M.$$

Warning: "Contraction" is not used is a uniform way in the literature and the property asked for in the above definition is called 'strongly contractive' or 'κ-contractive' in some books.

Theorem 1.1 (Contraction Mapping Theorem)**.** *Let* (X, d) *be a* complete *metric space, let* $M \subseteq X$ *be non-empty and closed and* $T : M \subset X \to M$ *a contraction. Then*

1. *the equation* $Tx = x$ *has a unique solution* $\overline{x} \in M$*;*

2. *the sequence* (x_n) *defined via* $x_{n+1} = Tx_n$ *converges for every* $x_0 \in M$ *to* $\overline{x}$*.*

Proof. Note that X is complete, and $M \subset X$ is non-empty and closed, therefore (M, d) is also a complete metric space. So it is sufficient to consider the case $M = X$.

- Uniqueness. Suppose that there are $a, b \in E$, $a \neq b$ such that $Ta = a$ and $Tb = b$. Then $d(Ta, Tb) = d(a, b)$, and since T is contractive $d(Ta, Tb) < d(a, b)$, which is a contradiction.

- Existence. Given $x_0 \in X$, consider the iterated sequence $x_{n+1} = Tx_n$. By assumption,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leqslant kd(x_n, x_{n-1}),$$

  and by induction

$$d(x_{n+1}, x_n) \leqslant k^n d(x_1, x_0).$$

  The previous inequality and the triangle inequality yield, for any $m \geqslant 0$,

$$\begin{aligned} d(x_{n+m}, x_n) &\leqslant \sum_{p=0}^{m-1} d(x_{n+p+1}, x_{n+p}) \\ &\leqslant \sum_{p=0}^{m-1} k^{n+p} d(x_0, x_1) \\ &= k^n \frac{1 - k^m}{1 - k} d(x_0, x_1) \\ &\leqslant \frac{k^n}{1 - k} d(x_0, x_1). \end{aligned}$$

  Since $k < 1$, we find $d(x_n, x_{n+m}) \to 0$ as $n \to \infty$. Hence (x_n) is a Cauchy sequence in X and since X is complete it has a limit $\overline{x}$. Next, note that a contractive map is continuous, therefore

$$T\overline{x} = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = \overline{x}.$$

□

Remarks.

(i) All assumptions in the Theorem are necessary (see Problem Sheet 0).

(ii) The most difficult part in applications is often to show that T maps M into itself.

(iii) We also have the following error estimates (see Problem Sheet 0):

$$\begin{aligned} d(x_n, \overline{x}) &\leqslant \frac{k^n}{1-k} d(x_1, x_0) && \text{(a-priori error estimate)}, \\ d(x_{n+1}, \overline{x}) &\leqslant \frac{k}{1-k} d(x_{n+1}, x_n) && \text{(a-posteriori error estimate)}, \\ d(x_{n+1}, \overline{x}) &\leqslant k d(x_n, \overline{x}) && \text{(linear convergence rate)}. \end{aligned}$$

This result is also called Banach's Fixed Point Theorem, and Picard Iteration Theorem (depending on the country).

1.1.2 Application to Initial value problems

Let X be a Banach space endowed with the norm $\|\cdot\|_X$ and let $t_0 \in \mathbb{R}$ and $y_0 \in X$ be given. We consider the following problem: for $t_0 \in \mathbb{R}$ and a Banach space X we look for a function $y \colon [t_0 - c, t_0 + c] \to X$ such that

$$y'(t) = f(t, y(t)), \qquad t \in (t_0 - c, t_0 + c), \qquad y(t_0) = y_0, \tag{C}$$

where $f \colon \mathbb{R} \times X \to X$ is continuous and $y_0 \in X$. We will consider the equivalent integral equation problem given by

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, \mathrm{d}s, \tag{C1}$$

and view it as a fixed point problem in an appropriate space. In what follows, for any $T > 0$ and $r_0 > 0$, we write

$$\begin{aligned} Q(T, r_0) &= [t_0 - T, t_0 + T] \times \bar{B}(y_0, r_0) \\ &= \{(t, y) \in \mathbb{R} \times X \colon |t - t_0| \leqslant T, \|y - y_0\|_X \leqslant r_0\} \end{aligned}$$

Remark. Strictly speaking, we do not know yet how differentiation and integration of Banach space valued functions are defined. You can in the following assume that $X = \mathbb{R}^N$, but statement and proof of the following theorem do not change in the general case of a Banach space X.

Theorem 1.2 (Cauchy-Lipschitz). *We assume that for some positive numbers a, b, L, K, the following assumptions hold:*

(a) $f \colon Q(a,b) \subset \mathbb{R} \times X \to X$ *is continuous*

(b) f *is L-Lipschitz-continuous with respect to* y *in* $Q(a,b)$*, that is*

$$\|f(t,y) - f(t,\tilde{y})\|_X \leqslant L \|y - \tilde{y}\|_X \quad \forall (t,y), (t,\tilde{y}) \in Q(a,b),$$

(c) f *is bounded by* K *on* $Q(a,b)$*, that is,*

$$\sup_{(t,y)\in Q} \|f(t,y)\|_X \leqslant K.$$

Then we have

(i) There exists a unique solution y *to* (C1) *in* $[t_0 - c, t_0 + c]$ *with* $c := \min\left(a, \frac{b}{K}\right)$*;*

(ii) The sequence

$$y_{n+1}(t) = y_0(t) + \int_{t_0}^{t} f(s, y_n(s)) ds$$

converges uniformly on $[t_0 - c, t_0 + c]$ *to* y*.*

Remark. Also called Picard-Lindelöf Theorem. If X is R^N (finite dimensional), assumption (c) follows from (a) by compactness of closed bounded sets in R^{N+1}. If f is not Lipschitz continuous locally, the solution may not be unique (see Problem Sheet 1).

Proof. We introduce the space $Z := C([t_0 - c, t_0 + c], X)$. Z is a Banach space endowed with the norm

$$\|y\|_0 := \max_{t\in[t_0-c,t_0+c]} \|y(t)\|_X .$$

We also introduce

$$\|y\|_1 := \max_{t\in[t_0-c,t_0+c]} \mathrm{e}^{-L|t-t_0|} \|y(t)\|_X .$$

Then

$$\mathrm{e}^{-Lc} \|y\|_0 \leqslant \|y\|_1 \leqslant \|y\|_0 \quad \forall y \in Z$$

and hence $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent and $(Z, \|\cdot\|_1)$ is also a Banach space.
Let $M := \{y \in Z \mid \|y - y_0\|_0 \leqslant b\}$ and define

$$T\colon M \subset (Z, \|\cdot\|_1) \to (Z, \|\cdot\|_1)$$

via

$$(Ty)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s.$$

We are going to show that T and M satisfy the conditions of the Contraction Mapping Theorem.

(1) *M is closed.* Let $(y_n) \subset M$ and $y_n \to y$ in $(Z, \|\cdot\|_1)$. Then due to the equivalence of the norms $y_n \to y$ in $(Z, \|\cdot\|_0)$. Since $(y_n) \subset M$ we have $\|y_n - y_0\|_0 \leqslant b$ for all $n \in \mathbb{N}$ and passing to the limit $n \to \infty$ in this inequality we find $\|y - y_0\|_0 \leqslant b$. Hence $y \in M$ and M is closed.

(2) $T\colon M \to M$. We have for $y \in M$ that

$$\begin{aligned} \|Ty - y_0\|_0 &= \max_{t\in[t_0-c,t_0+c]} \left\| \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s \right\|_0 \\ &\leqslant \max_{t\in[t_0-c,t_0+c]} \int_{t_0}^{t} \|f(s, y(s))\|_0 \,\mathrm{d}s \\ &\leqslant cK \\ &\leqslant b. \end{aligned}$$

Hence $Ty \in M$.

(3) *T is strongly contractive.* Using the Lipschitz continuity of f in y we find

$$\begin{aligned} \|Ty - T\tilde{y}\|_1 &= \max_{t\in[t_0-c,t_0+c]} \mathrm{e}^{-L|t-t_0|} \left\| \int_{t_0}^{t} \left(f(s, y(s)) - f(s, \tilde{y}(s))\right) \mathrm{d}s \right\|_X \\ &\leqslant \max_{t\in[t_0-c,t_0+c]} \mathrm{e}^{-L|t-t_0|} \int_{t_0}^{t} L \,\|y(s) - \tilde{y}(s)\|_X \,\mathrm{d}s \\ &= \max_{t\in[t_0-c,t_0+c]} \mathrm{e}^{-L|t-t_0|} \\ &\qquad \cdot \int_{t_0}^{t} L \underbrace{\|y(s) - \tilde{y}(s)\|_X \,\mathrm{e}^{-L|s-t_0|}}_{\leqslant \|y-\tilde{y}\|_1} \mathrm{e}^{L|s-t_0|} \,\mathrm{d}s \\ &\leqslant L\,\|y - \tilde{y}\|_1 \max_{t\in[t_0-c,t_0+c]} \int_{t_0}^{t} \mathrm{e}^{L|s-t_0|} \mathrm{e}^{-L|t-t_0|} \,\mathrm{d}s \\ &= L\,\|y - \tilde{y}\|_1 \max_{t\in[t_0-c,t_0+c]} \frac{1}{L} \left(\mathrm{e}^{L|t-t_0|} - 1\right) \mathrm{e}^{-L|t-t_0|} \\ &\leqslant \left(1 - \mathrm{e}^{-Lc}\right) \|y - \tilde{y}\|_1 \end{aligned}$$

Hence T is strongly contractive on M in $(Z, \|\cdot\|_1)$ with $k = 1 - \mathrm{e}^{-Lc}$.

The Contraction Mapping Theorem implies existence and uniqueness of a solution and convergence of y_n to y in $(Z, \|\cdot\|_1)$ and consequently also in $(Z, \|\cdot\|_0)$. □

Remark (Error estimates). We have

(i) $\|y_n - y\|_1 \leqslant \dfrac{k^n}{1-k} \|y_1 - y_0\|_1$.

(ii) $\|y_{n+1} - y\|_1 \leqslant \dfrac{k}{1-k} \|y_{n+1} - y_n\|_1$.

(iii) $\|y_{n+1} - y\|_1 \leqslant k \|y_n - y\|_1$.

These estimates follow from Remark (iii) in 1.1.1.

1.2 Brouwer's FPT and Calculus of Variations

1.2.1 The Fixed point Theorem of Brouwer

The Contraction Mapping Theorem only required few prerequisites on the space X but strong conditions on the map T.
For Brouwer's (and later Leray-Schauder's) Fixed Point Theorem we only need continuity of T but strong conditions on the space X. We will also lose uniqueness of a fixed point and (x_n), defined via $x_{n+1} = Tx_n$ does not necessarily converge.

Theorem 1.3 (Brouwer's Fixed Point Theorem)**.** *Let $K \subset \mathbb{R}^n$ be homeomorphic to $\overline{B_R(0)} \subset \mathbb{R}^n$ and let $T\colon K \to K$ be continuous. Then T has a fixed point.*

An important special case of this Theorem is

Corollary 1.4. *Let $\overline{B_R(0)} \subset \mathbb{R}^n$ and let $T\colon \overline{B_R(0)} \to \overline{B_R(0)}$ be continuous. Then T has a fixed point.*

Remarks.

(i) The proof is very simple for $n = 1$: Consider a continuous $f\colon [a,b] \to [a,b]$. Then f has a fixed point $\overline{x}$.

Proof. Define $g(x) := f(x) - x$. Then $g(a) \geqslant 0$ and $g(b) \leqslant 0$. Since g is continuous, there exists a zero $\overline{x}$ of g in $[a,b]$. □

(ii) Brouwer's Fixed Point Theorem is false in infinite-dimensional spaces. The reason behind this is that a closed ball is not compact. A counter-example is provided Problem Sheet 2, but a Theorem due to Kakutani shows that counter-examples exists in any infinite-dimensional separable Hilbert space. Note also that in an infinite dimensional space, a continuous function may well be unbounded on closed and bounded sets (and this in fact makes the study of nonlinear equations much more difficult).

The proof of Brouwer's Fixed Point Theorem in dimensions larger than 1 is not simple. There are many different ways of proving it and you might have seen a proof already in another lecture (e.g. Topology). We choose a relatively simple analytical proof which connects to elements from the Calculus of Variations.

1.2.2 Calculus of Variations

We consider so-called energy functionals of the form

$$I(v) := \int_\Omega L(\nabla v(x), v(x), x)\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^n$ is a smooth domain,
$v\colon \Omega \to \mathbb{R}^m$,
$L\colon \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \Omega \to \mathbb{R}$ is smooth,
and we use the notation $L = L(p, z, x) = L(p_{11}, \dots, p_{mn}, z_1, \dots, z_m, x_1, \dots, x_n)$ where $p = (p_{ij}) \in \mathbb{R}^{m\times n}$, $z = (z_i) \in \mathbb{R}^m$ and $x = (x_j) \in \Omega$.
L is called the *Lagrangian*.

Example. If $p = \nabla v$ then $p_{ij} = \partial_j v_i$, $i = 1, \dots, m$, $j = 1, \dots, n$. We also use the notation $L_{p_{ij}} = \partial_{p_{ij}} L$, $L_{z_i} = \partial_{z_i} L$.

Let now $g\colon \partial\Omega \to \mathbb{R}^m$ be a given smooth function. We are looking for the minimiser u of I in the set

$$M = \{v\colon \overline{\Omega} \to \mathbb{R}^m, v \text{ smooth}, v = g \text{ on } \partial\Omega\}.$$

Let u be a smooth minimiser of I in M (the existence of such a function is in general not clear). Then u solves the corresponding Euler–Lagrange equations which we will compute now.
Let $\phi \in C_0^\infty(\Omega)^m$ (where $C_0^\infty(\Omega)^m = \big\{\phi\colon \overline{\Omega} \to \mathbb{R}^m, \phi \in C^\infty(\overline{\Omega})^m, \mathrm{supp}\phi \subset\subset \Omega\big\}$). Then $u + \varepsilon\phi \in M$ for all $\varepsilon \in \mathbb{R}$. We consider $\mathscr{I}(\varepsilon) := I(u + \varepsilon\phi)$ and since u is a minimiser, we

have $\mathscr{I}'(0) = 0$. We compute $\mathscr{I}'(\varepsilon)$:

$$\begin{aligned}\mathscr{I}'(\varepsilon) &= \int_\Omega \sum_{j=1}^n \sum_{i=1}^m L_{p_{ij}}(\nabla u + \varepsilon \nabla \phi, u + \varepsilon\phi, x)\partial_j \phi_i \,\mathrm{d}x \\ &+ \int_\Omega \sum_{i=1}^m L_{z_i}(\nabla u + \varepsilon \nabla \phi, u + \varepsilon\phi, x)\phi_i \,\mathrm{d}x.\end{aligned}$$

Hence $\mathscr{I}'(0) = 0$ implies

$$\begin{aligned}0 &= \int_\Omega \sum_{j=1}^n \sum_{i=1}^m L_{p_{ij}}(\nabla u, u, x)\partial_j \phi_i \,\mathrm{d}x + \int_\Omega \sum_{i=1}^m L_{z_i}(\nabla u, u, x)\phi_i \,\mathrm{d}x \\ &= -\int_\Omega \sum_{j=1}^n \partial_j \left(\sum_{i=1}^m L_{p_{ij}}(\nabla u, u, x)\right) \phi_i \,\mathrm{d}x + \int_\Omega \sum_{i=1}^m L_{z_i}(\nabla u, u, x)\phi_i \,\mathrm{d}x.\end{aligned}$$

Since this equation is true for all $\phi \in C_0^\infty(\Omega)^m$ we find that the minimiser u satisfies the following system of PDE:

$$\begin{aligned}-\sum_{j=1}^n \partial_j \left(L_{p_{ij}}(\nabla u, u, x)\right) + L_{z_i}(\nabla u, u, x) = 0 \quad &\text{in } \Omega \qquad \text{for all } i = 1, \ldots, m \\ u = g \quad &\text{on } \partial\Omega.\end{aligned}$$

These are the *Euler–Lagrange equations for I*.

Example.

$$I(v) = \int_\Omega \frac{1}{2} |\nabla v|^2 - F(v)$$

with $v \colon \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^m$ and a differentiable function $F \colon \mathbb{R}^m \to \mathbb{R}$. Here

$$L(P, z) = \frac{1}{2}|P|^2 - F(z) = \frac{1}{2}\left((p_{11})^2 + \cdots + (p_{mn})^2\right) - F(z).$$

Hence $\partial_{p_{ij}} L(P, z) = p_{ij}$ and thus $\partial_{p_{ij}} L(\nabla u) = \partial_j u_i$. Furthermore $L_{z_i} = -\partial_{z_i} F$ so that the EL equations are given by

$$0 = -\sum_j \partial_j(\partial_j u_i) - \partial_{z_i} F(u) \quad \text{for all } i = 1, \ldots, m$$

or

$$0 = -\Delta u - \nabla F(u).$$

1.2.3 Null-Lagrangians

Definition 3. L is called a *Null-Lagrangian* if the corresponding Euler-Lagrange equations are satisfied by all smooth functions (i.e. all C^∞-functions).

Example. Suppose $u:\Omega\to\mathbb{R}^n$. The Lagrangian $L(\nabla u,u,x)=u\cdot\partial_1 u$ is a Null-Lagrangian since

$$-\sum_{j=1}^{n}\partial_j\left(\partial_{p_j}L(\nabla u,u,x)\right)+L_z(\nabla u,u,x)=-\partial_1\left(u\right)+\partial_1 u=0.$$

Proposition 1.5. *Let L be a Null-Lagrangian, let $u,v\colon\overline{\Omega}\to\mathbb{R}^m$ be two smooth functions with $u=v$ on $\partial\Omega$. Then $I(u)=I(v)$, i.e. for a Null-Lagrangian I depends only on the boundary values.*

Proof. Let $\mathscr{I}\colon[0,1]\to\mathbb{R}$, $\mathscr{I}(\tau)=I(\tau u+(1-\tau)v)$. Then

$$\begin{aligned}\mathscr{I}'(\tau) &= \int_\Omega\sum_{j=1}^{n}\sum_{i=1}^{m}L_{p_{ij}}(\tau\nabla u+(1-\tau)\nabla v,\tau u+(1-\tau)v,x)(\partial_j u_i-\partial_j v_i)\,\mathrm{d}x\\ &\quad+\int_\Omega\sum_{i=1}^{m}L_{z_i}(\tau\nabla u+(1-\tau)\nabla v,\tau u+(1-\tau)v,x)(u_i-v_i)\,\mathrm{d}x\\ &= \sum_{i=1}^{m}\int_\Omega\Big\{-\sum_{j=1}^{n}\partial_j L_{p_{ij}}(\tau\nabla u+(1-\tau)\nabla v,\tau u+(1-\tau)v,x)\\ &\qquad+L_{z_i}(\tau\nabla u+(1-\tau)\nabla v,\tau u+(1-\tau)v,x)\Big\}(u_i-v_i)\,\mathrm{d}x\end{aligned}$$

since $u-v=0$ on $\partial\Omega$. As $\tau u+(1-\tau)v$ is smooth and hence a solution to the EL equation we find that $\mathscr{I}'(\tau)=0$ and consequently $\mathscr{I}\equiv\text{const}$ on $[0,1]$, hence $\mathscr{I}(0)=\mathscr{I}(1)$ and thus $I(u)=I(v)$. □

We are going to show

Proposition 1.6. *The determinant is a Null-Lagrangian.*

To prove this we need a few technical Lemmas.

Lemma 1.7. *Let $g\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^{n-1}$ be a C^2 function and let $B^{(i)}$ be the $(n-1)\times(n-1)$ matrix obtained from dg by removing the i-th column (i.e. $\partial_i g$).*
Then, we have

$$\sum_{i=1}^{n}(-1)^i\,\partial_i\big(\det(B^{(i)})=0. \tag{1}$$

Proof. We can write

$$\partial_i\big(\det(B^{(i)})\big) = \sum_{\substack{j=1\\ j\neq i}}^{n} \det(C^{(i,j)})$$

where the matrix $C^{(i,j)}$ is obtained from dg by removing the i-th column and by replacing the j-th column by $\partial_i\partial_j g$.
As $\partial_i\partial_j g = \partial_j\partial_i g$ the matrices $C^{(i,j)}$ and $C^{(j,i)}$ have the same column vectors, though not appearing in the same order and one easily checks that

$$\det(C^{(i,j)}) = (-1)^{j+i-1}\det(C^{(j,i)}).$$

Thus

$$\alpha := \sum_{i=1}^{n} (-1)^i\,\partial_i\big(\det(B^{(i)}\big) = \sum_{\substack{i,j=1\\ i\neq j}}^{n} (-1)^i\det(C^{(i,j)}) = \sum_{\substack{i,j=1\\ i\neq j}}^{n} (-1)^{2i+j+1}\det(C^{(j,i)}) = -\alpha$$

so $\alpha = 0$ as claimed. □

Recall that the inverse of a matrix A can be expressed in terms of $\det(A)$ and the cofactor matrix $\mathrm{cof}(A)$ where

$$(\mathrm{cof}A)_{ij} = (-1)^{i+j}\det A^{ij}, \quad i,j = 1,\dots,n,$$

$A^{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ the matrix obtained from A by deleting the i-th row and the j-th column.

Lemma 1.8. *Let $u\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 function. Then*

$$\sum_{j=1}^{n} \partial_j(\mathrm{cof}\nabla u)_{ij} = 0 \quad \textit{for } i = 1,\dots,n.$$

Proof. We have

$$(\mathrm{cof}\nabla u)_{ij} = (-1)^{i+j}\det\big(\partial_p u^l\big)_{p\neq j, l\neq i}$$

For a given j, applying Lemma 1.7 to the function

$$g = \big(u^1,\dots,u^{i-1},u^{i+1},\dots,u^n\big)$$

gives the announced result. □

Proof of Proposition 1.6. Let $L(P) = \det P$. From Cramer's rule we know that

$$(\det P)\mathrm{Id} = P^T \mathrm{cof} P. \tag{2}$$

In particular, for every k

$$(\det P) = (P^T \mathrm{cof} P)_{kk} = \sum_{l=1}^{n} p_{lk} (\mathrm{cof} P)_{lk}. \tag{3}$$

Given $1 \leqslant i, j \leqslant n$ we choose $k = j$ and obtain

$$\frac{\partial \det P}{\partial p_{ij}} = \sum_{l=1}^{n} \delta_{li} (\mathrm{cof} P)_{lj} + p_{lj} \frac{\partial (\mathrm{cof} P)_{lj}}{\partial p_{ij}} = (\mathrm{cof} P)_{ij}. \tag{4}$$

We need to show that $\sum_{j=1}^{n} \partial_j (L_{p_{ij}}(\nabla u)) = 0$ for all smooth functions. Since $L_{p_{ij}}(\nabla u) = (\mathrm{cof} \nabla u)_{ij}$ due to (4) the statement follows from the previous Lemma. □

Another example for a Null-Lagrangian is

$$L(P) = \mathrm{tr}(P^2) - (\mathrm{tr}(P))^2,$$

see Problem Sheet 2.

1.2.4 Proof of Brouwer's FPT and the retraction principle

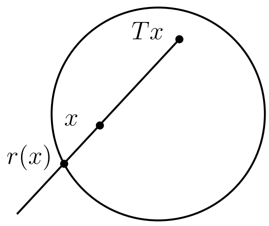
The proof of Brouwer's FPT relies on the retraction principle:

Definition 4. A retraction from $A \subset \mathbb{R}^n$ to $B \subset A$ is a continuous map $r : A \to B$ so that $r(x) = x$ for every $x \in B$.

Theorem 1.9 (Retraction Principle (for balls))**.** *There exists no retraction from $B = \overline{B_1(0)} \subset \mathbb{R}^n$ to ∂B, i.e. there is no continuous map $r : B \to \partial B$ so that $r|_{\partial B} = id$.*

We prove this result below and first explain how it leads to the proof of Brouwer's FPT.

Proof of Brouwer's Fixed Point Theorem. .
It is enough to consider the case that $\Omega = B := \overline{B_1(0)} \subset \mathbb{R}^n$ for if $T : K \to K$ is continuous and $h \colon \overline{B_1(0)} \to K$ is a homeomorphism (that is, h is bijective and both h and h^{-1} are continuous) then $\tilde{T} := h^{-1} \circ T \circ h$ is a continuous map from B to B and any fixed point $\tilde{x}_0$ of $\tilde{T}$ yields a fixed point $x_0 = h(\tilde{x}_0)$ of T.



We argue by contradiction:
Assume that $T : B \to B$ is so that for all $x \in B$, $Tx \neq x$. We may therefore, for a given x define the line l_x passing through Tx and x,

$$Tx + \lambda(x - Tx),\ \lambda \in \mathbb{R}$$

This line cuts ∂B in two points, given by the two solutions of quadratic equation in λ

$$\lambda^2 \|x - Tx\|^2 + 2\lambda(Tx, x - Tx) + \|Tx\|^2 - 1 = 0,$$

Choose the positive root, $\lambda(x)$, a continuous function of x. Then

$$r : x \to Tx + \lambda(x)(x - Tx)$$

is continuous, and contradicts Theorem 1.9.
Explicitly, introducing $Qx = (x - Tx)/\|x - Tx\|$, we have

$$r(x) = Tx + Qx \left(\sqrt{(Qx, Tx)^2 + 1 - \|Tx\|^2} - (Qx, Tx) \right).$$

□

Remark. Theorem 1.9 (the retraction principle) is in fact equivalent to Brouwer's Fixed Point Theorem. See Problem Sheet 2 for the other implication.

Proof of the Retraction Principle (Thm. 1.9. Step 1. We first show that there is no smooth function $r\colon B \to \partial B$ such that $r(x) = x$ for all $x \in \partial B$.
Assume there exists such a r. Define $\tilde{r}(x) = x$. Then $\tilde{r} = r$ on ∂B and, due to Propositions 1.5 and 1.6 in 1.2.3,

$$\int_B \det \nabla r \,\mathrm{d}x = \int_B \det \nabla \tilde{r} \,\mathrm{d}x = \int_B 1 \,\mathrm{d}x = |B| \neq 0. \tag{$*$}$$

By assumption $|r(x)|^2 = 1$ for all $x \in B$ and hence $(\nabla r)^T r = 0$. Since $|r| = 1$ it follows that $(\nabla r)^T$ has eigenvalue 0. But then $\det \nabla r(x) = 0$ for all $x \in B$ which contradicts $(*)$.

Step 2. We show by approximation that there is no continuous function $r\colon B \to \partial B$ such that $r(x) = x$ for all $x \in \partial B$.
Assume there exists such a function r and extend it via $r(x) = x$ for all $x \in \mathbb{R}^n \backslash B$. Then $r(x) \neq 0$ for all $x \in \mathbb{R}^n$.
Define $r_\varepsilon(x) := (\phi_\varepsilon * r)(x)$ where $\phi_\varepsilon = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$ with $\phi(x) = C\mathrm{e}^{\frac{1}{|x|^2-1}}$ for $|x| < 1$ and C is such that $\int_{\mathbb{R}^n} \phi_\varepsilon(x)\,\mathrm{d}x = 1$. In particular, ϕ_ε is radially symmetric.
We know that $r_\varepsilon \to r$ locally uniformly on $\mathbb{R}^n$ (as r_ε is a mollification of r see C5.1a Lecture notes, Section 4.4) and hence $r_\varepsilon(x) \neq 0$ for all $x \in \mathbb{R}^n$ for sufficiently small $\varepsilon > 0$.
Let $x \in \partial B_2(0)$ and ε be small (smaller that 1/2). Then, due to the properties of ϕ,

$$r_\varepsilon(x) = \int_{B_\varepsilon(0)} \phi_\varepsilon(y)(x-y)\,\mathrm{d}y = x.$$

Define

$$\tilde{r}_\varepsilon(x) = \frac{r_\varepsilon(2x)}{|r_\varepsilon(2x)|}.$$

Then $\tilde{r}_\varepsilon\colon B \to \partial B$ is smooth and $\tilde{r}_\varepsilon(x) = x$ for all $x \in \partial B$ which is a contradiction, thanks to step 1. □

Corollary 1.10. *If $K \subset \mathbb{R}^n$ is homeomorphic to $\overline{B_1(0)}$ then there exists no retraction from K to ∂K.*

Proof. If $h\colon \overline{B_1(0)} \to K$ is a homeomorphism (that is, h is bijective and both h and h^{-1} are continuous), then any retraction r from K to ∂K would yield a retraction $\tilde{r} := h^{-1} \circ r \circ h$ of B to ∂B which would then contradict Theorem 1.9.

□

Proposition 1.11. *Let $g\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field such that $g(x) \cdot x \geqslant 0$ for all x with $|x| = R$. Then there exists an $x_0 \in \mathbb{R}^n$ with $|x_0| \leqslant R$ and $g(x_0) = 0$.*

Proof. Assume that there exists no such x_0. Then we can define

$$f(x) = -R\frac{g(x)}{|g(x)|}.$$

f is continuous and $f\colon \overline{B_R(0)} \to \overline{B_R(0)}$. Brouwer's FPT implies that there exists $x_1 \in \overline{B_R(0)}$ such that $f(x_1) = x_1$. Then $|x_1| = |f(x_1)| = R$ such that the assumption on g implies $g(x_1) \cdot x_1 \geqslant 0$.
On the other hand

$$g(x_1) \cdot x_1 = -f(x_1) \cdot x_1 \frac{|g(x_1)|}{R} = -\frac{|x_1|^2\,|g(x_1)|}{R} < 0,$$

which is a contradiction. □

This application is in fact equivalent to the contraction principle, and therefore to Brouwer's Fixed Point Theorem. Indeed, given r a continuous retraction on the closed ball $\overline{B_R(0)}$, define g by $g = r$ on $\overline{B_R(0)}$ and $g(x) = x$ elsewhere. It is continuous, and when $|x| = R$ we have $g(x) \cdot x = R^2 > 0$. But g never cancels since $|g| \geqslant R$, which is a contradiction.

Corollary 1.12. *Let $K \subset \mathbb{R}^n$ be convex, compact, and non-empty, and $T\colon K \to K$ be continuous. Then T has a fixed point.*

Proof. Step 1: If K has non-empty interior. We will prove that a convex compact subset of $\mathbb{R}^n$ with non-empty interior is homeomorphic to the closed unit sphere. The conclusion then follows. We may assume without loss of generality that the ball $B_r(0) \subset K \subset B_R(0)$ for some $0 < r < R < \infty$ — true up to a translation (an homeomorphism), since K has non empty interior. We then define the map (the *gauge* of K)

$$j(x) = \inf \left\{ t > 0 \text{ such that } \frac{x}{t} \in K \right\}$$

You will show the following properties on Problem Sheet 3,

(i) The map $j\colon \mathbb{R}^n \to R$ is continuous,

(ii) For all $\lambda \geqslant 0$, $j(\lambda x) = \lambda j(x)$,

(iii) For all $x \in \mathbb{R}^n$, $\|x\|/R \leqslant j(x) \leqslant \|x\|/r$,

(iv) $j(x) \leqslant 1$ if and only if $x \in K$.

and show that these properties allow you to define the desired homeomorphism $g : K \to \overline{B_1(0)}$ and its inverse h as

$$g(x) = \begin{cases} \frac{x}{\|x\|} j(x) & \text{when } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad \text{and } h(y) = \begin{cases} \frac{\|y\|}{j(y)} y & \text{when } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Step 2: General case. We can assume again that $0 \in K$. Either $K = \{0\}$, in which case the result is trivial, or there exists a maximum of $x_1, \ldots, x_m$ independent vectors in K, with $m \leqslant n$. By convexity, the m-simplex $(0, x_1, \ldots, x_m)$, which contains an m dimensional ball, is contained in K. If $m = n$, so K has non empty interior. If $m < n$, C is contained in a space of dimension m, and we apply the previous step in that space. □

We will use this last result to prove Schauder's Fixed Point Theorem.

1.3 Schauder's Fixed Point Theorem

1.3.1 Three versions of the theorem and their proof

We will introduce Schauder's Theorem in three different formulation, together with the closely related Leray-Schauder Theorem.

Theorem 1.13 (Schauder's FPT, version I)**.** *Let X be a Banach space, $K \subset X$ be non-empty, convex and compact and $T\colon K \subset X \to K$ be continuous. Then T has a fixed point in K.*

For the proof we use

Lemma 1.14. *Let K, T and X be as in Theorem 1.13. Then for any $\varepsilon > 0$ there exist a finite dimensional subspace L_ε of X and a continuous map $T_\varepsilon : K \to K \cap L_\varepsilon$ so that*

$$\|Tx - T_\varepsilon x\| < \varepsilon \textit{ for all } x \in K.$$

Proof of Lemma 1.14. Since T is continuous and K is compact, T is uniformly continuous. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in K$ we have $\|Tx - Ty\| \leqslant \varepsilon$ provided $\|x - y\| \leqslant \delta$. Furthermore, there exists a finite set of points $\{x_1, \dots, x_N\} \subset K$ such that $K \subset \underset{1 \leqslant i \leqslant N}{\cup} B_\delta(x_i)$, where $B_\delta(x_i)$ is the open ball of centre x_i and radius δ. The vector space $L_\varepsilon = \operatorname{span}\{Tx_i,\, 1 \leqslant i \leqslant N\}$ is finite dimensional, and therefore $K \cap L_\varepsilon$ is non-empty, compact, convex and finite dimensional.
For $i \leqslant j \leqslant N$, we define the continuous function $\psi_j\colon X \to \mathbb{R}$ by

$$\psi_j = \begin{cases} 0 & \text{if } \|x - x_j\| \geqslant \delta, \\ 1 - \frac{1}{\delta}\|x - x_j\| & \text{otherwise.} \end{cases}$$

We see that ψ_j is strictly positive on $B(x_j, \delta)$ and vanishes elsewhere. Therefore we have $\sum_{j=1}^{N} \psi_i(x) > 0$ for all $x \in K$, and this continuous function is therefore bounded below on K. We can therefore define a partition of unity,

$$\phi_i(x) = \frac{\psi_i(x)}{\sum_{j=1}^{N} \psi_i(x)} \text{ for all } 1 \leqslant i \leqslant N,$$

which satisfy $\sum_{i=1}^{N} \phi_i(x) = 1$ for all $x \in K$. We now define an approximation of T on K by

$$T_\varepsilon x = \sum_{i=1}^{N} \phi_i(x) T x_i \text{ for all } x \in K.$$

We have, for every $x \in K$,

$$T_\varepsilon x - Tx = \sum_{i=1}^{N} \phi_i(x) \, (Tx_i - Tx)$$

Whenever $\phi_i \neq 0$, we have $\|x - x_i\| < \delta$ and therefore $\|Tx - Tx_i\| < \varepsilon$. Thus,

$$\|T_\varepsilon x - Tx\| \leqslant \sum_{i=1}^{N} \phi_i(x) \, \|T_\varepsilon x - Tx\| \leqslant \varepsilon \sum_{i=1}^{N} \phi_i(x) = \varepsilon.$$

Finally note that the map $T_\varepsilon x$ is continuous, and takes its values in $K \cap L_\varepsilon$, since K is convex and $\tilde{T}x$ is a weighted average of Tx_i, $i = 1, \ldots, N$. □

Proof of Theorem 1.13. Take any sequence $\varepsilon_n \to 0$ and let T_{ε_n} and L_{ε_n} be as in Lemma 1.14. We can then apply Corollary 1.12 to conclude that

$$T_{\varepsilon_n} : L_{\varepsilon_n} \cap K \to L_{\varepsilon_n} \cap K$$

has a fixed point x_n. As K is compact a subsequence x_{n_j} of the x_n converges to a point x^* which is a fixed point of the original map T since

$$\|Tx^* - x^*\| \leqslant \left\|Tx^* - Tx_{n_j}\right\| + \left\|Tx_{n_j} - T_{\varepsilon_{n_j}} x_{n_j}\right\| + \left\|x_{n_j} - x^*\right\| \to 0$$

thanks to the continuity of T and Lemma 1.14. □

Remark. For a set M its *convex hull* $\operatorname{conv}(M)$ is defined as

$$\operatorname{conv}(M) := \left\{ y \in X \mid \exists y_i \in M, \alpha_i \in [0,1], \sum_{i=1}^{m} \alpha_i = 1 \text{ s.t. } y = \sum_{i=1}^{m} \alpha_i y_i \right\}.$$

We often need an alternative version to this Theorem,

Theorem 1.15 (Schauder's FPT, version II)**.** *Let X be a Banach space, $M \subset X$ be non-empty, convex and closed and $T \colon M \subset X \to M$ be a continous operator such that $T(M)$ is precompact. Then T has a fixed point.*

For the proof we need the following lemma whose proof can be found in the appendix.

Lemma 1.16 (Mazur)**.** *Let X be a Banach space and $M \subseteq X$ be precompact. Then* $\operatorname{conv}(M)$ *is precompact.*

Proof of Theorem 1.15. The previous Lemma implies that $\operatorname{conv}(T(M))$ is precompact. Hence $\overline{\operatorname{conv}(T(M))}$ is compact. Furthermore, it is non-empty and convex. Note that $T(M) \subset M$, and since M is convex, $\operatorname{conv}(T(M)) \subset M$, and because M is closed $\overline{\operatorname{conv}(T(M))} \subset M$. Thus, Schauder's Fixed Point Theorem can be applied to the continuous map $T\colon \overline{\operatorname{conv}(T(M))} \to \overline{\operatorname{conv}(T(M))}$, and T has a fixed point in $\overline{\operatorname{conv}(T(M))}$. □

Remark. In applications to nonlinear problems, this result will be used as follows. First, the problem is reformulated as a fixed-point problem, for some T. Then we select a space X where T is continuous, and a closed convex set M such that $T : M \to M$, where either M is compact, or $T(M)$ is precompact. To show this last property, it is sufficient to show that for any sequence $x_n \in M$, there exists a sub-sequence $T(x_{n_k})$ which converges in X —in particular we need not show that x_{n_k} itself converges.

In its second form, Schauder's Fixed Point Theorem implies a statement about *compact operators*. The general definition of compact operators is as follows. Let X, Y be Banach spaces.

Definition 5. Let $T\colon M \subseteq X \to Y$, T is a *compact operator* if

(i) T is continuous;

(ii) $\overline{T(B)}$ is compact in Y for all bounded subsets $B \subseteq M$.

Compact operators on bounded sets can be equivalently characterised as being those operators that can be approximated by continuous operators to finite dimensional spaces as described in Lemma 1.14, namely

Remark (Approximation of compact operators by "finite-dimensional" operators)*.* Let $M \subset X$ be bounded and $T\colon M \subset X \to Y$. Then the following are equivalent:

(1) T is a compact operator.

(2) For all $n \in \mathbb{N}$ there exists a compact operator $T_n\colon M \to Y$ such that $T_n(M) \subset Y_n$ with $\dim Y_n < \infty$ and $T_n \to T$ uniformly in M.

The proof, more or less contained in the proof of Schauder's Theorem is left as an (optional) exercise. In terms of compact operators, Schauder Theorem can be written as follows.

Theorem 1.17 (Schauder's FPT, version III)**.** *Let X be a Banach space, $M \subset X$ be non-empty, convex, bounded and closed and $T\colon M \subset X \to M$ be a compact operator. Then T has a fixed point.*

To conclude this section, let us mention the following variant of Schauder's Theorem, known as Leray-Schauder Fixed Point Theorem, or Schaefer's Theorem.

Theorem 1.18 (Leray-Schauder/Schaefer Theorem). *Let X be a Banach space and $T\colon X \to X$ be a compact map with the following property: there exists $R > 0$ such that the statement ($x = \tau T x$ with $\tau \in [0,1)$) implies $\|x\|_X < R$. Then T has a fixed point x^* such that $\|x^*\| \leqslant R$.*

Proof. For the proof, see Problem Sheet 3. □

Theorem 1.18 says that a sequence of *a priori* estimates on fixed points of τT, whether they exist or not, implies in fact the existence of a fixed point, a surprising result.

Corollary 1.19. *Let X be a Banach space and let $T : X \to X$ be compact so that there exist $b > 0$ and $a \in [0,1)$ so that*

$$\|Tx\| \leqslant a\,\|x\| + b \text{ for all } x \in X$$

then T has a fixed point.

1.4 Application: Peano's Existence Theorem for ODEs

Consider again as in Section 1.1.2 the initial value problem

$$y\colon [t_0 - a, t_0 + a] \to \mathbb{R}^N, \quad y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

We assume now that $f\colon [t_0 - a, t_0 + a] \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous but not necessarily Lipschitz-continuous w.r.t. y. We know that under these conditions solutions are in general not unique. The following theorem, however, shows that solutions always exist.

Theorem 1.20 (Peano). *Let $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^N$, $a, b > 0$ and*

$$Q := \{(t,y) \in \mathbb{R} \times \mathbb{R}^N \mid |t - t_0| \leqslant a, |y - y_0| \leqslant b\}.$$

Let $f\colon Q \to \mathbb{R}^N$ be continuous. Let $K = \max_Q |f(t,y)|$. Then there exists a continuous solution $y\colon [t_0 - c, t_0 + c] \to \mathbb{R}^N$, with $c = \min(a, b/K)$ of the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s.$$

Proof. We define as in Section 1.1.2 the map T via

$$(Ty)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s$$

and with $I := [t_0 - c, t_0 + c]$

$$M = \Big\{ y \in C^0(I, \mathbb{R}^N) \mid \|y\| = \max_{t \in I} |y(t)|\,, \|y - y_0\| \leqslant b \Big\}.$$

Then M is non-empty, convex, closed (and bounded). We have

$$\|Ty - y_0\| \leqslant \max_{t \in I} \Big| \int_{t_0}^{t} |f(s, y(s))|\,\mathrm{d}s \Big| \leqslant cK \leqslant b,$$

therefore $T(M) \subset M$. We show that T is continuous: let $y_n \to y$ in M. Then

$$\max_{t \in I} |(Ty_n)(t) - (Ty)(t)| \leqslant \int_{t_0 - c}^{t_0 + c} |f(s, y_n(s)) - f(s, y(s))|\,\mathrm{d}s \to 0$$

as $n \to \infty$, since f is uniformly continuous on Q.
We show that $T(M)$ is pre-compact using Arzéla–Ascoli's Theorem (see Appendix). Using the assumptions we find $\sup_{y \in M} |(Ty)(t)| \leqslant |y_0| + aK$ and

$$\sup_{y \in M} |(Ty)(t_1) - (Ty)(t_2)| \leqslant K\,|t_1 - t_2| \to 0 \text{ as } |t_1 - t_2| \to 0.$$

Hence Schauder's FPT (version II) implies that T has a fixed point. □

Chapter 2

Applications to semilinear PDEs

Our goal in this section is to use Schauder's FPT to prove existence of solutions of equations of the form

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Remark. For simplicity we restrict ourselves to scalar equations and zero boundary conditions.

In the following we always assume that $\Omega \subset \mathbb{R}^n$ is open, bounded and smooth.

2.1 Some results for linear PDEs and Sobolev spaces

Here we recall some results from the course C4.3: Functional Analytic Methods for PDEs (and from A.04 Integration) that are needed later on in this section.

Proposition 2.1. *Let $g \in H^{-1}(\Omega)$ and $\mu \in [0, \infty)$. There exists a unique weak solution v_g of*

$$\begin{aligned} -\Delta v + \mu v &= g && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega. \end{aligned},$$

i.e. a unique function $v = v_g \in H_0^1(\Omega)$ that solves the variational problem

$$\forall\, w \in H_0^1(\Omega), \quad \int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x + \mu \int_\Omega vw \,\mathrm{d}x = \langle g, w \rangle. \tag{1}$$

Furthermore, the map

$$\begin{aligned} (-\Delta + \mu I_d)^{-1} : \quad H^{-1}(\Omega) &\to H_0^1(\Omega) \\ g &\mapsto v_g \end{aligned}$$

is continuous.

This is a consequence of Riesz Representation Theorem, and of Poincaré's inequality.

Notation. Note that by $(-\Delta + \mu I_d)^{-1} g$ we always mean the solution of the equation with zero boundary data.

The version of Poincaré inequality that we will use in these notes is as follows

Theorem 2.2 (Poincaré)**.** *For $p \in [1, \infty)$, there exists a constant $C = C(\Omega, p)$ such that*

$$\forall u \in W_0^{1,p}(\Omega) \quad \|u\|_{L^p(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega:\mathbb{R}^n)}$$

A key tool to obtain the compactness of the fixed point maps we will consider is the following theorem.

Theorem 2.3 (Rellich-Kondrachov)**.** *For $p \in [1, \infty)$,*

- *If $1 \leqslant p < n$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leqslant q < \frac{np}{n-p}$,*
- *If $p = n$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leqslant q < \infty$,*
- *If $p > n$, $W^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}(\Omega)$ for $0 \leqslant \gamma < 1 - \frac{n}{p}$,*

and all these embbedings are compact.

An important consequence for us is

Corollary 2.4. *Let $\mu \geqslant 0$. Then the map $g \mapsto (-\Delta + \mu I_d)^{-1} g$ is*

- *continuous as map from $L^2(\Omega)$ to $H_0^1(\Omega)$ in other words,*

$$\|v\|_{H_0^1(\Omega)} \leqslant C(\Omega)\|g\|_{L^2(\Omega)}$$

- *compact as map from $L^2(\Omega)$ to $L^2(\Omega)$*

Proof. The first part is due to the fact that $L^2(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$. The second part follows as $(-\Delta + \mu I_d)^{-1} : L^2(\Omega) \to L^2(\Omega)$ can be viewed as composition of the continuous map $(-\Delta + \mu I_d)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ and the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and as the composition of a compact linear operator and a continuous linear operator is again compact. □

Remark. As we shall see later also the first embedding is compact, though the proof of this is more involved.

Finally, let us clarify what $f(u)$ means when u is not a continuous function.

Lemma 2.5. *For $f \in C(\mathbb{R})$, and any two measurable functions u_1 and u_2 on Ω also $f(u_1)$ and $f(u_2)$ are measurable and if furthermore if $u_1 = u_2$ almost everywhere in Ω, then $f(u_1) = f(u_2)$ almost everywhere.*

Proof. If u_1 is measurable, so is $f(u_1)$ since the preimage of open sets under u is measurable and since f is continuous so the the preimage of any open set under f is again open. If $u_1 = u_2$ almost everywhere, there exists a set of zero measure N such that if $x \in \Omega \setminus N$ $u_1(x) = u_2(x)$. Then, certainly $f(u_1(x)) = f(u_2(x))$, therefore $f(u_1) = f(u_2)$ almost everywhere. □

Lemma 2.6. *Given $f \in C(\mathbb{R})$ such that $|f(t)| \leqslant a + b|t|^r$, where $a > 0$, $b > 0$ and $r > 0$ are positive constants. Then the map $u \mapsto f(u)$ is continuous from $L^p(\Omega)$ to $L^{p/r}(\Omega)$ for $p \geqslant \max(1, r)$ and maps bounded subsets of $L^p(\Omega)$ to bounded subsets of $L^{p/r}(\Omega)$*

Proof. Thanks to Jensen's inequality,

$$(a + b|t|^r)^{p/r} \leqslant 2^{p/r-1} a^{p/r} + 2^{p/r-1} b^{p/r} |t|^p \leqslant C(1 + |t|^p),$$

where C is a positive constant depending on a, b, p and r only. Since $u \in L^p(\Omega)$, we have

$$\int_\Omega |f(u)|^{p/r} \, \mathrm{d}x \leqslant C(a, b, p, r) \left(|\Omega| + \int_\Omega u^p \, \mathrm{d}x \right) < \infty,$$

therefore $f(u) \in L^{p/r}(\Omega)$. Let u_n be a sequence converging to u in $L^p(\Omega)$. There exists a subsequence $u_{n'}$ and a function $g \in L^p(\Omega)$ such that $u_{n'}$ converges almost everywhere to u, that is, for all $x \in \Omega \setminus N$ where N is a negligible set, $u_{n'}(x) \to u(x)$, and $|u_{n'}(x)| \leqslant g(x)$ almost everywhere. This is sometimes called the generalized DCT, or the partial converse of the DCT, or the Riesz-Fisher Theorem. From the continuity of f, $|f(u(x)) - f(u_{n'}(x))| \to 0$ on $\Omega \setminus N$, and

$$|f(u(x)) - f(u_{n'}(x))|^{p/r} \leqslant C(1 + g(x)^p + |f(u)|^p),$$

where C is another positive constant depending on a, b, p and r only. The left-hand-side is independent of n', and is in $L^1(\Omega)$. We can apply the Dominated Convergence Theorem to conclude that $\int_\Omega |f(u(x)) - f(u_{n'}(x))|^{p/r} \, \mathrm{d}x \to 0$, or in other words,

$$\|f(u(x)) - f(u_{n'}(x))\|_{L^{p/r}(\Omega)} \to 0.$$

Since the limit does not depend on the subsequence, this convergence holds for u_n. □

We now turn to applications.

2.2 Application I

We look for a weak solution $u\colon \Omega \to \mathbb{R}$ of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

under suitable conditions on $f\colon \mathbb{R} \to \mathbb{R}$.

Theorem 2.7. *Let $f \in C(\mathbb{R})$ and $\sup_{x\in R} |f(x)| = a < \infty$. Then (1) has a weak solution $u \in H_0^1(\Omega)$, i.e.*

$$\int_\Omega \nabla u \cdot \nabla \phi \,\mathrm{d}x = \int_\Omega f(u)\phi \,\mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega).$$

Proof. Our strategy is to apply Schauder's Fixed Point Theorem (version III) to the map

$$\begin{array}{rcl} T\colon L^2(\Omega) & \to & L^2(\Omega) \\ u & \mapsto & (-\Delta)^{-1}(f(u)) \end{array}$$

T is continuous. Lemma 2.6 shows that $u \to f(u)$ is continuous from $L^2(\Omega)$ into itself. Corollary 2.4 shows that $(-\Delta)^{-1}$ is continuous from $L^2(\Omega)$ into $H_0^1(\Omega)$, which is continuously embedded in $L^2(\Omega)$.

Find a closed, non-empty bounded convex set such that $T : M \to M$. Given $u \in L^2(\Omega)$, Tu satisfies

$$\int_\Omega \nabla Tu \cdot \nabla Tu \,\mathrm{d}x = \int_\Omega f(u) Tu \,\mathrm{d}x \leqslant a|\Omega| \|Tu\|_{L^2(\Omega)} \text{ Cauchy-Schwarz} \tag{2}$$

Therefore, using Poincaré's inequality

$$\|Tu\|_{L^2(\Omega)}^2 \leqslant C(\Omega) \|\nabla Tu\|_{L^2(\Omega)}^2 \leqslant a|\Omega| C(\Omega) \|Tu\|_{L^2(\Omega)}$$

Thus if we set $R = a|\Omega|C(\Omega)$ and choose

$$M = \{u \text{ s.t. } \|u\|_{L^2(\Omega)} \leqslant R\},$$

we have established that $T : M \to M$.

T is compact. Using Poincaré's inequality on the right-hand-side in (2), we obtain

$$\|\nabla Tu\|_{L^2(\Omega)}^2 \leqslant R \|\nabla Tu\|_{L^2(\Omega)}$$

Thus $T(M) \subset \{u \text{ s.t. } \|u\|_{H^1(\Omega)} \leqslant R\}$, and since the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, T is compact. □

The proof could also have been articulated differently, if we had defined $T : H_0^1(\Omega) \to H_0^1(\Omega)$ instead.

2.3 Application II

We look for a weak solution $u\colon \Omega \to \mathbb{R}$ of

$$\left.\begin{array}{rcll} -\Delta u + g(u, \nabla u) + \mu u & = & h & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array}\right\} \tag{1}$$

where $\mu \geqslant 0$, $b \in C(\mathbb{R} \times \mathbb{R}^n : \mathbb{R})$ grows at most linearly at infinity, i.e. there exists $M_1 > 0$,$M_2 > 0$ such that $|g(z,p)| \leqslant M_1 + M_2(|z| + |p|)$ for all $z \in \mathbb{R}$ and $p \in \mathbb{R}^n$, and $h \in L^2(\Omega)$. The following generalization of Lemma 2.6 shows that $B : u \mapsto b(u, \nabla)$ is continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$.

Lemma 2.8. *Let $g \in C(\mathbb{R} \times \mathbb{R}^n)$ be such that $|g(z,p)| \leqslant a + b\,|z|^\alpha + c\,|p|$, where a,b and c are non negative constants, and $0 < 2\alpha < 2^*$, where $2^* = 2n/(n-2)$ if $n \geqslant 3$, and $2* = \infty$ if $n = 1,2$. Then the map $u \mapsto g(u, \nabla u)$ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and maps bounded subsets of $H_0^1(\Omega)$ to bounded subsets of $L^2(\Omega)$.*

Proof. See Problem Sheet 4. □

Theorem 2.9. *If $M_2 = 0$, that is, if g is bounded, there exists a weak solution $u \in H_0^1(\Omega)$ of (1), i.e.*

$$\int_\Omega \nabla u \cdot \nabla \phi + g(u, \nabla u)\phi + \mu u \phi \,\mathrm{d}x = \int_\Omega h\phi \,\mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega).$$

If $M_2 \neq 0$ the result holds provided μ is larger than a constant depending on Ω and M_2 only.

Remark. If $g(u, \nabla u)$ is not bounded, there might not be a solution for an arbitrary μ. Take for example $g(u, \nabla) = -(\lambda + \mu)u$, where λ is a simple eigenvalue of the Δ, with a corresponding eigenvector ψ and choose h such that $\int_\Omega h\psi \,\mathrm{d}x \neq 0$. Then the Fredholm alternative shows that there is no solution, and it can be checked directly: if u is a solution, then we have

$$\int_\Omega \nabla u \cdot \nabla \psi \,\mathrm{d}x = \int_\Omega \lambda u \psi + h\psi \,\mathrm{d}x \text{ and } \int_\Omega \nabla \psi \cdot \nabla u \,\mathrm{d}x = \int_\Omega \lambda \psi u \,\mathrm{d}x$$

which implies $\int_\Omega h\psi \,\mathrm{d}x = 0$.

Proof of Theorem 2.9. We are going to use the Leray-Schauder-Schaefer Theorem, in the form detailed in Problem Sheet 3, namely: find a continous, compact operator T on $X = H_0^1(\Omega)$, a Banach space endowed the norm

$$\|u\|_X = \sqrt{\int_\Omega |\nabla u|^2 + |u|^2 \,\mathrm{d}x},$$

which satisfies for all $u \in X$ the *a priori* estimate

$$\|T\|_X \leqslant a \|u\|_X + b,$$

for some $b \in \mathbb{R}$ and $0 \leqslant a < 1$ independent of u. This is sufficient to obtain the existence of a fixed point (at it incidentally gives a bound on its norm). If T is chosen such that a fixed point corresponds to a weak solution of (1), we have a proof.
Choice for T. As before, place the linear part on the left, and non linear and source part on the right, and write

$$\begin{aligned} T\colon X = H_0^1(\Omega) &\to X \\ u &\to (-\Delta + \mu I_d)^{-1}(-g(u, \nabla u) + h) \end{aligned}$$

Clearly, T is well defined and continuous. Lemma 2.8 shows that $u \mapsto -g(u, \nabla u)$ is continuous from X to $L^2(\Omega)$, and Corollary 2.4 says that $v \to (-\Delta + \mu I_d)^{-1} v$ is continuous from $L^2(\Omega)$ to X.
A priori estimates Given $u \in X$, we write $v = Tu$. We have, for any $\phi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla v \cdot \nabla \phi + \mu v \phi \, \mathrm{d}x = \int_\Omega -g(u, \nabla u)\phi + h\phi \, \mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega).$$

Choosing $\phi = v$, and bounding the right-hand side with Cauchy-Schwarz, we obtain

$$\begin{aligned} \int_\Omega |\nabla v|^2 + \mu |v|^2 \, \mathrm{d}x &\leqslant \|g(u, \nabla u)\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leqslant (M_1 + \|h\|_{L^2(\Omega)}) \|v\|_{L^2(\Omega)} + M_2 \|u\|_X \|v\|_{L^2(\Omega)}, \end{aligned}$$

where in the second line we used the bound on g. Thanks to Poincaré inequality, we have

$$C(\Omega) \|v\|_X^2 \leqslant \int_\Omega |\nabla v|^2 .$$

For some positive constant $C(\Omega) > 0$ depending on Ω only. On the right-hand side, noting that for all $\kappa, a, b, > 0$ there holds $ab \leqslant \frac{\kappa}{2} a^2 + \frac{1}{2\kappa} b^2$, we have

$$\begin{aligned} M_2 \|u\|_X \|v\|_{L^2(\Omega)} &\leqslant \frac{1}{2} C(\Omega) \|u\|_X^2 + \frac{M_2^2}{2C(\Omega)} \|v\|_{L^2(\Omega)}^2, \\ (M_1 + \|h\|_{L^2(\Omega)}) \|v\|_{L^2(\Omega)} &\leqslant \frac{C(\Omega)}{4} \|v\|_{L^2(\Omega)}^2 + \frac{(M_1 + \|h\|_{L^2(\Omega)})^2}{C(\Omega)} \end{aligned}$$

Since $\|v\|_{L^2(\Omega)} \leqslant \|v\|_X$, We have therefore obtained, dividing both sides by $C(\Omega)$,

$$\frac{3}{4} \|v\|_X^2 + \frac{\mu}{C(\Omega)} \|v\|_{L^2(\Omega)}^2 \leqslant \frac{1}{2} \|u\|_X^2 + \frac{(M_1 + \|h\|_{L^2(\Omega)})^2}{C(\Omega)^2} + \frac{M_2^2}{2C(\Omega)^2} \|v\|_{L^2(\Omega)}^2 .$$

if $\mu \geqslant M_2^2/(2C(\Omega)^2)$, this implies,

$$\|v\|_X \leqslant \sqrt{\frac{2}{3}}\,\|u\|_X + \frac{(M_1 + \|h\|_{L^2(\Omega)})}{C(\Omega)}$$

which is the desired estimate.
Compactness of T In this case, it is not a direct application of Rellich-Kondrachov embedding Theorem: g is only continuous, it has no reason to be compact (it could be, for example, $\|u\|_X$). We show below that $v \mapsto (-\Delta + \mu I_d)^{-1}v$ is in fact compact from $L^2(\Omega)$ into X. With this result, T is the composition of a continuous map for which the images of bounded subsets are again bounded and a compact map; it is therefore compact, and the proof is complete. □

Lemma 2.10. *The map $v \mapsto (-\Delta + \mu I_d)^{-1}v$ is compact from $L^2(\Omega)$ into $H_0^1(\Omega)$.*

Proof. Take a bounded sequence $v_n \in L^2(\Omega)$. We can extract a weakly converging subsequence $v_{n'} \rightharpoonup v$. Let us show that the sequence of solution of weak solutions in $H_0^1(\Omega)$ of

$$-\Delta w_{n'} + \mu w_{n'} = v_{n'} \text{ in } \Omega$$

converges strongly in $H_0^1(\Omega)$. Then, we will have shown that from every bounded sequence in $L^2(\Omega)$, we can extract a sequence which converges in $H_0^1(\Omega)$, which is what compactness means. The variational formulation is

$$\int_\Omega \nabla w_{n'} \cdot \nabla \phi + \mu w_{n'} \phi \,\mathrm{d}x = \int_\Omega v_{n'} \phi \,\mathrm{d}x$$

Choose $\phi = w_{n'}$ to obtain that

$$\int_\Omega |\nabla w_{n'}|^2 \,\mathrm{d}x \leqslant \int_\Omega v_n w_{n'} \,\mathrm{d}x \leqslant \|v_{n'}\|_{L^2(\Omega)} \|w_{n'}\|_{L^2(\Omega)} .$$

Using Poincaré inequality, this gives

$$\|w_{n'}\|^2_{H_0^1(\Omega)} \leqslant \int_\Omega v_n w_{n'} \,\mathrm{d}x \leqslant \|v_{n'}\|_{L^2(\Omega)} \|w_{n'}\|_{H_0^1(\Omega)} ,$$

Therefore the sequence $w_{n'}$ is bounded in $H_0^1(\Omega)$. Thus there exists another subsequence, $w_{n''}$ which converges weakly in $H_0^1(\Omega)$ to a limit w''. Passing to the limit in the variational formulation along that subsequence, we obtain

$$\int_\Omega \nabla w'' \cdot \nabla \phi + \mu w'' \phi \,\mathrm{d}x = \int_\Omega v \phi \,\mathrm{d}x$$

Since Corollary 2.4 shows that this has a unique solution, $w'' = w$ does not depend on the extraction of a subsequence, therefore $w_{n'} \rightharpoonup w$ in $H_0^1(\Omega)$. The Rellich-Kondrachov Theorem shows that $w_{n'} \to w$ in $L^2(\Omega)$. We therefore have the following two variational formulations: for all $\phi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla w \cdot \nabla \phi + \mu w \phi \, \mathrm{d}x = \int_\Omega v \phi \, \mathrm{d}x,$$

and

$$\int_\Omega \nabla w_{n'} \cdot \nabla \phi + \mu w_{n'} \phi \, \mathrm{d}x = \int_\Omega v_{n'} \phi \, \mathrm{d}x.$$

Substracting these two identities, and choosing $\phi = w - w_{n'}$, we have

$$\int_\Omega |\nabla (w - w_{n'})|^2 + \mu \, |(w - w_{n'})|^2 \, \mathrm{d}x = \int_\Omega (v - v_{n'})(w - w_{n'}) \, \mathrm{d}x.$$

The right-hand-side is the product of a weakly converging sequence by a strongly converging one, so

$$\lim_{n' \to \infty} \int_\Omega (v - v_{n'})(w - w_{n'}) \, \mathrm{d}x = 0.$$

Since $w_{n'} - w \to 0$ in $L^2(\Omega)$,

$$\lim_{n' \to \infty} \int_\Omega \mu \, |(w - w_{n'})|^2 \, \mathrm{d}x = 0.$$

Therefore the last term on the right hand side has a limit, and

$$\lim_{n' \to \infty} \int_\Omega |\nabla (w - w_{n'})|^2 \, \mathrm{d}x = 0.$$

In other words, $w_{n'} \to w$ in $H_0^1(\Omega)$. □

The compactness of $(-\Delta)^{-1}$ could have also been used in the first application, instead of the Sobolev embeddings.

2.4 A Glimpse into positive operators : the maximum principle

It often happens in applications that finding a non-negative solution is particularly interesting — for instance, when the unknown u models a density. Working in the positive cone (a convex, closed set) of a given function space is thus sometimes a natural choice. It

is particularly judicious whenever the non linearity is unbounded in general, but bounded when limited to positive functions.
A key tool, to ensure that we indeed stay inside the positive cone is a weak form of the maximum principle.

Proposition 2.11. *Let $\Omega \subset \mathbb{R}^n$ be open, smooth and bounded, let $u \in H^1(\Omega)$ and assume that $u \geqslant 0$ on $\partial\Omega$. Furthermore assume that $-\Delta u \geqslant 0$ in Ω in a weak sense, i.e.*

$$\int_\Omega \nabla u \cdot \nabla v \, dx \geqslant 0 \qquad \text{for all } v \in H_0^1(\Omega), v \geqslant 0 \quad \text{a.e. in } \Omega.$$

Then $u \geqslant 0$ a.e. in Ω.

One way of making sense of saying that $u \geqslant 0$ on $\partial\Omega$ is to say that $\min(u, 0) \in H_0^1(\Omega)$.
In this section we will focus on the generic example

$$-\Delta u = f(u) \qquad \text{in } \Omega \quad u \in H_0^1(\Omega), \tag{1}$$

An application. Consider the equation (1) with $f(x) = \mathrm{e}^{-x}$. The function f is continuous, but grows very rapidly as $u \to -\infty$ and thus we cannot directly apply the previous technique. However, f is bounded on

$$C = \{u \in H_0^1(\Omega) : u \geqslant 0\} \subset H_0^1(\Omega).$$

Therefore we may consider the map $T : C \subset H_0^1(\Omega) \to H_0^1(\Omega)$ given by $Tu = (-\Delta)^{-1} f(u)$. The set C is convex and closed. Thanks to the maximum principle, $T(C) \subset C$. We now check that $T(C)$ is precompact. Note that for all f is continuous on $H_0^1(\Omega)$ and when $u \in C$, $f(u) \leqslant 1$, therefore

$$T(C) \subset (-\Delta)^{-1} \left(\{v \in L^2(\Omega) \text{ s. t. } \|v\|_{L^2(\Omega)} \leqslant \sqrt{|\Omega|}\} \right).$$

Thanks to Lemma 2.10 $T(C)$, is therefore contained in a (bounded) closed compact set, and is therefore precompact, and the existence of a positive solution is obtained by Schauder's Fixed Point Theorem (version 2).
We also notice that a solution of (1) is unique, c.f. problem sheet 3.

Definition 6. We say that $\underline{u} \in H^1(\Omega)$ is a weak-subsolution of (1) when

$$\int_\Omega \nabla \underline{u} \cdot \nabla v \leqslant \int_\Omega f(\underline{u}) v \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega),\ v \geqslant 0.$$

We say that $\bar{u} \in H^1(\Omega)$ is a weak-supersolution of (1) when

$$\int_\Omega \nabla \bar{u} \cdot \nabla v \geqslant \int_\Omega f(\bar{u}) v \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega),\ v \geqslant 0.$$

When sub and super solutions to the problem exists, the weak maximum principle provides a constructive method to find a solution, as we will see.

Theorem 2.12. *Assume there exist a subsolution and a supersolution in the weak sense to* (1)*, satisfying*

$$\underline{u} \leqslant 0 \leqslant \bar{u} \text{ in the sense of traces on } \partial\Omega \text{ and } \underline{u} \leqslant \bar{u} \text{ a.e. in } \Omega.$$

Suppose that $f \in C(\mathbb{R})$, that there exists $\lambda \geqslant 0$ such that $x \to f(x) + \lambda x$ is non-decreasing, and that $|f(t)| \leqslant C(1 + |t|)$ for all $t \in \mathbb{R}$. Then, there exists a solution of (1) *satisfying $\underline{u} \leqslant u \leqslant \bar{u}$.*

Proof. Without loss of generality, we may assume that f is non decreasing, since we can consider otherwise the problem

$$-\Delta u + \lambda u = f(u) + \lambda u, \quad u \in H_0^1(\Omega),$$

and replace in everything that follows Δ by $\Delta+\lambda$. Define $u_0 := \underline{\text{u}}$ and for all $k \in \{0, 1, 2, \ldots\}$ define inductively $u_{k+1} = Tu_k$ where

$$T(u) := (-\Delta)^{-1} f(u).$$

We have already established that T is a compact map from $H_0^1(\Omega)$ to itself — however, it is not clear that T takes a bounded closed convex set in $H_0^1(\Omega)$ into itself so one cannot directly apply one of the FPTs.
The sequence u_k is monotone increasing. We claim that $u_0 \leqslant u_1 \leqslant \ldots \leqslant u_k \leqslant \ldots$ a.e. in Ω. We argue by induction. When $k = 0$ u_1 and $-u_0$ satisfies (in the weak sense)

$$-\Delta u_1 = f(u_0), \quad -\Delta(-u_0) \geqslant (-f(u_0)), \text{ in } \Omega$$

And $u_1 - u_0 \geqslant 0$ on $\partial\Omega$ therefore the weak maximum principle shows that $u_1 - u_0 \geqslant 0$ a.e. in Ω. For an arbitrary $k \geqslant 1$, if $u_k \geqslant u_{k-1}$, since f is non decreasing we have, in the weak sense,

$$-\Delta(u_{k+1} - u_k) = f(u_k) - f(u_{k-1}) \geqslant 0$$

and $u_{k+1} - u_k = 0 \geqslant 0$ on $\partial\Omega$, so the weak maximum principle shows that $u_{k+1} \geqslant u_k$.
The sequence u_k is bounded from above by $\bar{u}$. We argue again by induction. It is true for $u_0 = \underline{u} \leqslant \bar{u}$. If it is true for u_k, then we have

$$-\Delta\bar{u} \geqslant f(\bar{u}), \quad -\Delta(-u_{k+1}) = (-f(u_k)), \text{ in } \Omega,$$

thus $-\Delta(\bar{u}-u_{k+1}) \geqslant f(\bar{u})-(-f(u_k)) \geqslant 0$ since f is non decreasing and $\bar{u}-u_{k+1}=\bar{u} \geqslant 0$ on $\partial\Omega$. Thus, $\bar{u} \geqslant u_{k+1}$ almost everywhere.
The sequence $u_k \to u$ solution of (1) in $H_0^1(\Omega)$. Since $u_0 \leqslant u_1 \leqslant \ldots \leqslant u_k \leqslant \ldots \leqslant \bar{u}$ a.e , the limit

$$u(x) := \lim_{k\to\infty} u_k(x)$$

exists for almost every x, and from the DCT, $u_k \to u$ in $L^2(\Omega)$. Next, note that for all k, $|f(u_k)| \leqslant C(1+|\bar{u}|+|u_0|)$, thus f maps the sequence u_k into the $L^2(\Omega)$ bounded set $A := \{v \text{ s.t. } \|v\|_{L^2(\Omega)} \leqslant \|C(1+|\bar{u}|+|u_0|+)\|_{L^2(\Omega)}\}$. Therefore $T(u_k) \in \Delta^{-1}(A)$, a bounded set in $H_0^1(\Omega)$. Therefore, there exists a weakly converging subsequence u_{k_n} in $H_0^1(\Omega)$. Necessarily, $u_{k_n} \rightharpoonup u$ in $H_1^0(\Omega)$. Since T is continuous, and $u_{k+1}=Tu_k$, $u_k \to u$ in $H_0^1(\Omega)$, and passing to the limit in the relationship $u_{k+1}=Tu_k$ we conclude the proof. $\square$

Chapter 3

Variational inequalities and monotone operators

So far, we have considered semi-linear problems: the nonlinearity only appears in terms for which the number of derivatives is strictly less than the maximal order of derivatives appearing in the equation. We used to a great extent that the principal part of the operator was a linear operator. The second part of this course is devoted to more general situations, the typical case under consideration being the case of quasi-linear problems. In this chapter we will prove a general existence theorem for variational inequalities while the following chapter will then be devoted to applications of this result to quasilinear PDEs and variational problems.
Let X be a Banach space and X^* its dual. If $x \in X$ and $x^* \in X^*$ we use the notation $\langle x^*, x\rangle := x^*(x)$.

3.1 Differentiation in Banach spaces

Definition 7. Let X, Y be Banach spaces, $M \subseteq X$, $x \in M$ and $F\colon M \subseteq X \to Y$. We say

(1) F has a *directional derivative* at x in direction $e \in X$ if

$$\partial_e F(x) := \lim_{t\to 0, t>0} \frac{F(x+te) - F(x)}{t} \in Y \quad \text{exists.}$$

(2) F is *Gâteaux differentiable* at x, if $\partial_e F(x)$ exists for all $e \in X$ and is continuous, i.e. there exists $L \in L(X,Y)$ such that $\partial_e F(x) = Le$.

(3) F is *Fréchet differentiable* at x if there exists a continuous linear map $DF(x)\colon X \to Y$ such that

$$F(x+h) = F(x) + DF(x)h + \mathrm{o}(\|h\|_X) \quad \text{as } \|h\|_X \to 0.$$

(4) F is *continuously differentiable* at x, if $y \mapsto DF(y)$ is continuous in x. If DF is continuous for all $x \in M$, then we write $F \in C^1(M, Y)$.

Remarks.

(i) If F is Fréchet differentiable at x then F is Gâteaux differentiable at x.

[$h = t\tilde{h}$, $\tilde{h}$ fixed, $t \to 0$.]

(ii) Gâteaux differentiability does *not* imply Fréchet differentiability.

Theorem 3.1 (Fundamental Theorem of Calculus)**.** *For $t \in [0,1]$ let $F(x+te)$ be Gâteaux differentiable and $t \mapsto \partial_e F(x+te)$ continuous. Then*

$$F(x+e) - F(x) = \int_0^1 \partial_e F(x+te)\,\mathrm{d}t.$$

Proof. Let $y^* \in Y^*$ be arbitrary, $g\colon [0,1] \to Y$, $g(t) := F(x+te)$, $h\colon [0,1] \to \mathbb{R}$, $h(t) := \langle y^*, g(t)\rangle$.
By assumption $g'(t) = \partial_e F(x+te)$, $h'(t) = \langle y^*, g'(t)\rangle$ exist. The Fundamental Theorem of Calculus in $\mathbb{R}$ gives $h(1) - h(0) = \int_0^1 h'(t)\,\mathrm{d}t$, that is

$$\begin{aligned}\langle y^*, F(x+e)\rangle - \langle y^*, F(x)\rangle &= \int_0^1 \langle y^*, \partial_e F(x+te)\rangle\,\mathrm{d}t \\ &= \langle y^*, \int_0^1 \partial_e F(x+te)\,\mathrm{d}t\rangle\,.\end{aligned}$$

Since $y^* \in Y^*$ is arbitrary the claim follows. □

Examples.

(1) $F\colon H^1(\Omega) \to \mathbb{R}$, $u \mapsto \int_\Omega |\nabla u|^2\,\mathrm{d}x$. Expanding, we find

$$F(u+v) - F(v) = 2\int_\Omega \nabla u \nabla v\,\mathrm{d}x + \int_\Omega |\nabla v|^2\,\mathrm{d}x = 2\int_\Omega \nabla u \nabla v\,\mathrm{d}x + \mathrm{o}(\|v\|_{H^1})$$

and thus F is Fréchet differentiable with $DF(u)v = 2\int_\Omega \nabla u \nabla v\,\mathrm{d}x$ (and even in $C^1(H^1(\Omega))$.

(2) $F\colon H^1(\Omega) \to \mathbb{R}$, $u \mapsto \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x$, Ω bounded.

The directional derivative is given by

$$\partial_e F(u) = \int_\Omega \frac{\nabla u \cdot \nabla e}{\sqrt{1+|\nabla u|^2}}\,\mathrm{d}x.$$

Is F Gâteaux differentiable on $H^1(\Omega)$? We need to show that $L\colon H^1(\Omega) \to \mathbb{R}$, $e \mapsto \int_\Omega \frac{\nabla u \cdot \nabla e}{\sqrt{1+\nabla u}^2}\,\mathrm{d}x$ is continuous. Indeed,

$$|Le| \leqslant \int_\Omega \underbrace{\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}}_{\leqslant 1} |\nabla e|\,\mathrm{d}x \leqslant \int_\Omega |\nabla e|\,\mathrm{d}x \leqslant |\Omega|^{1/2} \|\nabla e\|_{L^2(\Omega)}$$

and thus F is Gâteaux differentiable on $H^1(\Omega)$.

Is F Fréchet differentiable on $H^1(\Omega)$? Let

$$DF(u)h := \int_\Omega \frac{\nabla u \cdot \nabla h}{\sqrt{1+|\nabla u|^2}}\,\mathrm{d}x.$$

Then

$$\begin{aligned}
&|F(u+h) - F(u) - DF(u)h| \\
\leqslant\ & \int_\Omega \left| \sqrt{1+|\nabla u + \nabla h|^2} - \sqrt{1+|\nabla u|^2} - \frac{\nabla u \cdot \nabla h}{\sqrt{1+|\nabla u|^2}} \right| \mathrm{d}x \\
\leqslant\ & \int_\Omega |\nabla h| \left| \frac{2|\nabla u|}{\sqrt{1+|\nabla u+\nabla h|^2} + \sqrt{1+|\nabla u|^2}} - \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} \right| \mathrm{d}x \\
\leqslant\ & \|h\|_{H^1(\Omega)} \left\| \frac{2|\nabla u|}{\sqrt{1+|\nabla u+\nabla h|^2} + \sqrt{1+|\nabla u|^2}} - \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} \right\|_{L^2(\Omega)}
\end{aligned}$$

Since $2|\nabla u|/(\sqrt{1+|\nabla u+\nabla h|^2} + \sqrt{1+|\nabla u|^2}) - |\nabla u|/\sqrt{1+|\nabla u|^2} \to 0$ a.e. (for a subsequence) when $h \to 0$ in $H^1(\Omega)$, and this quantity is bounded by $3|\nabla u| \in L^2(\Omega)$, it follows from the DCT that

$$|F(u+h) - F(u) - DF(u)h| = \mathrm{o}\left(\|h\|_{H^1}\right),$$

and F is Fréchet differentiable on $H^1(\Omega)$.

(3) $F\colon X = L^1(\Omega) \to \mathbb{R}$, $u \mapsto \int_\Omega |u(x)|\, \mathrm{d}x$. For $t > 0$ we find

$$\frac{F(u+tv) - F(u)}{t} = \int_\Omega \frac{|u+tv| - |u|}{t}\, \mathrm{d}x \to \int_\Omega \left\{ \begin{array}{cc} (\mathrm{sign} u)v & u \neq 0 \\ |v| & u = 0 \end{array} \right\} \mathrm{d}x$$

and thus the directional derivative exists and is given by

$$\partial_v F(u) = \int_{\Omega \setminus \{u=0\}} (\mathrm{sign} u)v + \int_{\Omega \cap \{u=0\}} |v| \,.$$

Hence F is only Gâteaux differentiable if $u(x) \neq 0$ for a.a. x, since otherwise $\partial_v F(u) \notin L(X)$.

See Problem Sheet 3 for more examples.
We remark that the differentials DF in the examples (i) and (ii) are *monotone operators*.

3.2 Monotone Operators

In the following, unless specified otherwise, X will always be a reflexive separable Banach space, X^* its dual and $M \subseteq X$ non-empty, convex and closed.

Definition 8. $A\colon M \to X^*$ is a *monotone operator* if

(i) A is *monotone*, i.e. $\langle A(u) - A(v), u - v \rangle \geqslant 0$ for all $u, v \in M$;

(ii) A is *hemicontinuous*, i.e. for all $u, v \in M$ and $w \in X$ the map

$$t \mapsto \langle A((1-t)u + tv), w \rangle \quad \text{is continuous on } [0,1].$$

A monotone operator is called *strictly monotone* if $\langle A(u) - A(v), u - v \rangle \geqslant 0$, and $\langle A(u) - A(v), u - v \rangle = 0$ if and only if $u = v$.

Note that if A is continuous from M (strong) to X^* weak, then A is hemicontinuous.

Lemma 3.2 (Minty). *Let $A\colon M \to X^*$ be a monotone operator. Then the following are equivalent:*

(i) $\langle A(u) - \xi, u - v \rangle \leqslant 0 \qquad \forall v \in M.$

(ii) $\langle A(v) - \xi, u - v \rangle \leqslant 0 \qquad \forall v \in M.$

This lemma is useful for example when $A(u)$ and u converge weakly to a limit: for a fixed v, while it is not clear what the limit in (1) will be, because it involves the duality bracket between two weakly convergent sequences, in (2) only one appears, and the limit is straightforward.

Proof. Let us first show that (1) implies (2).

$$\langle A(v) - \xi, u - v\rangle = \underbrace{\langle A(v) - A(u), u - v\rangle}_{\leqslant 0 \text{ by monotonicity}} + \underbrace{\langle A(u) - \xi, u - v\rangle}_{\leqslant 0 \text{ by assumption}} \leqslant 0.$$

Let us now show the other implication. Given $v \in M$, let $v_\varepsilon := (1-\varepsilon)u + \varepsilon v \in M$. We have, by assumption

$$\langle A(v_\varepsilon) - \xi, \varepsilon(u - v)\rangle = \langle A(v_\varepsilon) - \xi, u - v_\varepsilon\rangle \leqslant 0.$$

On the other hand, thanks to the hemicontinuity, $\langle A(v_\varepsilon) - \xi, u - v\rangle \overset{\varepsilon \to 0}{\longrightarrow} \langle A(u) - \xi, u - v\rangle$, and we have proved (1). □

Problem Sheet 6 is devoted to the study of interesting properties of monotone operators. In particular, you will show:

- If A is monotone, then A satisfies condition (H3) encountered later on in Theorem 3.5
- Suppose that $F\colon X \to \mathbb{R}$ is Gâteaux differentiable in every u in X, with derivative $F'(u)$. Then F is convex if and only if $F'\colon X \to X^*$ is a monotone operator.
- If A is strictly monotone, then the solution of the variational inequality (1) (with $F = A$) is unique.

On Problem Sheet 3 you also find several necessary and sufficient conditions for a function to be monotone.

3.3 Variational inequalities

Consider $F\colon M \to X^*$. In this section, we wish to address the following problem
Model Problem. Find $u \in M$ such that

$$\langle F(u), u - v\rangle \leqslant 0 \quad \forall v \in M. \tag{1}$$

Remarks.

- When $M = X$ then (1) is equivalent to $\langle F(u), \eta \rangle = 0 \ \forall \eta \in X$ —choose $v = u \pm \eta$. Thus, $F(u) = 0$ in X^*, thus variational equalities are included in this settings.

- F could come from a variational problem, and correspond to a critical point of a functional I — $F(u) = I'(u)$, with $I : X \to \mathbb{R}$ and I' is its Gâteaux derivative.

We will assume that F satisfies the following three assumptions

(H1) F maps bounded sets into bounded sets.

No pre-compactness is required for example.

(H2) F is coercive with respect to some $u_0 \in M$, that is, there exists $u_0 \in M$ such that

$$\frac{\langle F(u), u - u_0 \rangle}{\|u - u_0\|} \to \infty \text{ as } \|u\| \to \infty, u \in M.$$

This hypothesis is only relevant when M is unbounded.
If F is additionally a monotone operator then we find

Theorem 3.3. *Let $A : M \to X^*$ be a monotone operator (where as usual M is convex, non-empty and closed and X is a seperable reflexive Banach space) so that conditions (H1) and (H2) are satisfied. Then the variational inequality* (1) *has a solution.*

We will obtain this result as a consequence of a more general theorem in which the assumption of F being a monotone operator is replaced with the weaker condition (c.f. Problem sheet 3)

(H3) F is satisfies the following *weak sequential lower semi-continuity condition*: Given a sequence (u_n) $n \in \mathbb{N}, u_n \in M$, such that $u_n \rightharpoonup u$ in X with $u \in M$, and $F(u_n) \rightharpoonup \xi$ in X^*, then

$$\langle \xi, u \rangle \leqslant \varliminf_{n \to \infty} \langle F(u_n), u_n \rangle.$$

Furthermore, if $\langle \xi, u \rangle = \varliminf_{n \to \infty} \langle F(u_n), u_n \rangle$ then

$$\langle F(u) - \xi, u - v \rangle \leqslant 0 \quad \forall v \in M.$$

As above this implies $F(u) = \xi$ if $M = X$. This last assumption seems more mysterious that the other two, but as we will see is very natural in the context of calculus of variations. It seems we do not require F to be continuous, but it is partially an illusion as the following proposition shows.

Proposition 3.4. *Suppose that F satisfies* (H1) *and* (H3) *and that M has a non-empty interior. Then, if $u_n \to u$ in* $\operatorname{Int}(M)$*, then $F(u_n) \rightharpoonup F(u)$ in X^*.*

Proof. Let u_n be a sequence such that $u_n \to u$ in $X \cap M$. Then, $\|u_n\|_X < \infty$ and since F maps bounded sets into bounded sets, $\|F(u_n)\|_{X^*}$ is bounded and X^* is reflexive, so we may extract a subsequence $u_{n'}$ such that $u_{n'} \to u$ in M and $F(u_{n'}) \rightharpoonup \xi$ in X'. Thus $\langle F(u_{n'}), u_{n'}\rangle \to \langle \xi, u\rangle$ (weak-strong duality bracket..), and (H3) implies that

$$\langle F(u) - \xi, u - v\rangle \leqslant 0 \quad \forall v \in M.$$

Since $u \in \operatorname{Int}(M)$, there exists a ball $B(u, \delta) \subset M$ with $\delta > 0$. Therefore, for any $w \in X \setminus \{0\}$, choose $v = u - \lambda_w w$, with $\lambda_w = \delta/(2\|w\|_X) > 0$, and divide by λ_w to obtain

$$\langle F(u) - \xi, w\rangle \leqslant 0 \quad \forall w \in X,$$

thus $F(u) = \xi$. Since the limit does not depend on the subsequence, the convergence holds globally. □

The motivation of the introduction of Assumptions (H1), (H2) and (H3) is the following result.

Theorem 3.5. *If F satisfies conditions* (H1), (H2) *and* (H3) *there exists $u \in M$ such that $\langle F(u), u - v\rangle \leqslant 0$ for all $v \in M$.*

Remark. In condition (H2) it is essential that $u_0 \in M$!
Consider e.g. $X = \mathbb{R}^2$, $M = \{x_2 = 1\}$, $I(x) = \mathrm{e}^{x_1} + \mathrm{e}^{x_1^2(x_2-1)}$.
Thus $I(x) = \mathrm{e}^{x_1} + 1$ on M and I does not attain its minimum on M.

$$F(x) = \nabla I(x) = \left(\mathrm{e}^{x_1} + 2x_1(x_2-1)\mathrm{e}^{x_1^2(x_2-1)}, x_1^2\mathrm{e}^{x_1^2(x_2-1)}\right).$$

For $x \in M$:

$$\frac{\langle F(x), x\rangle}{|x|} = \frac{x_1\mathrm{e}^{x_1} + x_1^2}{\sqrt{x_1^2+1}} \to \infty \text{ as } |x_1| \to \infty.$$

Hence F is coercive with respect to $0 \notin M$, but F does not have a zero on M.

3.4 Proof of Theorem 3.5 by the Galerkin Method

The Galerkin method is very general and robust. The idea is as follows. To tackle a problem posed in an infinite dimensional space, start with a studying its approximation on a nested sequence of finite dimensional sub-spaces. Solving the approximate problem is generally simpler than solving the infinite dimensional one. Passing to the limit, we construct a solution of the original problem. This method is very popular for numerical methods, because it is constructive. It is also interesting from a theoretical point of view. We proceed in several steps. We first recall a result concerning Banach spaces

Lemma 3.6. *Let X be a separable Banach space of infinite dimension. There exists a countable linearly independent family $(v_i)_{i\in\mathbb{N}}$, $v_i \in X$, such that the linear combinations of v_i are dense in X.*

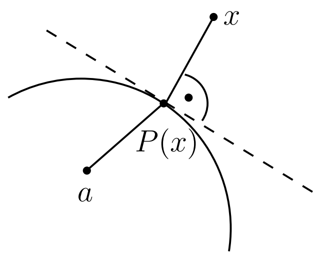
Another way of writing this is that if $X_i = \mathrm{span}(v_1, \ldots, v_i)$ then the $\cup_{i=0}^{\infty} X_i$ is dense in X. For problem, we will apply this result to both X and M and choose X_1 so that u_0, the coercivity point from (H2), is in X_1. To construct a finite dimensional approximation, simply restrict the variational problem (1) to X_i and $M_i = M \cap X_i$.
Let us first address the approximated problem.

Proposition 3.7. *If F satisfies the conditions* (H1), (H2), (H3) *and if* $\dim X < \infty$*, then there exists $u \in M$ such that $\langle F(u), u - v\rangle \leqslant 0$ for all $v \in M$.*

Proof. Without loss of generality we may assume that M does not have an empty interior (otherwise, reduce dimensions).
1. The interior case. Let us first prove the result for the restriction of F to a smaller closed convex set $\tilde{M}$ such that $\tilde{M} \subset \mathrm{Int}\,(M)$. Proposition 3.4 shows that F is continuous on $\tilde{M}$ — since in finite dimensional spaces, weak and strong convergences are equivalent. Using Riesz Theorem, there exists $\tilde{F} : \tilde{M} \to X$ such that $\langle F(u), v\rangle = (\tilde{F}(u), v)$ for all $v \in X$. Recall the *Projection Theorem:*

> Let X be real Hilbert space, $\tilde{M} \subset X$ non-empty, closed, convex. Then there exists a unique map $P\colon X \to \tilde{M}$ such that
>
> $$\|x - P(x)\|_X = \mathrm{dist}(x, \tilde{M}),$$
>
> which is equivalent to
>
> $$(x - P(x), a - P(x)) \leqslant 0 \quad \text{for all } a \in \tilde{M}.$$

Using the orthogonal projection P on $\tilde{M}$ we define a map $G\colon \tilde{M} \to \tilde{M}$ via $G := P \circ (\mathrm{Id} - \tilde{F})$. The map G is continuous, since it is the composition of continuous maps. Brouwer's Fixed



Point Theorem implies that that there exists $u \in \tilde{M}$ such that $G(u) = u$. Hence $P(w) = u$ with $w = u - \tilde{F}(u)$. Due to the properties of P we find

$$\langle F(u), u - v\rangle = -(v - u, \tilde{F}(u)) = (v - P(w), w - P(w)) \leqslant 0 \text{ for all } v \in \tilde{M},$$

<u>2. The bounded case.</u> Given $n > 0$, consider now $\widetilde{M}_n = M \cap \{x \in M \text{ s.t. } \operatorname{dist}(x, \partial M) \geqslant \frac{1}{n}\}$, a closed, convex set, inside Int (M) non-empty for n big enough (the convexity follows from that of M). From the previous step, there exists $u_n \in \widetilde{M}_n$ such that

$$\langle F(u_n), u_n - v\rangle \leqslant 0 \quad \text{for all } v \in \widetilde{M}_n. \tag{1}$$

Since M is bounded, there exists a subsequence such that $u_{n'} \to u$ and $F(u_{n'}) \to \xi$ as $n' \to \infty$. Passing to the the limit on the left-hand side in (1), and noting that the space $\widetilde{M}_{n_1} \subset \widetilde{M}_{n_2}$ if $n_1 < n_2$, we obtain $\langle \xi, u - v\rangle \leqslant 0 \quad$ for all $v \in \widetilde{M}_n$ for any n. Passing now to the limit in n, we have

$$\langle \xi, u - v\rangle \leqslant 0 \quad \text{for all } v \in \overline{\cup_n \widetilde{M}_n} = M.$$

On the other hand, Condition (H3) (and strong convergence) implies that

$$\langle F(u) - \xi, u - v\rangle \leqslant 0 \quad \text{for all } v \in M.$$

Therefore $\langle F(u), u - v\rangle \leqslant \langle \xi, u - v\rangle \leqslant 0$.

<u>3. The unbounded case.</u> Now let M be unbounded and define now $\widetilde{M}_R := M \cap \overline{B_R(0)}$. We have obtained that there exists u_R such that

$$\langle F(u_R), u_R - v\rangle \leqslant 0 \quad \text{for all } v \in \widetilde{M}_R.$$

The coercivity condition (H2) implies that there exists $u_0 \in M$ such that $\frac{\langle F(u), u - u_0\rangle}{\|u - u_0\|} \to \infty$ as $\|u\| \to \infty$. Hence there exists $C_0 > 0$ such that $\langle F(u), u - u_0\rangle > 0$ for all u with $\|u\| \geqslant C_0$. Choose now $R := \max(\|u_0\|, C_0) + 1$. Then, $u_0 \in \widetilde{M}_R$, and therefore

$$\langle F(u_R), u_R - u_0\rangle \leqslant 0$$

and consequently $\|u_R\|_X \leqslant C_0$. Then, for any $v \in M$ introduce $v_R = (1-\theta)u_R + \theta v$, with $\theta = (C_0 + \|v\|_X)/(C0 + \|v\|_X + 1)$ (or anything else small enough). Then, $\|v_R\|_X \leqslant R$, that is, $v_R \in M_R$ and therefore

$$\langle F(u_R), u_R - v\rangle = \theta^{-1}\langle F(u_R), u_R - v_R\rangle \leqslant 0,$$

Thus u_R is a solution. □

As the bound on $\|u_R\|$ depends only on the constant C_0 obtained from the coercivity condition (H2) we thus conclude

Corollary 3.8. *Let M, X, F be as in Theorem 3.5 and let $X_i = span(v_1, \dots, v_i)$ the finite dimensional subspaces obtained in Lemma 3.6.*
Then there exist solutions u_i of the variational inequalities

$$\langle F(u_i), u_i - v\rangle \leqslant 0, \qquad \textit{for all } v \in M_i$$

which are uniformly bounded, i.e. so that $\sup \|u_i\| < \infty$.

Conclusion of the proof of Theorem 3.5. Since the sequence u_i is bounded it follows from (H1) that the sequence $F(u_i)$ is also bounded. Therefore we may extract a subsequence u_n such that $u_n \rightharpoonup u$ weakly in M (since M is closed), and $F(u_n) \rightharpoonup \xi$ weakly in X^*.
Since the sets M_i and X_i are nested, that is, $M_i \subset M_{i+1}$ and $X_i \subset X_{i+1}$, we have, for all $n \geqslant i$,

$$\langle F(u_n), u_n\rangle \leqslant \langle F(u_n), v\rangle \quad \text{for all } v \in M_i.$$

Now, for any given $v \in M_i$ $\langle F(u_n), v\rangle \to \langle \xi, v\rangle$, and in turn

$$\liminf \langle F(u_n), u_n\rangle \leqslant \langle \xi, v\rangle \quad \text{for all } v \in \overline{\cup_{i=0}^{\infty} M_i} = M.$$

Since condition (H3) guarantees that $\langle \xi, u\rangle \leqslant \liminf\langle F(u_n), u_n\rangle$, choosing $v = u \in M$, we have in fact $\liminf\langle F(u_n), u_n\rangle = \lim\langle F(u_n), u_n\rangle = \langle \xi, v\rangle$. The second part of condition (H3) then shows that

$$\langle F(u), u - v\rangle \leqslant \langle \xi, u - v\rangle \quad \text{for all } v \in M.$$

Finally, notice that for any from condition (H3) again, since $u_n - v \rightharpoonup u - v$ for a given i and $v \in M_i$, and by construction $\langle F(u_n), u_n - v\rangle \leqslant 0$ for all $n \geqslant i$,

$$\langle \xi, u - v\rangle \leqslant \liminf\langle F(u_n), u_n - v\rangle \leqslant 0 \quad \text{for all } v \in \overline{\cup_{i=0}^{\infty} M_i} = M,$$

which is what we wished to show. □

.

Chapter 4

Applications to quasilinear PDEs and variational inequalities

We shall now detail various applications of Theorem 3.5.

4.1 Some Quasilinear problems in $W_0^{1,p}(\Omega)$

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $p \in (1, \infty)$, $X = W^{1,p}(\Omega)$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ continuous and monotonous, which satisfies the growth condition

$$F(\lambda) \leqslant C(1 + |\lambda|^{p-1}) \quad \text{for all } \lambda \in \mathbb{R}^n.$$

Suppose furthermore that there exists $\alpha > 0$ such that for all

$$F(\lambda) \cdot \lambda \geqslant \alpha |\lambda|^p \quad \text{for all } \lambda \in \mathbb{R}^n.$$

We denote by p' the conjugate exponent of p, i.e. $p^{-1} + (p')^{-1} = 1$.

Theorem 4.1. *For all $f \in W^{-1,p'}(\Omega)$, there exists $u \in W_0^{1,p}(\Omega)$ such that*

$$-div(F(\nabla u)) = f \text{ in the sense of } \mathcal{D}'(\Omega).$$

If F is strictly monotonous, this solution is unique.

Proof. Let us check that $A := -\text{div}(F(\nabla u)) - f$ is a map from X to X^* which satisfies (H1), (H2) and (H3). Let us first verify that $-f$ satisfies (H1) and (H3). By the definition of $X^* = W^{-1,p'}(\Omega)$, there exist a constant $C < \infty$ (the smallest such constant is $\|f\|_{X^*}$) such that for all $v \in X$ we have $|\langle f, v, \rangle| \leqslant C \|v\|_X$, so f maps bounded sets into bounded sets, so (H1) is satisfied. Hypothesis (H3) is obviously true: since f does not depend on u, since $\xi = f$. Let us now turn to $B(u) := -\mathrm{div}\,(F(\nabla u))$. By assumption

$$\left|F(\lambda)^{p'}\right| \leqslant C\left(1+|\lambda|^{p-1}\right)^{\frac{p}{p-1}} \leqslant C'\left(1+|\lambda|^{p}\right).$$

Therefore, if $\nabla u \in L^p(\Omega)$, then $\left|F(\nabla u)^{p'}\right| \in L^1(\Omega)$, and for any $w \in X = W_0^{1,p}(\Omega)$ we have

$$\langle B(u), w\rangle = \int_\Omega F(\nabla u)\cdot \nabla w \,\mathrm{d}x \leqslant C\left(1+\|u\|_X^{p-1}\right)\|w\|_X$$

Therefore $B : X \to X^*$ maps bounded sets into bounded sets. Let us now verify that B is monotone and hemicontinuous: this will imply that (H3) is also satisfied. From the monotonicity of F, we have

$$\langle B(u) - B(v), u - v\rangle = \int_\Omega \left(F(\nabla u) - F(\nabla v)\right)\cdot\left(\nabla u - \nabla v\right)\,\mathrm{d}x \geqslant 0,$$

and

$$\langle B(tu + (1-t)v), w\rangle = \int_\Omega F(t\nabla u + (1-t)\nabla v)\cdot \nabla w \,\mathrm{d}x.$$

Since F is continuous, the hemicontinuity follows from the Dominated Convergence Theorem.

Finally, let us verify that A is coercive at 0. We have

$$\langle A(u), u\rangle \geqslant \int_\Omega F(\nabla u)\cdot\nabla u\,\mathrm{d}x - \|f\|_{X^*}\|u\|_X \geqslant \alpha\|\nabla u\|_{L^p(\Omega)}^p - \|f\|_{X^*}\|u\|_X\,.$$

Thanks to Poincaré's inequality, $\alpha\|\nabla u\|_{L^p(\Omega)}^p \geqslant \alpha'\|u\|_X^p$, thus we have

$$\frac{|\langle A(u), u\rangle|}{\|u\|_X} \geqslant \alpha'\|u\|_X^{p-1} - \|f\|_{X^*}\,,$$

and this lower bound tends to infinity with $\|u\|_X$, since $p > 1$. □

Example. Find a solution to the p-Laplacian, that is

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + \mu u = f \quad \text{in } \Omega,\ \text{with } u = 0 \text{ on } \partial\Omega.$$

Where $\mu \in \mathbb{R}$, and the domain is smooth and bounded. For this equation to make sense, a natural choice looking at the principal terms seems to be $u \in W_0^{1,p}(\Omega)$. For the right-hand-side to make sense, we need $\int f\phi\,\mathrm{d}x$ to be well defined for $\phi \in W_0^{1,p}(\Omega)$. So choose $f \in L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, or better, using Sobolev embeddings choose $f \in L^{\tilde{p}}(\Omega)$ with

$\frac{1}{\bar{p}} + \frac{1}{p} = 1 + \frac{1}{n}$ if $p < n$. We also need $\int u\phi \,\mathrm{d}x$ to make sense. Since u and ϕ lie in the same space, the best we can do is

$$\left|\mu \int u\phi \,\mathrm{d}x\right| \leqslant |\mu| \, \|u\|_{L^2(\Omega)} \, \|\phi\|_{L^2(\Omega)} \, ,$$

that is, require that $u \in L^2(\Omega)$. Therefore a good space to work in is $X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ when $\mu \neq 0$, and $X = W_0^{1,p}(\Omega)$ if $\mu = 0$.

When $\mu = 0$. We can apply theorem 4.1 provided we check that $F(\lambda) := |\lambda|^{p-2} \zeta$ satisfies the required properties. The bound $|F(\lambda)| \leqslant |\lambda|^{p-1}$ comes immediately, and the lower bound $F(\lambda) \cdot \lambda \geqslant |\lambda|^p$ as well. We simply need to check that F is monotone and continuous, but that is clear since $p > 1$, F is the differential of $\frac{1}{p} |\lambda|^p$ a convex function since $p > 1$.

When $\mu > 0$. No change for the principal part. Let us check directly that the second term $\langle \mu u, \phi \rangle$ satisfies the (H1) (H3) requirements:

$$\langle \mu u, \phi \rangle \leqslant |\mu| \, \|u\|_{L^2(\Omega)} \, \|\phi\|_{L^2(\Omega)}$$

so a bounded set in $X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ is mapped into a bounded set. Next, suppose $u_n \rightharpoonup u$. We write

$$\langle \mu u_n, u_n \rangle = \mu \int_\Omega (u_n - u)^2 - u^2 + +2uu_n \,\mathrm{d}x \geqslant \mu \int_\Omega -u^2 + +2uu_n \,\mathrm{d}x,$$

and the right-hand side term has a limit, $\langle \mu u, u \rangle$. Thus $\underline{\lim\inf} \langle \mu u_n, u_n \rangle \geqslant \langle \mu u, u \rangle$. The second identity is obvious since $u \to \mu u$ is linear.
Finally, let us verify that (H2) is holds with 0 as a coercivity point. Let us choose the norm on $X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ to be $\|u\|_X = \|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}$. This norm is equivalent to the canonical norm, $X = \|\nabla u\|_{L^p(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$ by Poincaré's inequality. We have

$$\begin{aligned} & \langle -\mathrm{div} \left(|\nabla u|^{p-2} \nabla u \right) + \mu u, u \rangle \\ \geqslant \; & \int_\Omega |\nabla u|^p \,\mathrm{d}x + \mu \int_\Omega |u|^2 \,\mathrm{d}x \\ = \; & \theta \, \|\nabla u\|_{L^p(\Omega)}^{p-1} + (1 - \theta) \, \|\mu u\|_{L^2(\Omega)} \text{ with } \theta = \frac{\|\nabla u\|_{L^p(\Omega)}}{\|u\|_X}. \end{aligned}$$

If $\|u\|_{L^2(\Omega)} \to \infty$ while $\|\nabla u\|_{L^p(\Omega)}$ stays bounded, $\theta \to 0$ and the lower bound tends to infinity. Symmetrically, $\|u\|_{L^2(\Omega)}$ stays bounded while $\|\nabla u\|_{L^p(\Omega)} \to \infty$, $\theta \to 1$ and the lower bound tends to infinity. Finally, when both tend to infinity, $\theta a + (1 - \theta) b > \min(a, b)$ therefore the lower bound also tends to infinity.

When $\mu < 0$. No change for the principal part. We cannot just apply the same argument as before (the inequalities are in the wrong direction!). The coercivity will also be an issue, if we cannot dominate the $L^2(\Omega)$ part by the $W_0^{1,p}(\Omega)$. We therefore wish to use some compactness to bypass these difficulties. So we only consider the case when $p > 2$. Then, the Rellich-Kondrachov embeddings show that $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, and $L^2(\Omega) \subset L^p(\Omega)$ since Ω is bounded. Indeed, using Hölder,

$$\int_\Omega u^2 \, \mathrm{d}x \leqslant (\int_W u^{2r})^{\frac{1}{r}} (\int_W 1)^{\frac{r-1}{r}}$$

and thus for $r = p/2 > 1$, we obtain $\|u\|_{L^2(\Omega)} \leqslant C \|u\|_{L^p(\Omega)}$.
Now, given a sequence $u_n \rightharpoonup u$ in $X = W_0^{1,p}(\Omega) (= W_0^{1,p}(\Omega) \cap L^2(\Omega))$, we have $u_n \to u$ in $L^2(\Omega)$ by compactness, therefore $\langle \mu u_n, u_n \rangle \to \langle \mu u, u \rangle$, so (H3) is satisfied. Assumption (H1) is satisfied as before.
Finally, let us verify that (H2) holds, with 0 as coercivity point. We have

$$\begin{aligned}
& \langle -\mathrm{div}\left(|\nabla u|^{p-2} \nabla u\right) + \mu u, u \rangle \\
\geqslant \;& \int_\Omega |\nabla u|^p \, \mathrm{d}x + \mu \int_\Omega |u|^2 \, \mathrm{d}x \\
= \;& \|\nabla u\|_{L^p(\Omega)}^p - |\mu| \, \|u\|_{L^2(\Omega)}^2 \\
\geqslant \;& \|\nabla u\|_{L^p(\Omega)}^p - C |\mu| \, \|u\|_{L^p(\Omega)}^2 \\
\geqslant \;& \|u\|_X^p - C |\mu| \, \|u\|_X^2
\end{aligned}$$

and since $p > 2$, this tends to infinity with $\|u\|_X^p$.

The conclusion is that when $\mu = 0$, the solution is unique in $W_0^{1,p}(\Omega)$ for $p > 1$, when $\mu > 0$, the solution is unique in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ for $p > 1$, and when $\mu < 0$ the solution exists for $p > 2$.

4.2 A remark on the growth bound

In the theorem on the existence of solution to the variational inequality problem we proved, we did not impose a specific growth constraint, but we required that F maps bounded sets in M into bounded sets in X^*. In the two applications we just considered, we satisfied this hypothesis by imposing a growth bound on F, namely that $F(\lambda)$ grows at most polynomially in λ. In the case of exponential growth, the situation is much more delicate, as the following example shows.

Consider the following one-dimensional boundary value problem

$$-u''(x) = C\mathrm{e}^u \text{ in } (0,1), \text{ with } u(0) = u(1) = 0.$$

where $C > 0$ is a constant. Let us show that there is no $u \in C([0,1])$ solution to this problem for some values of C. Let us first show that there is no solution in $H_0^1(\Omega)$. Note that the function $\psi(x) = \sin(\pi x)$ satisfies $-\psi''(x) = \pi^2\psi(x)$, and $\psi(0) = \psi(1)$. Writing the weak form of the equation, we find

$$\int_0^1 u'w'dx = -\int_0^1 uw''dx = C\int_0^1 \mathrm{e}^u w dx,$$

for all $w \in H^2(0,1) \cap H_0^1(\Omega)$. Choosing $w = \psi$, and placing all u dependent terms on the right hand side, we have

$$\int_0^1 (\pi^2 u - C\mathrm{e}^u)\psi dx = 0.$$

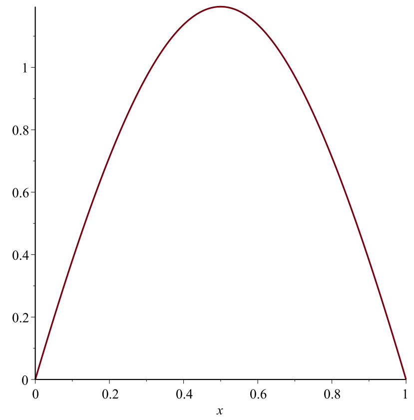
The function $x \to \pi^2 x - C\mathrm{e}^x$ is negative for all for $C > \frac{\pi^2}{e}$. Thus, for any such C this implies that $(\pi^2 u - C\mathrm{e}^u)\psi < c < 0$, which leads to a contradiction. To show that there is no continuous solution to the weak formulation

$$-\int_0^1 uw''dx = C\int_0^1 \mathrm{e}^u w dx,$$

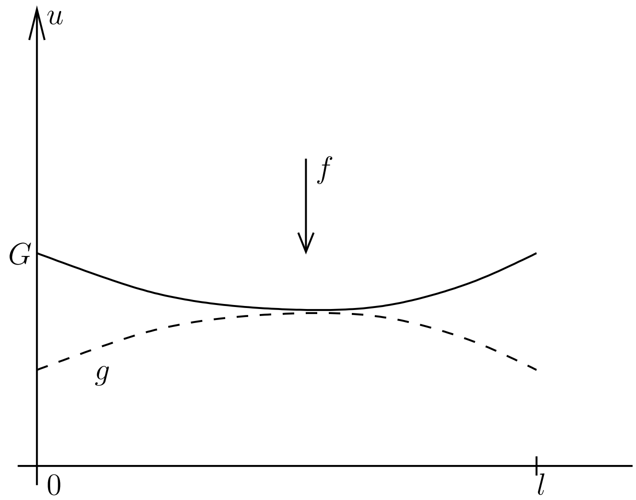
with $w \in C_c^\infty(0,1)$, we can argue similarly, since ψ can be obtained as the limit of such functions.

On the other hand for C small enough, one can prove existence of a solution using e.g. Schauder's FPT, and indeed it is possible to find a closed form solution, and the threshold value of non-existence by hand, $C = 8\left(\max_{x>0} \frac{x}{\cosh x}\right)^2 \approx 3.5138307$. Up to that critical value, the solution (represented below for $C = 3.5138306$) is perfectly smooth.

This non-existence proof relies crucially on the existence of a non-negative function $\psi \in H_0^1(\Omega)$ such that $-\Delta\psi = \lambda\psi$ with $\lambda > 0$. This fact is true in any dimension for smooth domains but this is out of the scope of this course.



4.3 A (simple) variational inequality: an elastic beam



We consider an elastic beam whose deviation from a flat state is described by a function $v\colon [0,l] \to \mathbb{R}$. We fix the beam at the end points $x = 0, l$ such that v satisfies the boundary conditions $v(0) = v(l) = 0$.
A simple model for the energy of the beam is

$$I(v) = \int_0^l \left\{ \frac{a_0}{2} |v''|^2 + fv \right\} \mathrm{d}x$$

where $f\colon [0,l] \to \mathbb{R}$ represents an outer force and $a_0 > 0$ is an elasticity constant.

Furthermore $g\colon [0,l] \to \mathbb{R}$, g smooth, $g(0), g(l) < 0$ represents a rigid obstacle.
Let $f \in L^2(0,l)$ and $X = H^2(0,l) \cap H^1_0(0,l)$. Then X is a reflexive separable Banach space and $\|u\| := \|u''\|_{L^2(0,l)} + \|u'\|_{L^2(0,l)}$ is a norm on X due to Poincaré's inequality.
We are interested in finding a minimiser u of I in the set

$$M := \left\{v \in H^2(0,l) \cap H^1_0(0,l) \mid v(x) \geqslant g(x)\ \forall x \in [0,l]\right\}$$

Remark. Since $H^1_0(0,l) \hookrightarrow C([0,l])$ it makes sense to say that $v(x) \geqslant g(x)$ for all x.

Claim. *M is a non-empty, closed and convex subset of $H^2(0,l) \cap H^1_0(0,l)$*

Proof.

(a) *Convexity:* if $u, v \in M$ then $tu(x) + (1-t)v(x) \geqslant g(x)$ for all $t \in [0,1]$ and hence $tu + (1-t)v \in M\ \forall t \in [0,1]$.

(b) *Closedness:* if $(u_n) \subset M$ such that $u_n \to u$ in $H^2(0,l) \cap H^1_0(0,l)$ then, since $H^1_0(0,l) \hookrightarrow C([0,l])$, we find that $u_n \to u$ uniformly in $C([0,l])$ Thus $u(x) \geqslant g(x)$ for all $x \in [0,1]$ and consequently $u \in M$.

(c) *M non-empty:* Since $g(0), g(l) < 0$ and g is smooth we can easily construct $u \in M$, e.g. by a parabola with sufficiently negative curvature. □

Next we derive the variational inequality which is satisfied by a minimiser u of I on M. If u is minimiser, then $I(u) \leqslant I(v)$ for all $v \in M$. If $v \in M$ then $u + \varepsilon(v-u) \in M$ for all $\varepsilon \in (0,1)$ and thus $I(u) \leqslant I(u + \varepsilon(v-u))$ for all $v \in M$ and $\varepsilon \in (0,1)$ and consequently $0 \leqslant \dfrac{I(u+\varepsilon(v-u)) - I(u)}{\varepsilon}$ for all $v \in M$ and $\varepsilon \in (0,1)$
Passing to the limit $\varepsilon \to 0$ we find $\partial_{v-u} I(u) \geqslant 0$ for all $v \in M$ such that

$$0 \leqslant \partial_{v-u} I(u) = \int_0^l a_0 u''(v'' - u'') + f(v-u)\,\mathrm{d}x =: \langle F(u), v-u\rangle.$$

We now introduce the operator

$$A\colon H^2(0,l) \cap H^1_0(0,l) \to (H^2(0,l))^*\,, \qquad \langle A(u), v\rangle := \int_0^l a_0 u'' v''\,\mathrm{d}x.$$

On Problem Sheet 4 you show that A *is a strongly monotone operator* on $H^2(0,l) \cap H^1_0(0,l)$ and hence coercive and satisfies the continuity condition (H3). To apply the existence Theorem 3.5 we still need to show that
A maps bounded into bounded sets: Let $u \in H^2 \cap H^1_0$ and $\|u\| \leqslant K$ Then

$$|\langle A(u), v\rangle| \leqslant a_0 \|u''\|_{L^2} \|v''\|_{L^2} \leqslant a_0 \|u\| \|v\| \leqslant a_0 K \|v\|$$

and thus $\sup\limits_{\|u\|\leqslant K} \|A(u)\|_{(H^2)^*} \leqslant a_0 K$.

Now consider $F\colon M \to (H^2)^*$ defined via $\langle F(u), v\rangle := \langle A(u), v\rangle + \int_0^l fv \,\mathrm{d}x$, i.e. we add a constant linear operator to A. This operator is well-defined and continuous, since

$$\left| \int_0^l fv \,\mathrm{d}x \right| \leqslant \|f\|_{L^2} \|v\|_{L^2} \leqslant C \|f\|_{L^2} \|v\|,$$

and as a constant operator it is also hemicontinuous.
We conclude that F is also a coercive monotone operator, (notice that the constant part drops out in the monotonicity condition: $\langle F(u) - F(v), u - v\rangle = \langle A(u) - A(v), u - v\rangle \geqslant c_0 \|u - v\|^2$), which maps bounded into bounded sets.
Hence, due to the Theorem 3.5 and the uniqueness result for strictly monotone operators there exists a unique $u \in M$ such that $\langle F(u), u - v\rangle \leqslant 0$ for all $v \in M$.

What else can we say about u?
Let $D = \{x \in (0, l) \mid u(x) > g(x)\}$. Notice that, since u and g are continuous, the set D is open. Let us also assume that f is smooth and that u is smooth in D – not that difficult to show, but outside the scope of this lecture where regularity is not mentioned. Then for any $x_0 \in D$ there exists $\delta > 0$ such that $B_\delta(x_0) \subset D$. Now choose $\eta \in C_0^\infty(B_\delta(x_0))$ such that we can take $v = u \pm \varepsilon\eta \in M$ for sufficiently small ε. We conclude that

$$\int_0^l a_0 u'' \eta'' + f\eta \,\mathrm{d}x = 0$$

and after an integration by parts that

$$\int_0^l (a_0 u'')'' + f\eta \,\mathrm{d}x = 0.$$

Since $x_0 \in D$ is arbitrary and $\eta \in C_0^\infty(B_\delta(x_0))$ is arbitrary, it follows that

$$(a_0 u'')'' + f = 0 \quad \text{in } D.$$

Thus, whenever the beam is strictly above the obstacle, the function v fulfils the Euler–Lagrange equation for I. Where the beam is sitting on obstacle, v is obviously equal to the obstacle and we can also conclude that $(a_0 u'')'' + f \geqslant 0$.

An important question in pratice is how smooth u is at the points where it touches the obstacle (less regular implies more prone to break. It is indeed quite smooth, but a proof is (far) beyond the scope of this lecture.

Remark. We could also consider a domain $\Omega \subset \mathbb{R}^n$, smooth and bounded, and a smooth function $g\colon \overline{\Omega} \to \mathbb{R}$ with $g \leqslant 0$ on $\partial\Omega$. The higher dimensional analogue of the energy I is

$$I(v) = \int_\Omega \frac{a_0}{2}|\Delta v|^2 + fv\,dx\,.$$

We are interested in finding a minimiser u of I on the set

$$M := \big\{v \in H^2(\Omega) \cap H_0^1(\Omega) \,|\, v \geqslant g \;\text{ a.e. on } \Omega\big\}\,.$$

Notice that now we can in general not request that $v(x) \geqslant g(x)$ for all $x \in \Omega$, since a function in M is not necessarily continuous (even though in the case of a plate, i.e. in case $n = 2$, it is, due to the embedding theorems).

We can now proceed analogously to the one-dimensional case (see Problem Sheet 7) to find that there exists a unique $u \in M$ that minimizes I on M. This u satisfies

$$\int_\Omega a_0 \Delta u \Delta(v-u) + f(v-u)\,dx \geqslant 0 \qquad \text{for all } v \in M\,.$$

Define $D := \{x \in \Omega \,|\, u(x) > g(x)\}$. One cannot immediately say that D is open, however, one can prove (which is again not possible within this lecture) that the minimiser u is sufficiently smooth, such that indeed D is open. Proceeding then as above, we find that u solves $a_0\Delta^2 u + f = 0$ in D. The set $\partial D \cap \Omega$ is an unknown in this problem, a so called *free boundary.* The advantage of variational inequalities is that one can study this problem without having to decide what this free boundary is.

In the next section we consider another physically motivated problem.

4.4 The Stationary Navier–Stokes equations

In this chapter we consider a container, whose interior is represented by a smooth, open and bounded domain $\Omega \subset \mathbb{R}^3$, which is filled with a fluid. We denote by

$\mathbf{u}\colon \Omega \to \mathbb{R}^3$	the velocity field of the fluid,
$p\colon \Omega \to \mathbb{R}$	the pressure,
$\mathbf{f}\colon \Omega \to \mathbb{R}^3$	an outer force density, e.g. gravity,
$\mathrm{Re} > 0$	the Reynolds number.

Our goal is to find a solution of the stationary Navier-Stokes equations, which are given by

$$\begin{aligned} -\frac{1}{\mathrm{Re}}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{u} \qquad \text{in } \Omega \qquad (1)\\ \mathrm{div}\mathbf{u} &= 0 \qquad \text{in } \Omega \qquad (2)\\ \mathbf{u} &= 0 \qquad \text{on } \partial\Omega \qquad (3) \end{aligned}$$

where

$$(\mathbf{u} \cdot \nabla) u_i = \sum_{j=1}^{3} u_j \partial_j u_i.$$

(*Recall also:* $\Delta \mathbf{u} \in \mathbb{R}^3$, $(\Delta \mathbf{u})_i = \sum_{j=1}^{3} \partial_{jj}^2 u_i$; $\mathrm{div} \mathbf{u} \in \mathbb{R}$, $\mathrm{div} \mathbf{u} = \sum_{j=1}^{3} \partial_j u_j$.)

Equation (1) comes from conservation of momentum, (2) means that the fluid is incompressible, and (3) is the so-called no-slip boundary condition.

Our first goal is to find an appropriate weak formulation of (1)-(3). First note the following algebraic identity

$$\mathrm{div}(\mathbf{u} \otimes \mathbf{u})_i = \sum_{j=1}^{3} \partial_j (u_i u_j) = \sum_{j=1}^{3} u_j \partial_j u_i + u_j \mathrm{div}(\mathbf{u}).$$

Therefore, when $\mathrm{div}(\mathbf{u}) = 0$, we have

$$-\frac{1}{\mathrm{Re}} \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = -\mathrm{div} \left(\frac{1}{\mathrm{Re}} \nabla \mathbf{u} - \mathbf{u} \otimes \mathbf{u} \right) + \nabla p = -\mathrm{div} \left(F(\nabla \mathbf{u}, \mathbf{u}) \right) + \nabla p$$

with $F(\nabla \mathbf{u}, \mathbf{u}) := \frac{1}{\mathrm{Re}} \nabla \mathbf{u} - \mathbf{u} \otimes \mathbf{u}$.

Now we define the space $X = H_0^1(\Omega, \mathbb{R}^3)$ endowed with the norm $\|\mathbf{u}\|_X = \|\nabla \mathbf{u}\|_{L^2(\Omega)^{3\times 3}}$ and consider the incompressible subset

$$M = \{\mathbf{v} \in X \mid \mathrm{div} \mathbf{v} = 0 \text{ a.e. in } \Omega\},$$

that is, be the subset of divergence-free vector-fields. We have incorporated all the constraints coming from the equations.

Given $\mathbf{w} \in M$, we find

$$\int_\Omega -\mathrm{div} \left(F(\mathbf{u}) \right) \cdot \mathbf{w} + \nabla p \cdot \mathbf{w} \, \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}x$$

An integration by parts gives

$$\int_\Omega F(\mathbf{u}) \colon \nabla \mathbf{w} + p \mathrm{div} \mathbf{w} = \int_\Omega \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}x,$$

where for $A, B \subset \mathbb{R}^{n\times n}$ we denote $A : B = \sum_{i,j=1}^{n} a_{ij} b_{ij}$. Note that the second term on the left-hand side is nought, since $\mathbf{w} \in M$.

For the right-hand side to make sense, we choose $\mathbf{f} \in L^{\tilde{2}}$, with $\frac{1}{\tilde{2}} + \frac{1}{2} = 1 + \frac{1}{3}$, that is, $\tilde{2} = \frac{6}{5}$.

We now have a clearly defined problem : find $\mathbf{u} \in M$ such that

$$\langle F(\mathbf{u}), \mathbf{w} \rangle = \langle \mathbf{f}, \mathbf{w} \rangle \quad \text{for all } \mathbf{w} \in M. \tag{4}$$

Theorem 4.2. *Given* $\mathbf{f} \in L^{6/5}(\Omega, \mathbb{R}^3)$ *there exists a solution to* (4). *If* $\|\mathbf{f}\|_{L^{6/5}(\Omega)}$ *is small enough, the solution is unique.*

Proof. We need to show:

(a) M is a closed subspace of X.

Let $\mathbf{v}_n \to \mathbf{v}$ in M. Then there exists a subsequence $(\mathbf{v}_{n_k})$ such that $\nabla \mathbf{v}_{n_k} \to \nabla \mathbf{v}$ a.e. in Ω Thus $0 = \text{div}\mathbf{v}_{n_k} \to \text{div}\mathbf{v}$ a.e. in Ω. Thus $\text{div}v = 0$ a.e. in Ω, that is, $v \in M$.

(b) If $\mathbf{f} \in L^{6/5}(\Omega)$ then $\langle \mathbf{f}, \mathbf{w} \rangle$ is well-defined.

The embedding theorems give $H_0^1(\Omega, \mathbb{R}^3) \hookrightarrow L^6(\Omega, \mathbb{R}^3)$. If $p = 6$ then $p' = 6/5$ and thus $f \in L^{p'} \simeq (L^6)^*$ implies that $f \in X^*$. Hence $\langle \mathbf{f}, \mathbf{w} \rangle$ is indeed the representation of a map in X^*.

(c) $F : X \to X^*$ is well-defined and maps bounded into bounded sets. We estimate

$$|\langle F(\nabla \mathbf{u}, \mathbf{u}), \mathbf{w} \rangle| = \frac{1}{\text{Re}} \int_\Omega \nabla \mathbf{u} \colon \nabla \mathbf{w} \, \mathrm{d}x - \int_\Omega u \otimes u \colon \nabla \mathbf{w} \, \mathrm{d}x$$

$$\leqslant \frac{1}{\text{Re}} \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{w}\|_{L^2} + \left\| |\mathbf{u}|^2 \right\|_{L^2} \|\nabla \mathbf{w}\|_{L^2}$$

$$\leqslant \frac{1}{\text{Re}} \|\mathbf{u}\|_X \|\mathbf{w}\|_X + \|u\|_{L^4}^2 \|\nabla \mathbf{w}\|_{L^2}.$$

Thanks to the Sobolev embeddings, $H_0^1(\Omega, \mathbb{R}^3) \hookrightarrow L^4(\Omega, \mathbb{R}^3)$, thus

$$|\langle F(\nabla \mathbf{u}, \mathbf{u}), \mathbf{w} \rangle| \leqslant \left(\frac{1}{\text{Re}} \|\mathbf{u}\|_X + C \|\mathbf{u}\|_X^2 \right) \|\nabla \mathbf{w}\|_{L^2}.$$

which implies that F is well-defined and maps bounded into bounded sets.

(d) $F(\nabla \mathbf{u}, \mathbf{u}) - \mathbf{f}$ is coercive.

For any $\mathbf{w} \in C_c^\infty(\Omega) \cap M$, we have

$$\begin{aligned}
\int_\Omega \mathbf{w} \otimes \mathbf{w} \colon \nabla \mathbf{w} \, \mathrm{d}x &= \int_\Omega \sum_{i,j=1}^{3} w_j \partial_j w_i w_i \, \mathrm{d}x \\
&= \int_\Omega \frac{1}{2} \sum_{i,j=1}^{3} w_j \partial_j (w_i^2) \, \mathrm{d}x \\
&= -\int_\Omega \frac{1}{2} \sum_{i,j=1}^{3} (\partial_j w_j) w_i^2 \, \mathrm{d}x \\
&= -\int_\Omega \frac{1}{2} |\mathbf{w}|^2 \, \text{div}\mathbf{w} \, \mathrm{d}x = 0.
\end{aligned}$$

By approximation, this also holds for all $\mathbf{w} \in M$. Therefore

$$\langle F(\nabla \mathbf{u}, \mathbf{u}) - \mathbf{f}, \mathbf{u} \rangle \geqslant \frac{1}{\mathrm{Re}} \int_\Omega \nabla \mathbf{u} \colon \nabla \mathbf{u} \, \mathrm{d}x + 0 - \|\mathbf{f}\|_{L^{6/5}} \|\mathbf{u}\|_X = \|\mathbf{u}\|_X \left(\|\mathbf{u}\|_X - \|\mathbf{f}\|_{L^{6/5}} \right).$$

and the coercivity follows.

(e) $F(\nabla \mathbf{u}, \mathbf{u}) - \mathbf{f}$ satisfies (H3).

Suppose $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in X with $u \in M$. Then, $\nabla \mathbf{u}_n \rightharpoonup \nabla \mathbf{u}$ in L^2. Since the embedding $X \hookrightarrow L^4$ is compact, $\mathbf{u}_n \to \mathbf{u}$ in L^4, and $\mathbf{u}_n \otimes \mathbf{u}_n \to \mathbf{u} \otimes \mathbf{u}$ in L^2. In particular, $F(\nabla \mathbf{u}_n, \mathbf{u}_n) - \mathbf{f} \rightharpoonup F(\nabla \mathbf{u}, \mathbf{u}) - \mathbf{f}$. We now compute

$$\begin{aligned}
\langle F(\nabla \mathbf{u}_n, \mathbf{u}_n) - \mathbf{f}, \mathbf{u}_n \rangle &= \frac{1}{\mathrm{Re}} \int_\Omega \nabla \mathbf{u}_n \colon \nabla \mathbf{u}_n \, \mathrm{d}x - 0 - \int_\Omega \mathbf{f} \cdot \mathbf{u}_n \, \mathrm{d}x \\
&\geqslant \frac{1}{\mathrm{Re}} \int_\Omega |\nabla (\mathbf{u}_n - \mathbf{u})|^2 \, \mathrm{d}x - \frac{1}{\mathrm{Re}} \int_\Omega \nabla \mathbf{u} \colon \nabla \mathbf{u} \, \mathrm{d}x \\
&\quad + \frac{2}{\mathrm{Re}} \int_\Omega \nabla \mathbf{u} \colon \nabla \mathbf{u}_n \, \mathrm{d}x - \int_\Omega \mathbf{f} \cdot \mathbf{u}_n \, \mathrm{d}x \\
&\geqslant -\frac{1}{\mathrm{Re}} \int_\Omega \nabla \mathbf{u} \colon \nabla \mathbf{u} \, \mathrm{d}x + \frac{2}{\mathrm{Re}} \int_\Omega \nabla \mathbf{u} \colon \nabla \mathbf{u}_n \, \mathrm{d}x - \int_\Omega \mathbf{f} \cdot \mathbf{u}_n \, \mathrm{d}x \\
&\to \frac{1}{\mathrm{Re}} \int_\Omega \nabla \mathbf{u} \colon \nabla \mathbf{u} \, \mathrm{d}x - \int_\Omega \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \\
&= \langle F(\nabla \mathbf{u}, \mathbf{u}) - \mathbf{f}, \mathbf{u} \rangle,
\end{aligned}$$

and (H3) is verified.

We have obtained that there exists $u \in M$ such that for all $v \in M$,

$$\langle \operatorname{div}(F(\mathbf{u})) - \mathbf{f}, \mathbf{u} - \mathbf{w} \rangle \leqslant 0 \text{ for all } \mathbf{w} \in M.$$

Now, note that if $\eta = \mathbf{u} - \mathbf{w} \in M$, then so is $-\eta$, so the equality holds.
Let us now turn to uniqueness. Assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of (4). Let us first bound their norm. Using $\mathbf{w} = \mathbf{u}_1$ in (4), we obtain

$$\int_\Omega \nabla \mathbf{u}_1 \colon \mathbf{u}_1 = \int_\Omega \nabla \mathbf{u}_1 \colon \mathbf{u}_1 - \int_\Omega \mathbf{u}_1 \otimes \mathbf{u}_1 \colon \nabla \mathbf{u}_1 = \int_\Omega \mathbf{f} \cdot \mathbf{u}_1 \leqslant C \|\mathbf{f}\|_{L^{6/5}(\Omega)} \|\mathbf{u}_1\|_X ,$$

using Solobev embeddings, therefore $\|\mathbf{u}_1\|_+ \|\mathbf{u}_2\|_\leqslant C \|\mathbf{f}\|_{L^{6/5}(\Omega)}$. Next, we note that

$$\begin{aligned}
|\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2| &= \frac{1}{2} |(\mathbf{u}_1 + \mathbf{u}_2) \otimes (\mathbf{u}_1 - \mathbf{u}_2) + (\mathbf{u}_1 - \mathbf{u}_2) \otimes (\mathbf{u}_1 + \mathbf{u}_2)| \\
&\leqslant |\mathbf{u}_1 + \mathbf{u}_2| \, |\mathbf{u}_1 - \mathbf{u}_2| .
\end{aligned}$$

Subtracting (4) written for $\mathbf{u}_1$ and $\mathbf{u}_2$ we have

$$\int_\Omega \nabla(\mathbf{u}_1 - \mathbf{u}_2) \colon \nabla \mathbf{w} = \int_\Omega (\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2) \colon \nabla \mathbf{w} \leqslant \|\mathbf{u}_1 + \mathbf{u}_2\|_{L^4} \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^4} \|\mathbf{w}\|_X .$$

Applying this inequality to $\mathbf{w} = (\mathbf{u}_1 - \mathbf{u}_2)$ we obtain

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_X^2 \leqslant \|\mathbf{u}_1 + \mathbf{u}_2\|_{L^4} \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^4} \|\mathbf{u}_1 - \mathbf{u}_2\|_X .$$

Thanks to the embedding $X \hookrightarrow L^4$, we obtain

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_X^2 \leqslant C \|\mathbf{u}_1 + \mathbf{u}_2\|_X \|\mathbf{u}_1 - \mathbf{u}_2\|_X^2 \leqslant C \|\mathbf{f}\|_{L^{6/5}(\Omega)} \|\mathbf{u}_1 - \mathbf{u}_2\|_X^2 .$$

This implies $\mathbf{u}_1 = \mathbf{u}_2$ if $C \|\mathbf{f}\|_{L^{6/5}(\Omega)} < 1$. $\square$

Remark. To solve the full problem one also needs to show that there exists $p \in L^2(\Omega)$ such that

$$\frac{1}{\mathrm{Re}} \int_\Omega \{\nabla u : \nabla \phi + u \cdot \nabla \phi u + p \operatorname{div} \phi\} \, \mathrm{d}x = \int_\Omega f \cdot \phi \, \mathrm{d}x \quad \forall \phi \in X.$$

This can be done, but is outside the scope of this lecture.

4.5 Appendix: Convex hulls of precompact sets are precompact

Lemma 4.3 (Mazur)**.** *Let X be a Banach space and $M \subseteq X$ be precompact. Then* $\operatorname{conv}(M)$ *is precompact.*

Proof. Let $\varepsilon > 0$. Since M is precompact, there exist $x_1, \dots, x_N \in M$ such that for all $x \in M$ there exists $i \in \{1, \dots, N\}$ with

$$\|x - x_i\|_X < \frac{\varepsilon}{2}. \tag{$*$}$$

Define $v(x) := j$, where j is the smallest index such that $(*)$ is satisfied. If $y \in \operatorname{conv}(M)$ then $y = \sum_{i=1}^m \alpha_i y_i$ for some $y_i \in M$ and $\alpha_i \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$.
Then

$$\left\| y - \sum_{i=1}^m \alpha_i x_{v(y_i)} \right\|_X = \left\| \sum_{i=1}^m \alpha_i (y_i - x_{v(y_i)}) \right\|_X \leqslant \sum_{i=1}^m \alpha_i \left\| y_i - x_{v(y_i)} \right\|_X < \frac{\varepsilon}{2}.$$

Since $\sum_{i=1}^m \alpha_i x_{v(y_i)} \in K =: \operatorname{conv}(x_1, \dots x_N)$ we thus showed that

$$\operatorname{conv}(M) \subseteq \bigcup_{x \in K} B_{\frac{\varepsilon}{2}}(x).$$

But K is also the image of a compact set under the following continuous map:

$$\psi\colon [0,1]^n \to X : (\alpha_1, \dots, \alpha_n) \mapsto \sum \alpha_i x_i.$$

With $A := \{(\alpha_1, \dots, \alpha_n) \in [0,1]^n : \sum \alpha_i = 1\}$ we have $\psi(A) = K$. Hence K is compact and thus there exist $k_1, \dots, k_m \in K$ such that $K \subseteq \bigcup_{i=1}^M B_{\frac{\varepsilon}{2}}(K_i)$. As a consequence $\operatorname{conv}(M) \subseteq \bigcup_{i=1}^M B_\varepsilon(K_i)$. □

4.6 Appendix: Characterisation of compact sets in C and L^p

Theorem 4.4 (Arzela–Ascoli). *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and $F \subset C(\overline{\Omega})^N$ be a family of continuous functions on $\overline{\Omega}$. Then F is precompact if and only if F is bounded and equicontinuous, i.e.*

(i) $\sup_{f\in F} \sup_{x\in\overline{\Omega}} |f(x)| \leqslant C$*;*

(ii) $\sup_{f\in F} |f(x+h) - f(x)| \to 0$ *as* $|h| \to 0$ *(*$\forall x \in \Omega,\ x+h \in \Omega$*).*

Theorem 4.5 (Riesz–Kolmogorov). *Let $1 \leqslant p < \infty$ and $F \subset L^p(\mathbb{R}^n)$. Then F is precompact if and only if*

(i) $\sup_{f\in F} \|f\|_{L^p(\mathbb{R}^n)} \leqslant C$*;*

(ii) $\sup_{f\in F} \|f(\cdot + h) - f\|_{L^p(\mathbb{R}^n)} \to 0$ *as* $|h| \to 0$*;*

(iii) $\sup_{f\in F} \|f\|_{L^p(\mathbb{R}^n\setminus B_R(0))} \to 0$ *as* $R \to \infty$*.*

4.7 Appendix: Positive and negative parts of $W^{1,p}(\Omega)$ functions are in $W^{1,p}(\Omega)$

Given $u \in W^{1,p}(\Omega)$, define $u^+ = \max(0,u)$ and $u^- = max(0,-u)$.

Lemma 4.6. *Assume $u \in W^{1,p}(\Omega)$ and let H be the Heaviside function given by*

$$H(x) = 1 \text{ if } x > 0 \text{ and } 0 \text{ otherwise.}$$

Then, u^+, u^-, $|u| \in W^{1,p}(\Omega)$ *and*

$$\begin{aligned} \nabla\left(u^+\right) &= H(u)\nabla u \\ \nabla\left(u^-\right) &= -H(-u)\nabla u \\ \nabla|u| &= \nabla u\left(H(u) - H(-u)\right) \end{aligned}$$

Proof. It is sufficient to prove it for $p = 1$. Indeed, if $p > 1$ then in any open $\Omega_1$such that $\bar{\Omega}_1$ is compact in Ω, we have $\nabla u \in L^1(\Omega_1)$ thus $\nabla\left(u^+\right) = H(u)\nabla u$. Since Ω_1 is arbitrary, this shows that ∇u^+ in a distributional sense, is in $L^p(\Omega)$.
Let $j_\epsilon : \mathbb{R} \to \mathbb{R}$ be such that

$$j_\epsilon(t) = \sqrt{t^2 + \epsilon^2} - \epsilon \text{ for } t > 0 \text{ and } j_\epsilon(t) = 0 \text{ for } t \leqslant 0.$$

It is easy to see that j_ϵ converges uniformly towards $j(t) = t^+$ and that $j'_\epsilon(t)$ converges for all t towards $H(t)$. Let $u \in L^1_{\text{loc}}(\Omega)$. Thanks to the Dominated Convergence Theorem, $j_\epsilon(u)$ converges u^+ in $L^1_{\text{loc}}(\Omega)$.
Furthermore, $\nabla\left(j_\epsilon(u)\right) = \left(u^2 + \epsilon^2\right)^{-1/2}\left(u^+\nabla u\right)$ converges, for almost every x towards $H(u)\nabla u$ and is dominated by $|\nabla u|$. We deduce that in $L^1(\Omega)$, there holds

$$\lim_{\epsilon \to 0} \nabla\left(j_\epsilon(u)\right) = H(u)\nabla u.$$

Altogether this shows that $u^+ \in W^{1,p}(\Omega)$ and $\nabla\left(u^+\right) = H(u)\nabla u$. The other results follow from the fact that $u^- = (-u)^+$. □