PART A TOPOLOGY
HT 2022
EXERCISE SHEET 1

For tutorials in week 3

Exercise 1. Let $\mathcal{T}_{\text{left}}$ be the family of subsets U of $\mathbb{R}$ with the property that for every $x \in U$ there exists $\varepsilon > 0$ such that $(x-\varepsilon, x] \subseteq U$. Prove that $\mathcal{T}_{\text{left}}$ is a topology on $\mathbb{R}$.

What is the closure of $(0,1)$ with respect to this topology?

Closure, interior, accumulation points

Exercise 2. Let (X, d) be a metric space and A a non-empty subset of X. For every $x \in X$, define the distance from x to A, $\mathrm{dist}(x, A)$, by

$$\mathrm{dist}(x, A) = \inf_{a \in A} \mathrm{d}(x, a).$$

Prove that $x \in \overline{A}$ if and only if $\mathrm{dist}(x, A) = 0$.

Exercise 3. Let (X, d) be a metric space.

(1) Prove that a closed ball is a closed set in $(X, \mathcal{T}_{\mathrm{d}})$, where $\mathcal{T}_{\mathrm{d}}$ is the topology defined by the metric d. Prove that the closure of an open ball is contained in the closed ball.

(2) Draw the picture of the closed ball with centre $\left(\frac{1}{2}, 0\right)$ and of radius $\frac{1}{2}$ in $(\mathbb{R}^2, \|\cdot\|_\infty)$.

(3) Consider the subset $E = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$ of $\mathbb{R}^2$ endowed with the metric induced from $(\mathbb{R}^2, \|\cdot\|_\infty)$. What are the open and the closed ball of centre $\left(\frac{1}{2}, 0\right)$ and of radius $\frac{1}{2}$ in this new metric space ? Show that the closure of the open ball is a proper subset of the closed ball.

(4) Let $(V, \|\cdot\|)$ be a normed real vector space. Show that in this case the closure of any open ball is a closed ball.

Exercise 4. Let $f\colon X \to Y$ be a continuous map between topological spaces, and let A and B be subsets of X such that $\overline{A} = \overline{B}$. Prove that $\overline{f(A)} = \overline{f(B)}$.

Exercise 5. Give examples of subsets A_1, A_2 and A_3 of $\mathbb{R}^2$ (with its standard topology) such that

(1) $A_1' = \mathbb{R}^2$ and $(\mathbb{R}^2 \setminus A_1)' = \mathbb{R}^2$;

(2) $A_2' \not\subseteq A_2$ and $A_2 \not\subseteq A_2'$;

(3) $A_3''' \neq A_3'' \neq A_3' \neq A_3$.

Hausdorff spaces

Exercise 6. Let $(X, \mathcal{T})$ be a *Hausdorff* topological space and let A be a non-empty subset of X.

(1) Prove that an open set $U \in \mathcal{T}$ has non-empty intersection with A if and only if U has non-empty intersection with $\overline{A}$.

(2) Prove that a point x is an accumulation point of A (i.e. $x \in A'$) if and only if x is in $\overline{A \setminus \{x\}}$.

(3) Prove that A' is closed in X.

(4) Prove that $(\overline{A})' = A'$. Deduce that $(A')' \subseteq A'$.