Chapter 3

Composite Systems, Tensor Products, Entanglement

It is important to be able to build up the description of a quantum system from those of more elementary subsystems with a smaller number of degrees of freedom. In this section we introduce the basic mathematical machinery for doing this (the Hilbert space tensor product), and look at simple aspects of the phenomenon of quantum entanglement, which arises naturally as a consequence.

3.1 Hilbert space tensor product

Suppose we encounter two quantum systems that (at least in some idealisation) do not interact with one another, and that are to be taken together in a single description. (You might imagine two atoms kept far enough apart so as to be non-interacting.) The two systems, taken on their own, will have their states encoded by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, while as a composite system we should assign a single Hilbert space $\mathcal{H}_3$. How should this Hilbert space $\mathcal{H}_3$ be characterised? The following construction arises naturally from physical considerations:

- For state vectors $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$, there should exist a definite state vector $|\psi_1 \otimes \psi_2\rangle \in \mathcal{H}_3$. Such a vector is called a *pure tensor*, or alternatively, *decomposable*.[24] Evidently the set of pure tensors is just $\mathcal{H}_1 \times \mathcal{H}_2$.
- By general principles of linearity in quantum theory, we should be able to take linear superpositions of these pure tensors.

At this stage we have effectively reproduced the following definition:

Definition 3.1.1. The *free vector space* on the set $\mathcal{H}_1 \times \mathcal{H}_2$ is the vector space of all finite linear combinations of elements of $\mathcal{H}_1 \times \mathcal{H}_2$.

The free vector space overcounts in some obvious ways, and we introduce a number of identifications.

- Since state vectors only encode physical states up to overall scalar multiplication, the consequence of rescaling either tensor factor should be no different from rescaling the vector as a whole:
$$|\lambda\psi_1 \otimes \psi_2\rangle \sim \lambda\,|\psi_1 \otimes \psi_2\rangle \sim |\psi_1 \otimes \lambda\psi_2\rangle \text{ for } \lambda \in \mathbb{C}\,.$$
- If system two is definitely in state $|\psi_2\rangle$, then when system one is in a superposition of two states, the total system is in the superposition of the corresponding two decomposable states where the second system remains in $|\psi_2\rangle$.
$$|(\psi_1 + \varphi_1) \otimes \psi_2\rangle \sim |\psi_1 \otimes \psi_2\rangle + |\varphi_1 \otimes \psi_2\rangle\ .$$
- The same argument as above should hold with the two systems switched.
$$|\psi_1 \otimes (\psi_2 + \varphi_2)\rangle \sim |\psi_1 \otimes \psi_2\rangle + |\psi_1 \otimes \varphi_2\rangle\ .$$

Definition 3.1.2. The *vector space tensor product* is then defined as the quotient of the free vector space above by the equivalence relations given above,
$$\mathcal{H}_1 \otimes \mathcal{H}_2 := F(\mathcal{H}_1, \mathcal{H}_2)/\sim\ . \tag{3.1}$$

An inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is inherited from those on $\mathcal{H}_1$ and $\mathcal{H}_2$ by defining for pure tensors
$$\langle \psi_1 \otimes \psi_2 | \varphi_1 \otimes \varphi_2\rangle = \langle \psi_1|\varphi_1\rangle\,\langle\psi_2|\varphi_2\rangle\ , \tag{3.2}$$

[24] When not using *bra-ket* notation, it is common to write this vector as $\psi_2 \otimes \psi_2$. We may also sometime use $|\psi_1, \psi_2\rangle$ or $|\psi_1\rangle \otimes |\psi_2\rangle$ interchangeably. Hopefully the situation will always be clear in context.

and extending this to general elements by sesquilinearity. Note that this definition is compatible with the equivalence relations given above.

Definition 3.1.3. The Hilbert space tensor product is obtained by taking the *completion* of this vector space tensor product with respect to the norm induced by the inner product.

As usual, this completion is a technical detail that is relevant in the infinite dimensional case. It essentially means that we allow the possibility of taking infinite linear combinations of pure tensors whose norm is still finite. Finer aspects of this construction won't be important or examinable in this course, though we will see examples.

Remark 3.1.4 (Alternate construction of tensor product). There is an equivalent, in a sense much simpler, definition of the Hilbert space tensor product that is often used. Let $\{|\alpha_i\rangle\}$ and $\{|\beta_j\rangle\}$ denote bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be identified with the Hilbert space with given basis $\{|\alpha_i \otimes \beta_j\rangle\}$ (again, in the infinite dimensional case one requires completeness, which allows infinite linear combinations of these with finite norm). To a purist, the first definition has the advantage being explicitly basis-independent. For practical purposes, this latter definition is often the most useful.

If a system is described by a tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, then operators and observables that are defined to act separately on $\mathcal{H}_1$ and $\mathcal{H}_2$ naturally extend to the tensor product. If $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $A_2 : \mathcal{H}_2 \to \mathcal{H}_2$, then we can define

$$\begin{aligned} A_1 \otimes A_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 &\longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \left|\psi_1 \otimes \psi_2\right\rangle &\longmapsto \left|A_1\psi_1 \otimes A_2\psi_2\right\rangle . \end{aligned} \tag{3.3}$$

In particular, when either A_1 or A_2 is the identity operator, then this gives operators that act on the tensor product only through the second or first tensor factor, respectively. Such operators naturally commute,

$$(A_1 \otimes 1_{\mathcal{H}_2})(1_{\mathcal{H}_1} \otimes A_2) = (A_1 \otimes A_2) = (1_{\mathcal{H}_1} \otimes A_2)(A_1 \otimes 1_{\mathcal{H}_2}) , \tag{3.4}$$

which is in agreement with the physical criterion that making observations on one system should not impact another, in principle disjoint, system.

We can similarly form the n-fold tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$ with basis $\alpha_{i_1} \otimes \beta_{i_2} \otimes \ldots \otimes \gamma_{i_n}$ with i_j indexing a basis of $\mathcal{H}_j$. This is the Hilbert space for the composite of the n quantum mechanical systems described by $\mathcal{H}_i$, $i = 1, \ldots, n$. When the constituent Hilbert spaces are all identical to $\mathcal{H}$ we simply write $\otimes^n \mathcal{H}$ or $\mathcal{H}^{\otimes n}$.

A first important behaviour of Hilbert spaces under tensor product is that their dimensions (when finite) combine multiplicatively,

$$\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = \dim \mathcal{H}_1 \times \dim \mathcal{H}_2 . \tag{3.5}$$

This follows immediately from the second construction of the tensor product given above, where the number of basis elements clearly obeys this relation. It is worth pausing to compare this situation with what one encounters classically. If a two classical systems have configuration spaces of dimensions d_1 an d_2, say, then taken together their joint configuration space will be of dimension $d_1 + d_2$. In this sense, quantum mechanical state spaces get very big very fast compared to their classical analogues. Indeed, this is one of the properties that underlies the power of quantum computation.

There is a subspace of the tensor product Hilbert space that behaves a bit more classically: this is the subspace of pure tensors. Note that this is not a linear subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$, since the property of being a pure tensor is not preserved under addition. The dimensionality of the subspace of pure tensors does behave additively,

$$\dim(\mathcal{H}_1 \otimes \mathcal{H}_2)_{\text{decomposable}} = \dim \mathcal{H}_1 + \dim \mathcal{H}_2 - 1 , \tag{3.6}$$

where the correction by one comes from the equivalence of rescaling the two tensor factors in a pure tensor. Another way to see this result is to consider the relevant subspaces in projectivised Hilbert space. Here we have that subspace of decomposable states is of the form

$$\mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2) \subset \mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2) . \tag{3.7}$$

The dimensionality of the let hand side is $\dim \mathcal{H}_1 + \dim \mathcal{H}_2 - 2$, and deprojectivising to recover the subspace of the Hilbert space adds one dimension. The embedding describing how the left hand side of (3.7) sits inside the right hand side is known as the *Segre embedding*.

3.2 Example: tensor product of qubits; entanglement

Let's consider the tensor product in the simplest case of combining several qubits (see Chapter 1.2). We recall that the qubit has Hilbert space $\mathcal{H} \cong \mathbb{C}^2$; let us now (adopting Dirac notation) fixed an orthonormal basis $\{|0\rangle, |1\rangle\}$ for the qubit such that $\sigma_3 |1\rangle = 1$ and $\sigma_3 |0\rangle = -1$.[25] We can then take as a basis for the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ the following pure tensors

$$|0 \otimes 0\rangle \;, \quad |0 \otimes 1\rangle \;, \quad |1 \otimes 0\rangle \;, \quad |1 \otimes 1\rangle \;. \tag{3.8}$$

Within this vector space, the most general state takes the form

$$a\,|0 \otimes 0\rangle + b\,|0 \otimes 1\rangle + c\,|1 \otimes 0\rangle + d\,|1 \otimes 1\rangle \;. \tag{3.9}$$

while the most general pure tensor takes the form

$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes (\gamma\,|0\rangle + \delta\,|1\rangle) = \alpha\gamma\,|0 \otimes 0\rangle + \alpha\delta\,|0 \otimes 0\rangle + \beta\gamma\,|0 \otimes 0\rangle + \beta\delta\,|0 \otimes 0\rangle \tag{3.10}$$

One can check that a state of the form (3.9) can be written as in (3.10) if and only if $ad - bc = 0$, so indeed the set of pure states is a nonlinear subspace of $\mathcal{H}$ dimension $2 + 2 - 1 = 3$.

If we combine more qubits the dimension of the Hilbert space grows exponentially. In particular,

$$\otimes^n \mathbb{C}^2 \cong \mathbb{C}^{2^n} \;, \tag{3.11}$$

while the space of pure tensors is dramatically smaller (namely $2n - 1$). Indeed, entangled states are by a wide margin the generic ones in large composite quantum systems.

The two qubit system lets us introduce the simplest example of *quantum entanglement*. Suppose we have two qubit systems that are prepared (somehow) in the initial state[26]

$$|\mathrm{EPR}\rangle = \frac{|0 \otimes 1\rangle - |1 \otimes 0\rangle}{\sqrt{2}} \;. \tag{3.12}$$

Then suppose that Alice carries the first qubit with her to a faraway star system, while Bob remains on Earth with the second qubit. If Alice makes a measurement corresponding to the observable σ_3 on her qubit (so corresponding to the observable $\sigma_3 \otimes \sigma_0$ on the tensor product Hilbert space), there is a 50% probability that she will find the value $+1$ and a 50% probability that she will find the value -1. In either case, she should find that the quantum state collapses according to the wave function collapse postulate,

$$\begin{aligned} \text{Alice measures } \sigma_3 \text{ finds } +1 \quad &\longrightarrow \quad |\psi\rangle = |1 \otimes 0\rangle \;, \\ \text{Alice measures } \sigma_3 \text{ finds } -1 \quad &\longrightarrow \quad |\psi\rangle = |0 \otimes 1\rangle \;. \end{aligned} \tag{3.13}$$

In each of the collapsed states, the results of a σ_3 measurement by Bob of his qubit (corresponding to the observable $\sigma_0 \otimes \sigma_3$ on the combined system) should return a definite answer, with which answer is returned being dictated by the results of Alice's measurement. One might phrase this in a paradoxical-sounding way, as saying that when Alice makes her measurement, it instantaneously impacts the outcomes of Bob's experiment.

A still more surprising version of this situation occurs if we consider the possibility that Alice might either measure σ_3 or, say, σ_1, while Bob will definitely measure σ_3. In the former case, as we said above, the result of Bob's experiment is determined completely once Alice's measurement has been performed. However, if Alice performs a σ_1 measurement,

[25] These basis vectors are often denoted instead by $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively due to their interpretation in terms of spins, and sometimes also $|+\rangle$ and $|-\rangle$. We may use either or both of these when we revisit this system in later chapters.

[26] EPR here stands for Einstein–Podolsky–Rosen, the authors of a famous paper pointing out seemingly paradoxical properties of entangled quantum systems. This kind of a state is also sometimes called a *Bell pair*.

then the resulting state after acting with the appropriate projection operator is

$$
\begin{aligned}
&\text{Alice measures } \sigma_1 \text{ finds } +1 \quad \longrightarrow \quad |\psi\rangle = |1\otimes 0\rangle - |0\otimes 1\rangle \ , \\
&\text{Alice measures } \sigma_1 \text{ finds } -1 \quad \longrightarrow \quad |\psi\rangle = |1\otimes 0\rangle + |0\otimes 1\rangle \ .
\end{aligned}
\tag{3.14}
$$

In this case, depending on Alice's choice of what observable to measure, the probability distribution of outcomes for Bob's measurement changes completely. This sounds odd, especially in view of Einstein's theory of relativity, which says that there should be no communication faster than the speed of light. Upon additional scrutiny, however, the situation is perhaps not quite so paradoxical; though the result of Alice's measurement (and even her choice of what to measure) has an implication for what Bob might measure, Bob has no way of knowing what result Alice done or found. What we really get from the entangled state is an interesting set of *correlations* between the results of various experiments Alice and Bob might peform.[27]

3.3 Example: multi-particle systems of distinguishable particles

Another incarnation of the tensor product arises when we consider systems of several elementary particles. If our particles move in d dimensional space, then the Hilbert space for the i'th particle will be identified as $\mathcal{H}_i \cong L^2(\mathbb{R}^d)$, and for n particles we are supposed to be interested in the Hilbert space

$$
\mathcal{H} \cong L^2(\mathbb{R}^d)_1 \otimes L^2(\mathbb{R}^d)_2 \otimes \cdots \otimes L^2(\mathbb{R}^d)_n \ . \tag{3.15}
$$

The result of the Hilbert space tensor product turns out to just be the space of square-normalisable wave functions of the n particle positions $\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$, *i.e.*,

$$
\mathcal{H} \cong L^2(\mathbb{R}^{d\times n}) \ . \tag{3.16}
$$

At a technical level, this is a case where the final step of *completing* the Hilbert space is relevant. We identify a pure tensor of single-particle wave functions with a separable n-particle wave function,

$$
\psi_1(\mathbf{x}_1) \otimes \psi_1(\mathbf{x}_1) \otimes \cdots \otimes \psi_n(\mathbf{x}_n) \quad \longleftrightarrow \quad \psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \psi_1(\mathbf{x}_1)\psi_1(\mathbf{x}_1)\cdots\psi_n(\mathbf{x}_n) \ . \tag{3.17}
$$

A general n-particle wave function certainly can't be written as a finite linear combination of separable wave functions of the above form. However, given a basis $\psi_i(\mathbf{x})$, $i = 1, 2, \ldots, \infty$ for $L^2(\mathbb{R}^d)$, pure tensors formed from these basis elements do form an orthonormal basis for $L^2(\mathbb{R}^{d\times n})$.

Remark 3.3.1. The technical subtlety associated with completion of the Hilbert space is, at least formally, evaded when we choose to work with generalised position eigenstates. In this case, we introduce basis elements

$$
|\mathbf{x}_1, \ldots, \mathbf{x}_n\rangle = |\mathbf{x}_1\rangle \otimes \cdots \otimes |\mathbf{x}_n\rangle \ , \tag{3.18}
$$

which obey

$$
\langle \mathbf{x}_1, \ldots, \mathbf{x}_n | \mathbf{x}'_1, \ldots, \mathbf{x}'_n\rangle = \delta^d(\mathbf{x}_1 - \mathbf{x}'_1)\cdots\delta^d(\mathbf{x}_n - \mathbf{x}'_n) = \delta^{n\times d}((\mathbf{x}_1; \ldots; \mathbf{x}_n) - (\mathbf{x}'_1; \ldots; \mathbf{x}'_n)) \ , \tag{3.19}
$$

where in the last expression we are using the $n \times d$-dimensional Dirac delta function. Then the most general state takes the form

$$
|\psi\rangle = \int_{\mathbb{R}^{nd}} d^d\mathbf{x}_1 \cdots d^d\mathbf{x}_n \psi(\mathbf{x}_1, \ldots, \mathbf{x}_n) \, |\mathbf{x}_1, \ldots, \mathbf{x}_n\rangle \ , \tag{3.20}
$$

which is just an n-particle wave function in the usual sense as an element of $L^2(\mathbb{R}^{n\times d})$.

[27] There is a lot more to say here and we won't pursue it in this course. A further refinement of this hypothetical, due to John Stewart Bell, led to the famed *Bell's inequality*, which highlights the degree to which quantum physics diverges from what is possible in a classical world. The subject is worth investigating for one's own edification.