STRING THEORY I

Lecture 3

Chapter 2 Classical relativistic string

Last lecture:

2.1 Classical) relativistic point particle ✓

2.2 Classical relativistic string. ✓

This lecture

2.3 Classical solutions

2.3.1 EOM and boundary conditions; solutions for closed & open strings

2.3.2 Conserved charges

2.3.3 Conformal algebra (next lecture)

In summary

- Polyakov action in conformal gauge

$\partial_a = \frac{\partial}{\partial \xi^a} \qquad \{\xi^a\} = \{\tau, \sigma\}$

$$S_P^{\text{conf. gauge}}[X^\mu] = -\frac{T}{2} \int_\Sigma \underbrace{\partial_a X \cdot \partial^a X}_{-\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X} \, d\tau \, d\sigma$$

- EOM for γ_{ab}

$$T_{ab} = -\frac{2}{T} \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ab}} = 0$$

In the conformal gauge $\rightarrow$ $T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2} \eta_{ab} \eta^{cd} \partial_c X \cdot \partial_d X$

which in components:

$$T_{\tau\tau} = \frac{1}{2}(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X)$$

$$T_{\tau\sigma} = \partial_\tau X \cdot \partial_\sigma X$$

$$T_{\sigma\sigma} = \frac{1}{2}(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X) = T_{\tau\tau}$$

Note:
- T_{ab} is traceless $\qquad \eta^{ab} T_{ab} = -T_{\tau\tau} + T_{\sigma\sigma} = 0 \quad$ due to Weyl inv.
- T_{ab} is conserved $\qquad \eta^{ab} \partial_a T_{bc} = 0$

On the tracelessness of T (from BLT):

(off recorded lecture)

Let S be an action

$$S[\gamma, \phi]$$

metric on Σ — collection of fields ϕ^i on Σ

which is invariant under Weyl transformations, i.e.

$$S[\tilde{\gamma}, \tilde{\phi}] = S[\gamma, \phi]$$

where the Weyl transformation is

$$\gamma_{ab} \to \tilde{\gamma}_{ab} = e^{2\omega}\gamma_{ab}, \quad \phi^i \to \tilde{\phi}^i = e^{d_i\omega}\phi^i.$$

Then

$$0 = \delta S = \int d^2\xi \left\{ -2\frac{\delta S}{\delta\gamma^{ab}}\gamma^{ab} + \sum_i d_i \frac{\delta S}{\delta\phi^i}\phi^i \right\}\delta\omega$$

EOM for ϕ^i: $\frac{\delta S}{\delta\phi^i} = 0$; EOM for γ^{ab}: $\frac{\delta S}{\delta\gamma^{ab}} \propto T_{ab} \Longrightarrow \gamma^{ab}T_{ab} = 0$

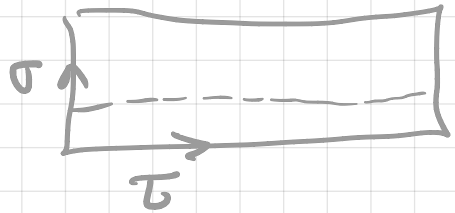
Remark: $\gamma^{ab}T_{ab} = 0$ true without using EOM for ϕ^i iff $d_i = 0$ which is the case of S_p where $\{\phi^i\} = \{X^\mu\}$

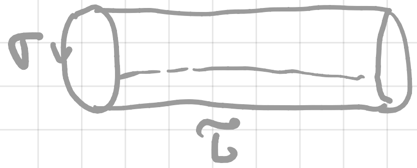
2.3 Classical solutions

We are interested in the equations of motion for the fields X^μ.

$\tau \to$ time coordinate on Σ

$-\infty \le \tau \le \infty$





$\sigma \to$ spatial coordinate on Σ

strings with finite spatial length $\quad \sigma \in [0,\pi]$

2.3.1 Equations of motion and boundary conditions

Writing the action as $S[X^\mu] = \int_\Sigma d\tau\, d\sigma\, \mathcal{L}[X^\mu, \partial_a X^\mu]$

a standard computation gives

$$\delta S = \int_\Sigma d\tau\, d\sigma \left\{ \frac{\partial \mathcal{L}}{\partial X^\mu} \delta X^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \delta \partial_a X^\mu \right\}$$

$$= \int_\Sigma d\tau\, d\sigma \left\{ \underbrace{\partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \delta X^\mu \right)}_{\text{total derivative}} + \left[\frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \right) \right] \delta X^\mu \right\}$$

$\delta S = 0$: • second term must vanish $\forall$ δX^μ of the motion

$\Longrightarrow$ Euler-Lagrange eqs $\frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \right) = 0$

• 1st term must vanish too

For the Polyakov action: $S_P^{\text{conf. gauge}}[X^\mu] = -\frac{T}{2}\int_\Sigma \underbrace{\partial_a X \cdot \partial^a X}_{\substack{\text{depends only on} \\ \partial_a X \text{ (not on } X^\mu)}} \, d\tau \, d\sigma$

- Euler Lagrange eqs for X^μ:

$\boxed{\frac{\partial \mathcal{L}}{\partial X^\mu} = 0}$

$$0 = \frac{\partial}{\partial \xi^a}\left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)}\right) = \frac{\partial}{\partial \xi^a}\left(-\frac{T}{2}\cdot 2\,\partial_a X^\mu\right)$$

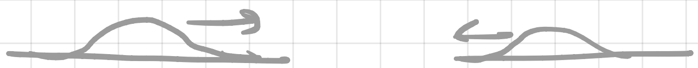
which give

$$\eta^{ab}\partial_a \partial_b X^\mu = -\partial_\tau^2 X^\mu + \partial_\sigma^2 X^\mu = 0$$

two dim wave eq
waves travelling at $c = 1$

General solution: $X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$

right moving + left-moving wavefronts



surface terms:

$$\int_\Sigma d\tau\, d\sigma\, \frac{\partial}{\partial \xi^a}\left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)}\, \delta X^\mu \right) = 0$$

$T(\partial_a X^\nu)\eta_{\mu\nu}$

$$0 = -T\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left\{ \cancel{\frac{\partial}{\partial \tau}\left(\eta_{\mu\nu}\partial_\tau X^\mu\, \delta X^\nu\right)} + \frac{\partial}{\partial \sigma}\left(\eta_{\mu\nu}\partial_\sigma X^\mu\, \delta X^\nu\right) \right\}$$

$$\downarrow$$

$$-T\int_0^\pi d\sigma\, \eta_{\mu\nu}\left(\partial_\tau X^\mu\, \delta X^\nu\right)\Big|_{\tau=\tau_i}^{\tau=\tau_f} = 0$$

analogy: particle

$\delta X(\tau_{i,f}) = 0$

choose: $\delta X^\mu(\tau_i, \sigma) = 0$, $\delta X^\mu(\tau_f, \sigma) = 0$

so

$$0 = -T\int_{\tau_i}^{\tau_f} d\tau \left(\eta_{\mu\nu}\partial_\sigma X^\mu\, \delta X^\nu\right)\Big|_{\sigma=0}^{\sigma=\pi}$$

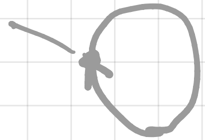
surface term: $0 = -T \int_{\tau_i}^{\tau_f} d\tau \left(\eta_{\mu\nu} \partial_\sigma X^\mu \delta X^\nu \right) \Big|_{\sigma=0}^{\sigma=\pi}$

$\boxed{\text{closed strings}}$: periodicity condition

identify $\sigma \sim \sigma + \pi$

(surface term vanishes)

$\sigma \sim \sigma+\pi$



Require

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma+\pi)$$

$$\partial_\sigma X^\mu(\tau, \sigma) = \partial_\sigma X^\mu(\tau, \sigma+\pi)$$

surface term: $0 = -T \int_{\tau_i}^{\tau_f} d\tau \left(\eta_{\mu\nu} \partial_\sigma X^\mu \delta X^\nu \right) \Big|_{\sigma=0}^{\sigma=\pi}$

$\boxed{\text{open strings}}$: boundary conditions on the string endpoints

Neumann: $\partial_\sigma X^\mu(\tau,\pi) = 0, \quad \partial_\sigma X^\mu(\tau,0) = 0$

ends of the string move freely in M

Dirichlet: $\delta X^\mu \big|_{\sigma=\pi,0} = 0$ ie $\underbrace{X^\mu(\tau,\pi) = c^\mu, \; X^\mu(\tau,0) = b^\mu}$

end points of the string are fixed in space-time

the choice of constant space-time vectors

$\Rightarrow$ breaks Poincaré invariance

One can impose Neumann boundary conditions on $d-(p+1)$ coordinates $(D = \dim M)$ and Dirichlet conditions on the rest $p+1$ coordinates.

In this case the ends of the string are fixed on subspace $Q \subset M$ of $\dim Q = p+1$.

These subspace is called a Dp-brane with c^μ & b^μ interpreted as the position of the brane.

These objects are very important in string theory and we will talk about them later

Closed strings

general soln $\quad X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$

Want solns which satisfy the periodicity conditions

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi)$$

$$\partial_\sigma X^\mu(\tau,\sigma) = \partial_\sigma X^\mu(\tau,\sigma+\pi)$$

$$X^\mu_R(\tau,\sigma) = \frac{1}{2}x^\mu + \frac{1}{2}\ell^2 p^\mu(\tau-\sigma) + \frac{i}{2}\ell \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)}$$

$$X^\mu_R(\tau,\sigma) - X^\mu(\tau,\sigma-\pi) = \frac{1}{2}\ell^2 p^\mu \pi$$

$$X^\mu_L(\tau,\sigma) = \frac{1}{2}x^\mu + \frac{1}{2}\ell^2 p^\mu(\tau+\sigma) + \frac{i}{2}\ell \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}$$

$$X^\mu_L(\tau,\sigma+\pi) - X^\mu(\tau,\sigma) = \frac{1}{2}\ell^2 p^\mu \pi$$

$\ell = (\pi T)^{-1/2}$

separately periodic up to a zero mode

where X^μ real requires $x^\mu, p^\mu \in \mathbb{R}$, $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$, $\tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^*$

Open strings

general soln $$X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$$

Want solns which satisfy Neumann boundary conditions

$$\partial_\sigma X^\mu(\tau,\pi) = 0 \,, \quad \partial_\sigma X^\mu(\tau,0) = 0$$

We find

$$X^\mu(\tau,\sigma) = \underbrace{x^\mu + \ell^2 p^\mu \tau} + i\ell \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n}\alpha_n^\mu \cos(n\sigma)\, e^{-in\tau}$$

av. position $\quad x^\mu(\tau) = \frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu(\tau,\sigma) = \downarrow$

Real X^μ : $\quad x^\mu, p^\mu \in \mathbb{R}\,, \quad \alpha_{-n}^\mu = (\alpha_n^\mu)^*$

2.3.2 Conserved charges for Poincaré symmetry

Recall & Noether's theorem: for ea symmetry in the action, there is a corresponding conserved current

$$J^a = \frac{\partial \mathcal{L}}{\partial(\partial_a \phi)} \delta\phi, \quad \partial_a J^a = 0, \quad \text{for} \quad \mathcal{L} = \mathcal{L}[\phi, \partial_a\phi, \gamma_{ab}]$$

$$S[\phi + \delta\phi] = S[\phi]$$

For a spacetime translation $\delta X^\mu = \underline{\epsilon}^\mu$

momentum density

$$J^\mu_a = T\,\partial_a X^\mu(\tau, \sigma) \qquad (\partial_a J^{a\mu} = 0)$$

conserved charge

$$P^\mu = \int_0^\pi \underline{J^\mu_\tau(\tau,\sigma)}\, d\sigma = p^\mu$$

two mode momentum

$$= T\ell^2 \pi p^\mu$$

$$\partial_\tau X^\mu = \ell^2 p^\mu + \cdots$$

true for both open & closed strings

terms that vanish upon σ-integration

For a space-time rotation $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$:

antisymmetric matrix

angular momentum density

$$J_a^{\mu\nu}(\tau,\sigma) = T(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu)$$

conserved charges

$$M^{\mu\nu} = \int_0^\pi J_\tau^{\mu\nu}(\tau,\sigma)\, d\sigma = \begin{cases} l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} & \text{closed string} \\ l^{\mu\nu} + E^{\mu\nu} & \text{open string} \end{cases}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

zero mode of angular momentum

$$E^{\mu\nu} = -i \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu)$$

$$\tilde{E}^{\mu\nu} = -i \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} (\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu)$$

oscillator modes angular momentum (spin)

Recall: we need to supplement S_P with the constraints from the stress tensor

$$\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X = 0 \qquad \& \qquad \partial_\tau X \cdot \partial_\sigma X = 0$$

Use light cone coordinates: $\sigma^{\pm} = \tau \pm \sigma$

In these coordinates: $\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

metric on Σ: $ds^2 = -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-$

$$\Rightarrow \quad \eta_{++} = \eta_{--} = 0 \qquad \eta_{+-} = \eta_{-+} = -\frac{1}{2}$$

inverse metric $\quad \eta^{++} = \eta^{--} = 0 \qquad \eta^{+-} = \eta^{-+} = -2$

In light cone coordinates,

$$T_{++} = \partial_+ X \cdot \partial_+ X \,, \qquad T_{--} = \partial_- X \cdot \partial_- X$$

and $$T_{+-} = \partial\, \partial_+ X \cdot \partial_- X = T_{-+}$$

Note T_{ab} is automatically symmetric

$$T_{+-} = T_{-+}$$

The tracelessness condition is : $T_{+-} + T_{-+} = 0$

Thus $$T_{+-} = 0$$

T_{ab} is conserved: $\eta^{ab} \partial_a T_{bc} = 0 \Rightarrow$ $\partial_+ T_{-+} + \partial_- T_{++} = 0$

$\partial_+ T_{--} + \partial_- T_{+-} = 0$

Combining these

$$\boxed{T_{+-} = 0 \,, \quad \partial_- T_{++} = 0 \,, \quad \partial_+ T_{--} = 0}$$

very powerfull

(Lorentzian version of conds for hol (antihol))

$\hookrightarrow$ next: these imply that there is an infinity of conserved charges

Closed strings

Let $f(\sigma^-)$ be an arbitrary function and consider

$$Q_f = \int d\sigma\, f(\sigma^-)\, T_{--}(\sigma^-)$$

$\partial_+ T_{--} = 0$

$$\Rightarrow \frac{\partial}{\partial\tau} Q_f = \int d\sigma\, (2\partial_+ - \partial_\sigma)(f(\sigma^-)\, T_{--}(\sigma^-))$$

$\partial_+ = \frac{1}{2}(\partial_\tau + \partial_\sigma)$

$$= - \int d\sigma\, \partial_\sigma (f(\sigma^-) T_{--}(\sigma^-)) = - (f(\sigma^-) T_{--}(\sigma^-)) \Big|_{\sigma=0,\ \tau \text{ fixed}}^{\sigma=\pi,\ \tau \text{ fixed}}$$

$= 0$ if $f(\sigma^-)$ is periodic

(Note that $T_{++} = T_{--}$ at end points)

That is: the current $f\, T_{--}$ is also conserved!

Since f is arbitrary $\Rightarrow$ there is an infinite set of conserved currents

Similarly: T_{++} is conserved and so is $g\, T_{++}$, $g = g(\sigma^+)$ periodic

We can get a complete set of conserved charges by taking $f(\sigma^-) = e^{2in\sigma^-}$ $\quad n \in \mathbb{Z}$

$$L_m = \frac{T}{2}\int d\sigma\, e^{2im\sigma^-}\, T_{--}(\sigma^-)$$

$\partial_- X \cdot \partial_- X = \partial_- X_R \cdot \partial_- X_R$

$$\tau = 0 \longrightarrow \; = \frac{T}{2}\int d\sigma\, e^{-2im\sigma}\, \partial_- X_R \partial_- X_R$$

$$\Longrightarrow \quad L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n \qquad \text{with} \quad \alpha_0^\mu = \frac{1}{2}l\, p^\mu$$

and $L_{-m} = (L_m)^*$ because T_{--} is real

similarly: for $T_{++}(\sigma^+)$, $g(\sigma^+) = e^{2im\sigma^+}$,

$$\tilde{L}_m = \frac{T}{2}\int d\sigma\, e^{2im\sigma^+}\, T_{++}(\sigma_+) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n\,, \qquad \tilde{\alpha}_0^\mu = \frac{l}{2}p^\mu$$

Notice that L_m & $\tilde{L}_m$ are the Fourier components of T_{--} & T_{++} respectively. Then setting

$$\boxed{L_m = 0 \quad , \quad \tilde{L}_m = 0} \qquad \forall \; m \in \mathbb{Z}$$

imposes the constraints $T_{++} = 0$ & $T_{--} = 0$.

The vanishing of these charges are equivalent to quadratic constraints on the oscillators α_n^μ & $\tilde{\alpha}_n^\mu$.

Consider these constraints for L_0 & $\tilde{L}_0$

$$L_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_n = \frac{\ell^2}{8} p^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 0$$

$p^2 = p^\mu p_\mu$
$= p \cdot p$

$$\tilde{L}_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = \frac{\ell^2}{8} p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = 0$$

Thm:

$$\frac{\ell^2}{8}p^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n = 0$$

$$\frac{\ell^2}{8}p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = 0$$

$$p^2 = p^\mu \cdot p_\mu$$

This implies

$$\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$$

level matching condition

Recall that p^μ = spacetime momentum $\Rightarrow M^2 = -p^2$

$\therefore$ we have a mass shell condition

$$M^2 = -p^2 = 4\pi T \sum_{n=1}^{\infty} \left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_{+n}\right)$$

$$4\pi T = \frac{2}{\alpha'} \qquad \left(\alpha' = \frac{1}{2\pi T}\right)$$

Open strings

Let $f(\sigma^+)$ & $g(\sigma^-)$ be arbitrary functions. Then

$$\partial_+ (g(\sigma^-) T_{--}) = 0 \quad \& \quad \partial_- (f(\sigma^+) T_{++}) = 0$$

Consider

$$Q_{f,g} = \int d\sigma \left(f(\sigma^+) T_{++} + g(\sigma^-) T_{--} \right)$$

Then

$$\partial_\tau Q_{fg} = \int d\sigma \left(\partial_\sigma (f(\sigma^+) T_{++}) - \partial_\sigma (g(\sigma^-) T_{--}) \right)$$

$\partial_\tau = 2\partial_- + \partial_\sigma$ $\quad$ $\partial_\tau = 2\partial_+ - \partial_\sigma$

$$= \left(f(\sigma^+) T_{++} - g(\sigma^-) T_{--} \right) \Big|_0^\pi$$

$\sigma^\pm = \tau \pm \sigma$

$\Rightarrow$ Q is conserved if $f(\sigma^+) = g(\sigma^-)$ at $\sigma = 0, \pi$

(i) $\sigma = 0 \Rightarrow$ $f(\tau) = g(\tau)$ so f & g are the same fun

(ii) $\sigma = \pi \Rightarrow$ $f(\tau + \pi) = g(\tau - \pi) = f(\tau - \pi)$

$\therefore$ $\underline{f(\tau) = f(\tau + 2\pi)}$ $\quad$ f periodic function with period 2π

Let $\quad f(\sigma^+) = e^{im\sigma^+} \qquad g(\sigma^-) = e^{im\sigma^-}$

and consider

$$Q_{f,g} \rightsquigarrow \quad L_m = \frac{T}{2} \int_0^{\pi} d\sigma \left(e^{im\sigma^+} T_{++} + e^{im\sigma^-} T_{--} \right) \qquad \text{conserved}$$

$$\tau = 0 \;\; \stackrel{}{=} \frac{T}{2} \int_0^{\pi} d\sigma \left(e^{im\sigma} T_{++} + e^{-im\sigma} T_{--} \right)$$

$$\boxed{L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n} \qquad \text{with} \quad \alpha_0^{\mu} = \ell p^{\mu}$$

<u>L_0</u> :

$$L_0 = \frac{1}{2} \ell^2 p^2 + \sum_{n=1} \alpha_{-n} \cdot \alpha_n$$

$L_0 = 0$ is :

$$M^2 = -p^2 = \underbrace{2\pi T}_{\frac{1}{\alpha'}} \sum_{n=1} \alpha_{-n} \cdot \alpha_n \qquad \text{mass-shell condition}$$

Next: conformal transformations.