STRING THEORY I

Lecture 4

$\boxed{\text{Chapter 2}}$ Classical relativistic string

2.1 Classical relativistic point particle

2.2 Classical relativistic string

2.3 Classical solutions

2.3.1 EOM and boundary conditions; solutions for closed & [illegible] strings

2.3.2 Conserved charges L_m & $\tilde{L}_m$

this lecture → 2.3.3 The Witt-algebra & conformal symmetries

2.3.3 The Witt-algebra & conformal symmetries

We constructed explicitly the space of solutions of the eqs of motion ie, the phase space.

This is an infinite dimensional affine space with coordinates

$$\{ x^\mu, p^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu \}$$

subject to quadratic constraints

$$\{ L_n = 0, \tilde{L}_n = 0, \forall n \in \mathbb{Z} \}$$

for the closed string (for the open string neglect $\tilde{\alpha}$ & $\tilde{L}$)

The L_n (& $\tilde{L}_n$) are the conserved charges corresponding to the conserved currents $\underbrace{e^{2im\sigma^-} T_{--} \; (\partial_+ T_{--} = 0)}$

$$L_n = \frac{T}{2} \int d\sigma \, e^{2im\sigma^-} \quad T_{--}(\sigma^-) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n$$

similarly for $\partial_- T_{++} = 0$

So far: working in Lagrangian formalism with

$$\mathcal{L} = \frac{T}{2}\left[\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X\right]$$

In a Hamiltonian formulation with canonical fields $X^\mu(\tau,\sigma)$ and conjugate momenta

$$\Pi^\mu(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau X_\mu(\tau,\sigma))} = T\,\partial_\tau X^\mu(\tau,\sigma)$$

we define a Hamiltonian

$$H = \int_0^\pi d\sigma \left(\partial_\tau X(\tau,\sigma)\cdot\Pi(\tau,\sigma) - \mathcal{L}\right)$$

$$= T\int_0^\pi d\sigma \left(\partial_+ X\cdot\partial_+ X + \partial_- X\cdot\partial_- X\right)$$

$$= \sum_m \left(\tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m + \alpha_{-m}\cdot\alpha_m\right) = 2\left(\underline{L_0 + \tilde{L}_0}\right)$$

Phase space is a Poisson manifold with Poisson brackets which in our case are given by

same τ

$$\{\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = \eta^{\mu\nu}\,\delta(\sigma-\sigma')$$

$$\Pi^\mu(\sigma) = \frac{\delta L}{\delta(\partial_\tau X^\mu)} = T\partial_\tau X^\mu$$

Also:

$$\{X^\mu(\sigma), X^\nu(\sigma')\}_{PB} = 0\,, \qquad \{\Pi^\mu(\sigma), \Pi^\nu(\sigma')\}_{PB} = 0$$

at same τ

These can be used to determine the Poisson bracket of the oscillator modes. Recall

$$X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)$$

where

$$X_R^\mu(\tau,\sigma) = \frac{1}{2}x^\mu + \frac{1}{2}\ell^2 p^\mu(\tau-\sigma) + \frac{1}{2}i\ell \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)}$$

$$X_L^\mu(\tau,\sigma) = \frac{1}{2}x^\mu + \frac{1}{2}\ell^2 p^\mu(\tau+\sigma) + \frac{1}{2}i\ell \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}$$

closed string

$$X^\mu(\tau,\sigma) = x^\mu + \ell^2 p^\mu \tau + i\ell \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha_n^\mu \cos(n\sigma) e^{-in\tau}$$

open string

We can extract the Fourier coefficients

For <u>closed</u> strings:

$$x^\mu + \frac{il}{2} \sum_{m\neq 0} \left(\alpha_m^\mu e^{2im\sigma} + \tilde{\alpha}_m^\mu e^{-2im\sigma} \right)$$

$n \neq 0$

$$\frac{1}{\pi} \int_0^\pi e^{-2in\sigma} X^\mu(0,\sigma)\, d\sigma = \frac{il}{2n} \left(\alpha_n^\mu - \tilde{\alpha}_{-n}^\mu \right)$$

$$\frac{1}{\pi} \int_0^\pi X^\mu(0,\sigma)\, d\sigma = x^\mu$$

$n \neq 0$

$$\int_0^\pi e^{-2in\sigma} \Pi^\mu(0,\sigma)\, d\sigma = T \int_0^\pi e^{-2in\sigma} \left(\partial_\tau X^\mu \right)\Big|_{\tau=0} d\sigma$$

$$= \frac{1}{l} \left(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu \right)$$

$$\int_0^\pi \Pi^\mu(0,\sigma)\, d\sigma = p^\mu$$

Then

$$\left.\begin{aligned} \alpha_n^\mu &= \int_0^\pi e^{-2in\sigma}\left(\frac{n}{\pi i \ell} X^\mu(0,\sigma) + \frac{\ell}{2}\Pi^\mu(0,\sigma)\right)d\sigma \\ \tilde{\alpha}_n^\mu &= \int_0^\pi e^{-2in\sigma}\left(-\frac{n}{\pi i \ell} X^\mu(0,\sigma) + \frac{\ell}{2}\Pi^\mu(0,\sigma)\right)d\sigma \end{aligned}\right\} n \neq 0$$

$$x^\mu = \frac{1}{\pi}\int_0^\pi X^\mu(0,\sigma)\,d\sigma$$

$$p^\mu = \int_0^\pi \Pi^\mu(0,\sigma)\,d\sigma$$

$$\Longrightarrow \boxed{\begin{aligned} \{\alpha_m^\mu, \alpha_n^\nu\}_{PB} &= i m\, \delta_{m+n,0}\, \eta^{\mu\nu} \\ \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\}_{PB} &= i m\, \delta_{m+n,0}\, \eta^{\mu\nu} \\ \{p^\mu, x^\nu\}_{PB} &= \eta^{\mu\nu} \end{aligned}}$$

$$\{\alpha_m^\mu, \tilde{\alpha}_n^\nu\}_{PB} = 0$$

(For open strings is the same but only one set of oscillators)

Poisson bracket of the constraints

Recall

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \qquad \text{with} \qquad \alpha_0^\mu = \frac{1}{2} \ell p^\mu$$

$$\tilde{L}_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n , \qquad \text{with} \qquad \tilde{\alpha}_0^\mu = \frac{\ell}{2} p^\mu$$

Then $\{L_m, L_n\}_{PB} = \frac{1}{4} \sum_{p\, q} \{ \alpha_{m-p} \cdot \alpha_p , \alpha_{n-q} \cdot \alpha_q \} = \cdots$

gives

$$\boxed{\begin{aligned} \{L_m, L_n\}_{PB} &= (m-n) L_{m+n} \\ \{\tilde{L}_m, \tilde{L}_n\}_{PB} &= (m-n) \tilde{L}_{m+n} \end{aligned}}$$

exercise PS 2

L_m & $\tilde{L}_n$ form a Lie algebra, the Witt algebra

This is the algebra of infinitesimal conformal transformations on the world-sheet. as we will see next.

A conformal transformation of a (Riemannian or Lorentzian) manifold Σ is a diffeomorphism $\xi \longmapsto \tilde{\xi}(\xi)$ that preserves the metric up to rescaling ie

$$\gamma_{..}(\xi) \longmapsto \tilde{\gamma}_{..}(\tilde{\xi}) = e^{2\Lambda(\tilde{\xi})}\,\gamma_{..}(\tilde{\xi})$$

($\Lambda = 0$ isometry)

Infinitesimal conformal transformations can be described explicitly.

let

$\sigma^\alpha \to (\sigma^+, \sigma^-)$

$$\sigma^\alpha \longmapsto \sigma^\alpha + \xi^\alpha(\sigma^\pm)$$

be a general infinitesimal diffeomorphism.

Then η transforms as

$\eta = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ η^{-1}

$$\eta^{\alpha\beta} \longmapsto \eta^{\alpha\beta} + \partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha$$

infinitesimal version of $\tilde{\eta}^{\alpha\beta}(\tilde{\sigma}^\alpha) = \partial_\gamma \tilde{\sigma}^\alpha \partial_\delta \tilde{\sigma}^\beta \eta^{\gamma\delta}$

This corresponds to a conformal transformation if ξ^α satisfy the equation

$\xi^\pm$

$$\boxed{\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda(\sigma^\pm)\,\eta^{\alpha\beta}}$$

infinitesimal $\tilde{\eta}^{\alpha\beta}(\tilde{\sigma}^\alpha) = e^{\eta(\tilde{\sigma})}\,\tilde{\eta}^{\alpha\beta}(\tilde{\sigma})$

(remark: $\Lambda = 0 \Rightarrow$ killing eq for isometries)

This is the <u>conformal</u> killing eq

a soln $\to$ conformal killing vector ξ^α

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda(\sigma^\pm)\,\eta^{\alpha\beta}$$

In the light cone coordinates: $\eta = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\eta^{-1} = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$(++)$ $\partial^+ \xi^+ = 0 \implies \partial_- \xi^+ = 0 \implies \xi^+ = \xi^+(\sigma^+)$

$(--)$ $\partial^- \xi^- = 0 \implies \partial_+ \xi^- = 0 \implies \xi^- = \xi^-(\sigma^-)$

$(+-)$ $\partial^+ \xi^- + \partial^- \xi^+ = \Lambda(\sigma^\pm)$ (no further constraints)

$\partial_\alpha \xi^\alpha = \Lambda$

The existence of these transformations means that our choice of gauge (conformal-unit gauge) does not fix all the gauge freedom!

In other words: even after fixing the reparametrization & Weyl gauge symmetries, there are still symmetries remaining which leave our gauge choice (eg the conformal/unit gauge) invariant.

Indeed, a conformal transformation of $\eta^{\alpha\beta}$

$$\delta\eta^{\alpha\beta} = (\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha) - \Lambda \eta^{\alpha\beta} = 0$$

where $\delta\sigma^+ = \xi^+(\sigma^+)$ $\quad \delta\sigma^- = \xi^-(\sigma^-)$, $\Lambda = \partial^- \xi^+ + \partial^+ \xi^-$

leaves η invariant.

Then we have found the classical bosonic string theory is invariant under a $\boxed{\text{larger}}$ group of symmetries

One can think of the transformations on the world sheet

$$\delta\sigma^{\pm} = \xi^{\pm}(\sigma^{\pm})$$

as generated by

$$V^{\pm} = -\frac{1}{2}\,\xi^{\pm}(\sigma^{\pm})\,\partial_{\pm}$$

generators of the group of conformal transformations on 2dim world-sheet with Minkowski metric

(correspondence between vectors, $\xi^{\pm}\partial_{\pm}$, and 1 parameter group of diffeomorphism, $\sigma^{\alpha} \mapsto \sigma^{\alpha} + \xi^{\alpha}$)

We can write a complex basis for these transformations (for closed strings) (open strings exercise)

$$V_n = -\frac{1}{2} e^{2in\sigma^-} \partial_- \,, \quad \tilde{V}_n = -\frac{1}{2} e^{2in\sigma^+} \partial_+ \qquad n \in \mathbb{Z}$$

These satisfy a Lie algebra with respect to the commutator of differential operators

$$[V_m, V_n] = i(m-n) V_{m+n}$$ & similar for $[\tilde{V}_m, \tilde{V}_n]$

$$+\frac{1}{4}[e^{2im\sigma^-}\partial_-, e^{2in\sigma^-}\partial_-] = \frac{1}{4}\left\{e^{2im\sigma^-}\partial_-(e^{2in\sigma^-}\partial_-) - \cdots\right\}$$

$$= \frac{1}{4} e^{2i(n+m)\sigma^-}(2in - 2im)\,\partial_- = i(m-n)\left(-\frac{1}{2} e^{2i(m+n)\sigma^-}\partial_-\right)$$

Then, the L_n's generate the residual gauge symmetries

The conformal symmetry is a residual gauge symmetry. This can be used to do some further gauge fixing.

But there is no space-time Lorentz invariant way to do this. At the expense of covariance one can use the light-cone gauge.

(em: Lorenz $\partial_\mu A^\mu = 0$, fix the gauge completely Coulomb gauge)

The appearence of conformal symmetry suggests that the 2dim field theory on the world sheet of the string is in fact a 2dim conformal field theory.

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(See CFT course in Trinity term or Polchinski Vol 1 Chapter 2)

As remarked earlier, the infinite-dimensional algebra associated to the conformal symmetry is special to 2-dims. In general, in $(1, d-1)$ dimensional Minkowski

$$\text{conformal algebra}_{(1,d-1)} \cong so(2,d) \supseteq \overset{\text{Lorentz}}{so(1,d-1)} \times so(1,1)$$

$\dim = \frac{1}{2}(d+2)(d+1)$ $\qquad$ $\dim = \frac{1}{2}d(d-1)$

(ex. in $d=4$ $\dim = 15$) $\qquad$ ($d=4 \to 6$)

For $d=2$: $\qquad so(2,2) \cong sl(2,\mathbb{R}) \times \widetilde{sl}(2,\mathbb{R})$

$\dim = 6$ $\qquad$ 3 $\qquad$ 3

$\{V_{0,\pm1}\}$ $\qquad$ $\{\tilde{V}_{0,\pm1}\}$

"global" part of the conformal algebra (which is $\text{Witt} \times \widetilde{\text{Witt}}$)

conformal transformations well defined on the WS

In particular $\qquad V_0 + \tilde{V}_0 \sim \partial_\tau \to H$

$\qquad V_0 - \tilde{V}_0 \sim \partial_\sigma$

Next: Quantization

There are several equivalent approaches

1. Covariant BRST quantization

Path integral:
$$Z = \int \frac{[\mathcal{D}X^\mu][\mathcal{D}\gamma]}{\mathrm{Vol}(\mathrm{Diff}\times\mathrm{Weyl})} e^{\frac{i}{\hbar} S_P[X^\mu,\gamma]}$$

best quantum mechanical treatment of a gauge theory

Fadeev-Popov-deWitt gauge fixing & identifying BRST symmetries & currents

$\rightsquigarrow$ Weyl anomaly $\Rightarrow$ $d = 26$

2 Old covariant quantization ← we will use this

quantize the classical conformally gauged systems

then one imposes the $T_{\pm\pm}$ constraints in quantum Hilbert space

3 Light-cone quantization

Fix all gauge symmetry but then not Poincaré invariant.

Then quantize the constrained string and check for consistency

↳ consistency involves subtleties eg $\sum_{n=1}^{\infty} n = -\frac{1}{12} = \zeta(-1)$

how?

Riemann ζ-function