B1 Set Theory:
Solutions to questions on problem sheet 0

1. $\varnothing$ has no elements. But $\{\varnothing\}$ has an element, namely $\varnothing$. So $\varnothing$ and $\{\varnothing\}$ do not have precisely the same elements, so they are not equal.

2. (i) For any x,

$$\begin{aligned} x \in A \cap (B \cup C) &\text{ iff } x \in A \text{ and } x \in B \cup C \\ &\text{ iff } x \in A \text{ and either } x \in B \text{ or } x \in C \\ &\text{ iff } x \in A \text{ and } x \in B\text{, or both } x \in A \text{ and } x \in C \\ &\text{ iff } x \in A \cap B \text{ or } x \in A \cap C \\ &\text{ iff } x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Hence $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$.

(ii) For any x,

$$\begin{aligned} x \in A \cup (B \cap C) &\text{ iff } x \in A \text{ or } x \in B \cap C \\ &\text{ iff } x \in A \text{ or both } x \in B \text{ and } x \in C \\ &\text{ iff } x \in A \text{ or } x \in B\text{, and either } x \in A \text{ or } x \in C \\ &\text{ iff } x \in A \cup B \text{ and } x \in A \cup C \\ &\text{ iff } x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

Hence $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$.

(iii) For any x,

$$\begin{aligned} x \in X \setminus (A \cup B) &\text{ iff } x \in X \text{ and } x \notin A \cup B \\ &\text{ iff } x \in X \text{ and } x \text{ does not belong to either } A \text{ or } B \\ &\text{ iff } x \in X \text{ and } x \notin A \text{ and } x \notin B \\ &\text{ iff } x \in X \setminus A \text{ and } x \in X \setminus B \\ &\text{ iff } x \in (X \setminus A) \cap (X \setminus B), \end{aligned}$$

so $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

(iv) For any x,

$$\begin{aligned} x \in X \setminus (A \cap B) &\text{ iff } x \in X \text{ and } x \notin A \cap B \\ &\text{ iff } x \in X \text{ and } x \text{ does not belong to both } A \text{ and } B \\ &\text{ iff } x \in X\text{, and } x \notin A \text{ or } x \notin B \\ &\text{ iff } x \in X \setminus A \text{ or } x \in X \setminus B \\ &\text{ iff } x \in (X \setminus A) \cup (X \setminus B), \end{aligned}$$

so $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

3. (i) Always true.

For any $y \in Y$,

$$\begin{aligned} y \in f(A) \cup f(B) &\text{ iff } y \in f(A) \text{ or } y \in f(B) \\ &\text{ iff there exists } x \in A \text{ such that } f(x) = y \\ &\quad\text{ or there exists } x \in B \text{ such that } f(x) = y \\ &\text{ iff there exists } x \in A \cup B \text{ such that } f(x) = y \\ &\text{ iff } y \in f(A \cup B). \end{aligned}$$

(ii) Not always true.

Let $X = \{0,1\}$, $Y = \{0\}$, f be the function taking all elements of X to 0, $A = \{0\}$, and $B = \{1\}$.

Then $f(A) = f(B) = \{0\}$, so $f(A) \cap f(B) = \{0\}$. However $A \cap B = \varnothing$, so $f(A \cap B) = \varnothing$.

(iii) Always true.

For any $x \in X$,

$$\begin{aligned} x \in f^{-1}(C) \cup f^{-1}(D) &\text{ iff } x \in f^{-1}(C) \text{ or } x \in f^{-1}(D) \\ &\text{ iff } f(x) \in C \text{ or } f(x) \in D \\ &\text{ iff } f(x) \in C \cup D \\ &\text{ iff } x \in f^{-1}(C \cup D). \end{aligned}$$

(iv) Always true.

For any $x \in X$,

$$\begin{aligned} x \in f^{-1}(C) \cap f^{-1}(D) &\text{ iff } x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \\ &\text{ iff } f(x) \in C \text{ and } f(x) \in D \\ &\text{ iff } f(x) \in C \cap D \\ &\text{ iff } x \in f^{-1}(C \cap D). \end{aligned}$$

(v) Always true.

For any $y \in Y$,

$$\begin{aligned} y \in f(f^{-1}(C)) &\Rightarrow \exists x\, (x \in f^{-1}(C) \text{ and } f(x) = y) \\ &\Rightarrow \exists x\, (f(x) \in C \text{ and } f(x) = y) \\ &\Rightarrow y \in C. \end{aligned}$$

(vi) Not always true.

Let $X = \{0,1\}$, $Y = \{0\}$, f be the function taking all elements of X to 0, and $A = \{0\}$. Then $f(A) = \{0\}$, and $f^{-1}(f(A)) = \{0,1\}$, which is not a subset of A.

(vii) Not always true.

Let $X = \{0\}$, $Y = \{0,1\}$, and let f be the function taking 0 to 0. Let $C = Y$. Then $f^{-1}(C) = X$, and $f(f^{-1}(C)) = \{0\}$.

(viii) Always true.

For any $x \in X$,

$$\begin{aligned} x \in A &\Rightarrow f(x) \in f(A) \\ &\Rightarrow x \in f^{-1}(f(A)). \end{aligned}$$

4. (ii) True if f is one-to-one. For then, for any $y \in \operatorname{ran} y$, if x is the unique element of x such that $f(x) = y$, then

$$\begin{aligned} y \in f(A \cap B) &\text{ iff } x \in A \cap B \\ &\text{ iff } x \in A \text{ and } x \in B \\ &\text{ iff } y \in f(A) \text{ and } y \in f(B) \\ &\text{ iff } y \in f(A) \cap f(B). \end{aligned}$$

(vi) True if f is one-to-one. For then, for any $x \in X$,

$$\begin{aligned} x \in f^{-1}(f(A)) &\Rightarrow f(x) \in f(A) \\ &\Rightarrow \exists z\,(f(z) = f(x) \text{ and } z \in A) \\ &\Rightarrow x \in A \qquad \text{since } f \text{ is one-to-one.} \end{aligned}$$

(vii) True if f is onto. For then, for any $y \in Y$,

$$\begin{aligned} y \in C &\Rightarrow \exists x\,(f(x) = y) && \text{since } f \text{ is onto} \\ &\Rightarrow \exists x\,(f(x) = y \text{ and } f(x) \in C) && \text{since } y \in C \\ &\Rightarrow \exists x\,(f(x) = y \text{ and } x \in f^{-1}(C)) \\ &\Rightarrow y \in f(f^{-1}(C)). \end{aligned}$$

5. There are a number of different correct ways to answer this question. We just give one method for each part.

(i) We first prove that $\mathbb{N} \times \mathbb{N}$ is countable, by defining a bijection between it and $\mathbb{N}$. (We will assume that $0 \in \mathbb{N}$.)

Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that

$$f(m, n) = 2^m(2n + 1) - 1.$$

First, we note that its range is included in $\mathbb{N}$, for if m is a non-negative integer, than 2^m is a positive integer; likewise if n is a non-negative integer, then $2n + 1$ is a positive integer. So $f(m, n) \in \mathbb{N}$.

Next, we argue that f is one-to-one. Suppose that $f(m, n) = f(m', n')$. Then $2^m(2n+1) = 2^{m'}(2n' + 1)$. If $m < m'$, then we can perform cancellation to show that $2n + 1 = 2^{m'-m}(2n'+1)$. But $2n+1$ is odd and $2^{m'-m}(2n'+1)$ is even, which is impossible. Likewise it is impossible that $m' < m$. So $m' = m$.

Now we cancel and see that $2n + 1 = 2n' + 1$. Then $n = n'$.

So $m = m'$ and $n = n'$, as required.

Finally we show that f is onto. Suppose k is any non-negative integer. Let m be the largest non-negative integer such that 2^m is a factor of $k + 1$; then if $k + 1 = 2^m q$, then q must be odd. Thus there is a non-negative integer n such that $q = 2n + 1$. So $k + 1 = 2^m(2n + 1)$, so $k = 2^m(2n + 1) - 1 = f(m, n)$.

The function f thus defined is the *Gödel pairing function*. There are of course many other bijections between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ that one can define.

Now suppose that A and B are countably infinite sets. Suppose $g : \mathbb{N} \to A$ and $h : \mathbb{N} \to B$ are both bijections. Then we define a bijection between $A \times B$ and $\mathbb{N}$ thus: define a function F so that

$$F(g(m), h(n)) = f(m, n).$$

Then F is clearly a bijection, and $A \times B$ is countable.

We can modify this argument to deal with the case when A or B is finite.

(ii) We show that $\mathbb{Z}$ is countable, using the previous part, by exhibiting a bijection between $\{0, 1\} \times \mathbb{N}$ and $\mathbb{Z}$. Define $f : \{0, 1\} \times \mathbb{N} \to \mathbb{Z}$ so that

$$f(0, n) = n$$

for all $n \in \mathbb{N}$, and

$$f(1, n) = -1 - n$$

for all $n \in \mathbb{N}$.

We show that $\mathbb{Q}$ is countable, again using part (i), by exhibiting a one-to-one function from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{N}$. Define $g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ as follows. If $q \in \mathbb{Q}$, then express q in lowest terms; that is, write it as m/n, where $m \in \mathbb{Z}$, $n \in \mathbb{N} \setminus \{0\}$, and m and n have no common factors. Let $f(q) = (m, n)$.

(iii) There is a slick way to answer this question using factorisation of natural numbers into primes. The method described here is a bit more sophisticated but uses less knowledge of algebra.

First, we define, by induction on n, a one-to-one function g_n from $\mathbb{N}^n$ to $\mathbb{N}$.

Let f be the pairing function defined in part (i).

There is only one element of $\mathbb{N}^0$, so let g_0 be the function mapping that element to 0.

Let $g_1 : \mathbb{N} \to \mathbb{N}$ be the identity.

If $k \geq 1$ and $n = k + 1$, define

$$g_n(m_1, m_2, \ldots, m_n) = f\big(g_k(m_1, \ldots, m_k), m_n\big).$$

Next, for each n, we define a one-to-one function from $\mathbb{N}^{[n]}$ to $\mathbb{N}$, where $\mathbb{N}^{[n]}$ is the set of n-element subsets of $\mathbb{N}$.

Suppose $A \in \mathbb{N}^{[n]}$. Write out the elements of A in increasing order of size, as $a_1 < a_2 < \cdots < a_n$. Then define $h_n(A)$ to be the ordered n-tuple $(a_1, a_2, \ldots, a_n)$.

Finally, let $\mathbb{N}^{<\omega}$ be the set of all finite subsets of $\mathbb{N}$, and define a one-to-one function F from it to $\mathbb{N}$ as follows.

Suppose A is a finite subset of $\mathbb{N}$. Suppose A has n elements.

Then define $F(A)$ to be $f(n, h_n(g_n(A)))$.

(iv) Suppose that $h : \mathbb{N} \to \wp\mathbb{N}$ is a bijection. Consider the set

$$E = \{n \in \mathbb{N} : n \notin h(n)\}.$$

Since h is onto, there exists $m \in \mathbb{N}$ such that $E = h(m)$. But then, if $m \in E$, then $m \in h(m)$ and so $m \notin E$, while if $m \in E$, then $m \notin h(m)$ so $m \notin E$. So we have a contradiction.