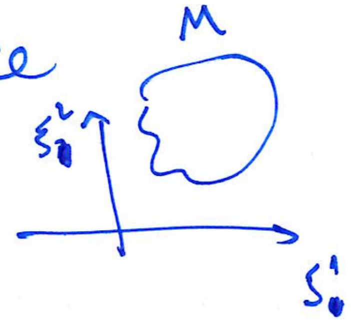
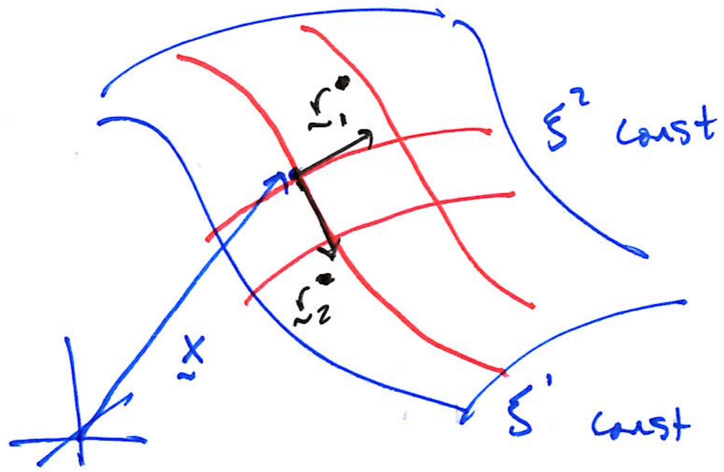
Geometry of Surfaces

Let Σ be an orientable surface





w/ parameterisation $\underset{\sim}{x}(\xi^1, \xi^2) \in \mathbb{R}^3$,
$(\xi^1, \xi^2) \in M \subset \mathbb{R}^2$

- We assume $\underset{\sim}{x}$ is at least C^2 & such that $\underset{\sim}{r}_i := \dfrac{\partial \underset{\sim}{x}}{\partial \xi^i}$ are lin. indep. $\forall\, (\xi^1, \xi^2) \in M$. Can define a unit normal $\underset{\sim}{n} = \dfrac{\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2}{\|\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2\|}$ and $\{\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{n}\}$ forms a basis.

Surface area $\quad A = \displaystyle\int_\Sigma dS \quad$ Recall (1^{st} yr): $d\underset{\sim}{S} = \underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2 \, d\xi^1 d\xi^2$, $dS = |d\underset{\sim}{S}|$

Using identity $(\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2)^2 = \underset{\sim}{r}_1^2 \underset{\sim}{r}_2^2 - (\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2)^2$, we have $dS = \sqrt{\underset{\sim}{r}_1^2 \underset{\sim}{r}_2^2 - (\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2)^2}\, d\xi^1 d\xi^2$

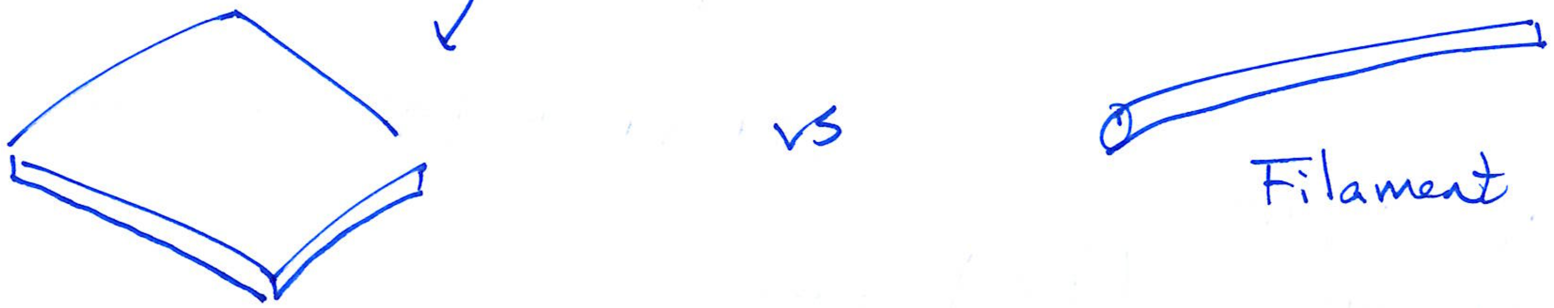
Defn Let $g_{ij} := \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j = \dfrac{\partial \underset{\sim}{x}}{\partial \xi^i} \cdot \dfrac{\partial \underset{\sim}{x}}{\partial \xi^j}$. This is metric tensor. Also define $G = (g_{ij})$ - the matrix of metric tensor

Then $dS = \sqrt{g_{11} g_{22} - g_{12}^2}\, d\xi^1 d\xi^2 = \sqrt{\det G}\, d\xi^1 d\xi^2$

and $A = \displaystyle\int_M \sqrt{\det G}\, d\xi^1 d\xi^2$

Biomembranes Overview

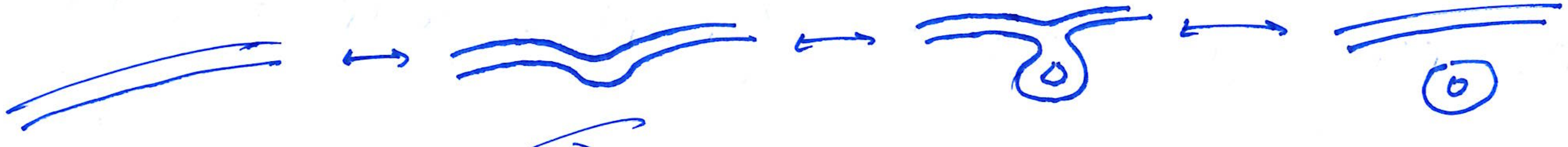
- Membrane - structure w/ one small dimension



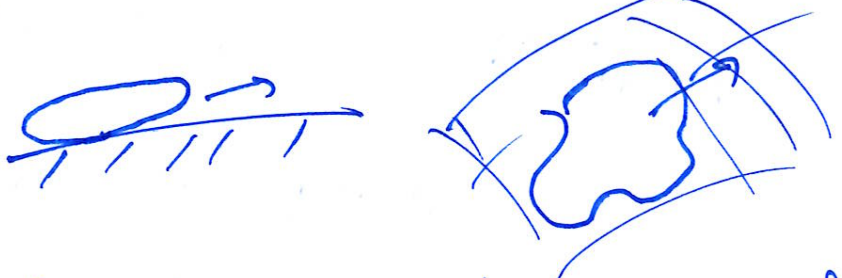
Motivation

- Micro : the diverse and fascinating behaviour of cells highly linked to mechanics of cell membrane

  Eg's
  - Permeability
  - Exo - endocytosis



  - Cell migration



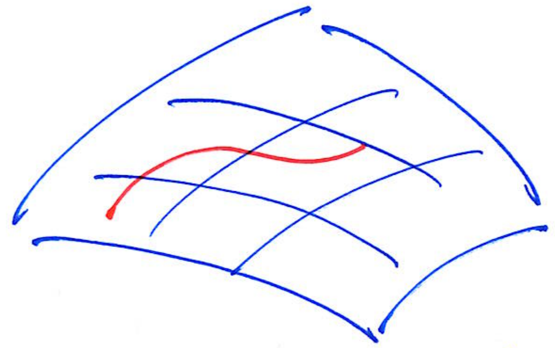
  ☆ Large shape changes ☆

- Macro
  - a sheet of cells → tissue mechanics, eg skin
  - in plants : leaves, petals Q: How do get shape?
    - Venus fly trap, rapid seed dispersal - rapid motion triggered by release of elastic energy

Plan :
- Geometry of surfaces → Membrane energy → Energy minimisation
- Alt: force, moment balance (for axisymmetric)

Surface $\underset{\sim}{x}(\xi^1, \xi^2)$, $\underset{\sim}{r}_i = \dfrac{\partial \underset{\sim}{x}}{\partial \xi^i}$, $\underset{\sim}{n} = \dfrac{|\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2|}{\|\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2\|}$, $g_{ij} = \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j$ metric tensor

Arclength



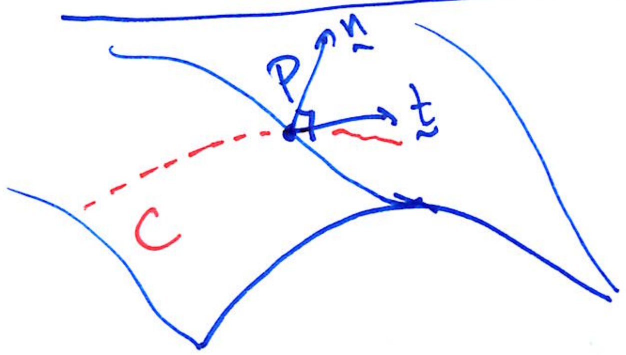
given a path on Σ, the infinitesimal length is $ds^2 = \left| \underset{\sim}{x}(\xi^1 + d\xi^1, \xi^2 + d\xi^2) - \underset{\sim}{x}(\xi^1, \xi^2) \right|^2$

$$\overset{\text{(expand)}}{=} |\underset{\sim}{r}_1 d\xi^1 + \underset{\sim}{r}_2 d\xi^2|^2 = \underset{\sim}{r}_1^2 (d\xi^1)^2 + \underset{\sim}{r}_2^2 (d\xi^2)^2 + 2 \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 \, d\xi^1 d\xi^2 = \boxed{g_{ij}\, d\xi^i d\xi^j}$$

(summation notation)

first fundamental form

So arclength $L = \displaystyle\int_a^b \sqrt{g_{ij} \frac{d\xi^i}{dt} \frac{d\xi^j}{dt}}\, dt$ is arclength of curve $\underset{\sim}{x}(\xi^1(t), \xi^2(t))$ for $t \in [a, b]$



Consider curve C on Σ passing through pt P and parameterised by arclength s. $\underset{\sim}{t}(s) := \dfrac{d\underset{\sim}{x}}{ds}$ is unit tangent.

Let $\underset{\sim}{\kappa} := \dfrac{d\underset{\sim}{t}}{ds}$. We know $\left|\dfrac{d\underset{\sim}{t}}{ds}\right| = |\kappa|$ ← curvature of C

Let $\underset{\sim}{\kappa} = -\kappa_n \underset{\sim}{n} + \underset{\sim}{\kappa}_g$ w/ $\underset{\sim}{\kappa}_g \cdot \underset{\sim}{n} = 0$

$\underbrace{\text{Normal curvature [sign so that sphere w/ outward normal has } \kappa_n > 0]}_{\text{Due to fact that surface is curved}}$

$\underbrace{|\underset{\sim}{\kappa}_g| = \kappa_g \text{ geodesic curvature}}_{\text{curve is curved relative to surface}}$

$$\underset{\sim}{t} = \frac{d\underset{\sim}{x}}{ds} = \frac{\partial \underset{\sim}{x}}{\partial \xi^i} \frac{d\xi^i}{ds} = \underset{\sim}{r}_i \frac{d\xi^i}{ds}$$

$$\Rightarrow \quad \underset{\sim}{K} = \frac{d\underset{\sim}{t}}{ds} = \frac{d}{ds}\left(\underset{\sim}{r}_i \frac{d\xi^i}{ds}\right) = \frac{\partial \underset{\sim}{r}_i}{\partial \xi^j} \frac{d\xi^j}{ds} \frac{d\xi^i}{ds} + \underset{\sim}{r}_i \frac{d^2\xi^i}{ds^2}$$

$\underset{\sim}{r}_i \cdot \underset{\sim}{n} = 0$

$$\text{So} \quad K_n = -\underset{\sim}{n}\cdot\underset{\sim}{K} = \underbrace{-\underset{\sim}{n}\cdot \frac{\partial \underset{\sim}{r}_i}{\partial \xi^j}}_{\parallel} \frac{d\xi^j}{ds} \frac{d\xi^i}{ds}$$

Define K_{ij} - symmetric

$$\frac{\partial \underset{\sim}{r}_i}{\partial \xi^j} = \frac{\partial^2 \underset{\sim}{x}}{\partial \xi^i \partial \xi^j} = \frac{\partial \underset{\sim}{r}_j}{\partial \xi^i}$$

so $K_{ij} = K_{ji}$

Then $K_n = K_{ij} \dfrac{d\xi^j}{ds} \dfrac{d\xi^i}{ds}$

K_{ij} is called <u>extrinsic curvature</u> tensor.

- $K_{ij}\, d\xi^j d\xi^i$ is <u>2nd fundamental form</u>

$g_{ij} = \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j$ metric tensor

$K_{ij} = -\underset{\sim}{n} \cdot \dfrac{\partial \underset{\sim}{r}_i}{\partial \xi_j}$ curv. tensor

$k_n = K_{ij} \dfrac{d\xi^i}{ds} \dfrac{d\xi^j}{ds}$ normal curvature

Extremal values of k_n

$$K_{ij}\, d\xi^i d\xi^j = k_n\, ds^2 = k_n g_{ij}\, d\xi^i d\xi^j$$

$$\Rightarrow \boxed{(K_{ij} - k_n g_{ij}) \frac{d\xi^i}{ds} \frac{d\xi^j}{ds} = 0}$$

g_{ij}, K_{ij} depend on surface only (& not on curve C), while k_n depends on C, such that $dk_n = 0$ at an extremal of varying the paths of curves through P.

$\therefore$ Differentiate wrt. $\dfrac{d\xi^p}{ds}$:

$$0 = (K_{ij} - g_{ij} k_n)\Big(\underbrace{\delta^i_p}_{=1 \text{ if } i=p} \frac{d\xi^j}{ds} + \frac{d\xi^i}{ds}\delta^j_p\Big) = \underbrace{(K_{pj} - g_{pj} k_n)\frac{d\xi^j}{ds} + (K_{ip} - g_{ip} k_n)\frac{d\xi^i}{ds}}_{\text{Same by symmetry of } K_{ij},\, g_{ij}}$$

$$= 2(K_{pj} - g_{pj} k_n)\frac{d\xi^j}{ds} \quad \therefore \quad 2(g^{\gamma p} K_{pj} - k_n g^{\gamma p} g_{pj})\frac{d\xi^j}{ds} = 0 \quad \text{where } g^{p\gamma} g_{pj} = \delta^\gamma_j$$

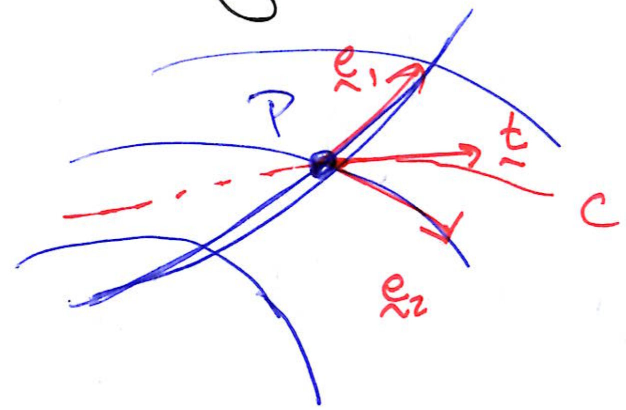
In matrix notation: $\boxed{\left(G^{-1}K - k_n \mathbb{1}\right)\dfrac{d\underset{\sim}{\xi}}{ds} = 0}$

$\therefore$ Extremal values of k_n eigenvalues of $L = G^{-1}K$

Let $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ denote the orthonormal eig'vectors of L

w/ associated eig'values $\underline{k_1, k_2}$

↑ principal curvatures



• can write $\underset{\sim}{t} = \cos\theta\, \underset{\sim}{e}_1 + \sin\theta\, \underset{\sim}{e}_2$ and (Euler 1760)

$$k_n = k_1 \cos^2\theta + k_2 \sin^2\theta$$

<u>Defn</u> $H = \dfrac{\mathrm{Tr}(L)}{2} = \dfrac{k_1 + k_2}{2}$ mean curvature

$K_G = \det L = k_1 k_2$ Gaussian curvature – intrinsic to surface

• $|H|$ is indep. of parameterisation, but H changes sign w/ dir. of normal vec.

<u>Classification</u> $K_G > 0$ Elliptic, $K_G < 0$ Hyperbolic, $K_G = 0$ Parabolic

<u>Gauss-Bonnet Thm</u> Let Σ be compact surface w/ b'dy $\partial\Sigma$. Then

$$\int_\Sigma K_G \, dS + \int_{\partial\Sigma} k_g \, ds = 2\pi \underline{\chi(\Sigma)}$$

Euler-characteristic $= 2-2p$ for surface of genus p
- sphere - genus 0
- torus genus 1

∴ For closed surface,

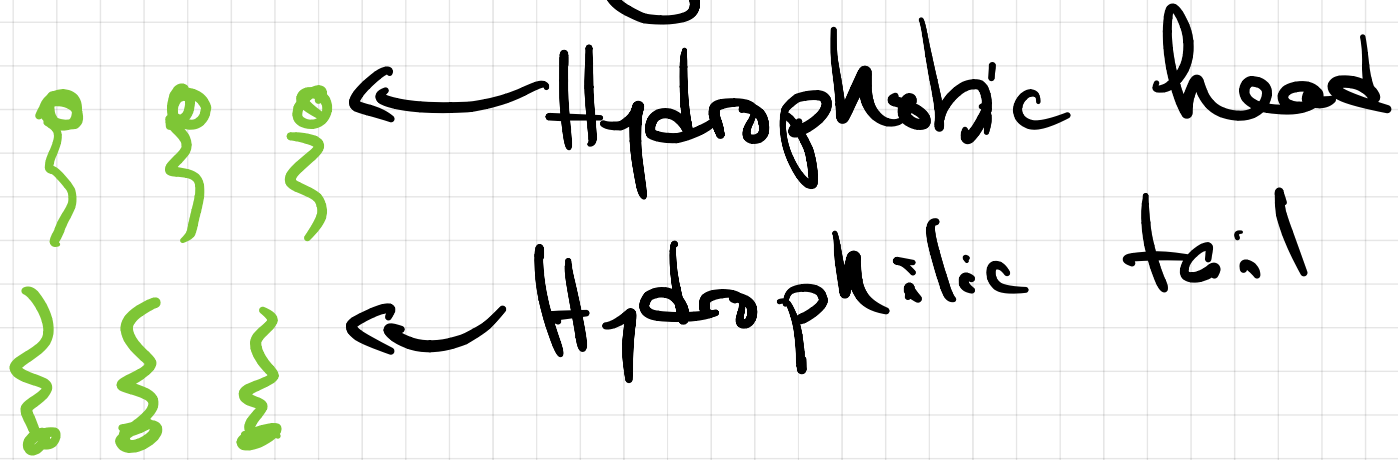
$$\int_\Sigma K_G \, dS = 4\pi \underbrace{(1-p)}_{\text{const.}}$$

Fluid biomembranes

Assumptions:

- sufficiently thin so that can be treated as surface Σ
- No resistance to shear, but resists bending & stretching

eg, lipid bilayer



- Free energy given by (Helfrich 1973):

$$\mathcal{E} = \int_{\Sigma} dS \left(\gamma + 2\kappa (H - H_0)^2 + \bar{\kappa} K_G \right)$$

surface tension

Bending modulus

intrinsic mean curv.

saddle-splay modulus

Note $2\kappa H^2 = \frac{\kappa}{2}(k_1 + k_2)^2 \Rightarrow$ if flat in one-dir.,

then $k_2 = 0 \Rightarrow K_G = 0 \rightarrow$ recover "beam energy"

<u>Goal</u> find shape that minimises $\mathcal{E}$, given ref. shape

- if extra constraints, add Lagrange multiplier
- eg. if fixed Volume - $V = V_0$, minimise $\mathcal{E} - P(V - V_0)$

Lagrange multiplier (pressure)

· For closed surface, Gauss-Bonnet $\Rightarrow$ can ignore κ_G

Estimates $\quad [H] = \frac{1}{\text{Length}} \Rightarrow [\kappa] = \text{Energy}$

$$[\gamma] = \frac{\text{Energy}}{\text{Length}^2} \Rightarrow \left(\frac{\kappa}{\gamma}\right)^{\frac{1}{2}} =: \lambda$$

characteristic length at which both bending & tension matter

If L is lengthscale of variation in membrane: $L \gg \lambda \Rightarrow$ surface tension dominant

$L \ll \lambda \Rightarrow$ bending dominant

eg lipid bilayer:

$$\kappa \approx 10^{-19}\,\text{J}, \quad \gamma \approx 10^{-3}\,\frac{\text{N}}{\text{m}} = 10^{-3}\,\frac{\text{J}}{\text{m}^2} \Rightarrow \lambda \approx 10^{-8}\,\text{m}$$

$\therefore$ For $L \gg 10\,\text{nm} \Rightarrow$ surface tension dominates

$\rightarrow$ will form approx. sphere

[Recipe: - parameterisation $\longrightarrow$ G, K $\rightarrow$ $L = G^{-1}K$

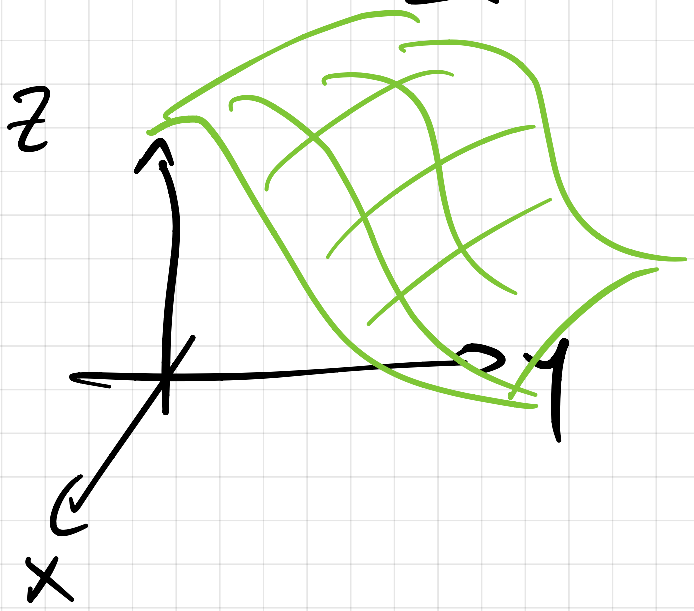
$\rightarrow K_1, K_2 \rightarrow H, K_G \rightarrow \mathcal{E} \rightarrow$ Euler - Lagrange

$\downarrow$

Shape]

<u>Monge Representation</u> - consider surface Σ as function $z = h(x,y) \in C^2$,

$(x,y) \in U \subset \mathbb{R}^2$



$$\underset{\sim}{r} = (x, y, h(x,y))$$

$$\rightarrow \underset{\sim}{r}_1 = \frac{\partial \underset{\sim}{r}}{\partial x} = (1, 0, h_x) \Longrightarrow \underset{\sim}{n} = -\frac{\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2}{\|\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2\|} = \frac{(h_x, h_y, -1)}{\sqrt{1 + h_x^2 + h_y^2}}$$

$$\underset{\sim}{r}_2 = (0, 1, h_y)$$

Metric tensor matrix

$$G = \begin{pmatrix} \underset{\sim}{r}_1^2 & \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 \\ \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 & \underset{\sim}{r}_2^2 \end{pmatrix} = \begin{pmatrix} 1+h_x^2 & h_x h_y \\ h_x h_y & 1+h_y^2 \end{pmatrix}$$

$$\Rightarrow \boxed{G^{-1} = \frac{1}{\det G} \begin{pmatrix} 1+h_y^2 & -h_x h_y \\ -h_x h_y & 1+h_x^2 \end{pmatrix}}, \quad \det G = 1 + h_x^2 + h_y^2$$

$$\underbrace{\det G}_{\text{call } g}$$

Now compute $K = (K_{ij})$, $K_{ij} = -\underset{\sim}{n} \cdot \frac{\partial \underset{\sim}{r}_i}{\partial x_j}$

$$\rightarrow \boxed{K = \frac{1}{\sqrt{g}} \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}}$$

Can now compute $L = G^{-1}K$, from which

we get $K_G = \det L = \det G^{-1} \det K = \dfrac{h_{xx}h_{yy} - h_{xy}^2}{g^2}$

& $H = \dfrac{\operatorname{tr}(L)}{2} = \dfrac{1}{2g^{3/2}}\left(h_{xx}(1+h_y^2) + h_{yy}(1+h_x)^2 - 2h_{xy}h_xh_y\right)$

Simple case – Surface tension only - set $\kappa = \kappa_G = 0$

γ const $\rightarrow$ $\mathcal{E} = \gamma \int_{\Sigma} dS$ (wants to minimise surface area)

$$\mathcal{E} = \gamma \int_M \sqrt{\det G}\, d\xi^1 d\xi^2 = \gamma \int \underbrace{\sqrt{1 + h_x^2 + h_y^2}}_{\text{call } \mathcal{L}(h, h_x, h_y)}\, dx\, dy$$

Calc. of variations - Euler Lagrange eqn:

$$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial h_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial h_y}\right) - \frac{\partial \mathcal{L}}{\partial h} = 0$$

$$\rightarrow \frac{\partial}{\partial x}\left(\frac{h_x}{\sqrt{1+h_x^2+h_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{h_y}{\sqrt{\quad\cdot}}\right) = 0$$

Note: Same as $\nabla\cdot\left(\frac{\nabla h}{\sqrt{g}}\right) = 0$ $\quad \left(\nabla \equiv \underset{\sim}{e}_x \frac{\partial}{\partial x} + \underset{\sim}{e}_y \frac{\partial}{\partial y}\right)$

Same as $\boxed{H = 0}$ or equiv., $\nabla\cdot\underset{\sim}{n} = 0$

↑ surface w/ zero mean curvature

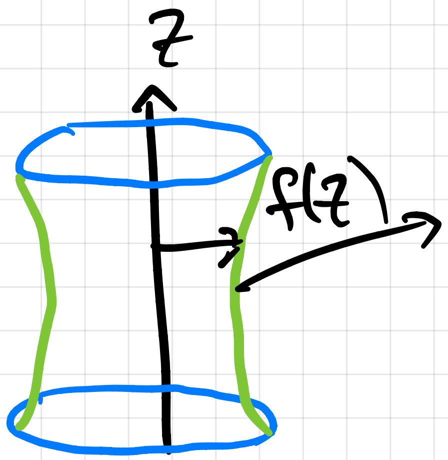
- called a <u>minimal surface</u> → locally minimizes surface at every pt.

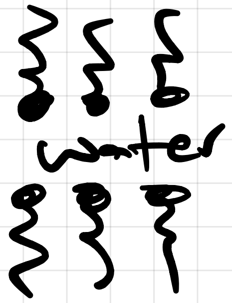
Note: $H = 0 \Rightarrow \kappa_1 = -\kappa_2 \Rightarrow K_G \leq 0$ (w/ equality only if $\kappa_1 = \kappa_2 = 0$ ie flat!)

$\Rightarrow$ minimal surfaces are saddle shaped

Ex. Soap film spanning any boundary

- consider soap film b/t two rings
- surface of revolution





Bilayer w/ $L \gg \lambda$

$$\underset{\sim}{X} = \begin{pmatrix} f(z)\cos\theta \\ f(z)\sin\theta \\ z \end{pmatrix}$$

Then $\underset{\sim}{r}_1 = \frac{\partial \underset{\sim}{x}}{\partial z} = \begin{pmatrix} f' \cos\theta \\ f' \sin\theta \\ 1 \end{pmatrix}$, $\underset{\sim}{r}_2 = \frac{\partial \underset{\sim}{x}}{\partial \theta} = \begin{pmatrix} -f\sin\theta \\ f\cos\theta \\ 0 \end{pmatrix}$

$\rightarrow G = \begin{pmatrix} 1+f'^2 & 0 \\ 0 & f^2 \end{pmatrix} \Rightarrow \mathcal{E} = \gamma \int_a^b \int_0^{2\pi} \sqrt{\det G} \, d\theta \, dz$

w/ $\sqrt{\det G} = f(z)\sqrt{1+f'^2} \overset{\text{call}}{=\!=} F(f, f')$

Euler-Lagrange $\longrightarrow$ $\frac{1+f'^2 - ff''}{(1+f'^2)^{3/2}} = 0$ [Same as $H = 0$]

Better : Beltrami identity $-\ \dfrac{\partial F}{\partial z} = 0$

$\rightarrow \quad F - f' \dfrac{\partial F}{\partial f'} = c$ (const) $\rightarrow$ 1st integral

$\rightarrow \quad \dfrac{f}{\sqrt{1+f'^2}} = c$ (can solve for f', separate and integrate

... $f = A \cosh(Bz)$)

Catenoid

Small Gradient Approx.

Keep $\bar{K} = H_0 = 0$ but w/ $K \neq 0$ and

Suppose $|h_x|, |h_y| \ll 1$

Then $\sqrt{g} = (1 + h_x^2 + h_y^2)^{\frac{1}{2}} \approx 1 + \frac{1}{2}(h_x^2 + h_y^2) + O(\nabla h^3)$

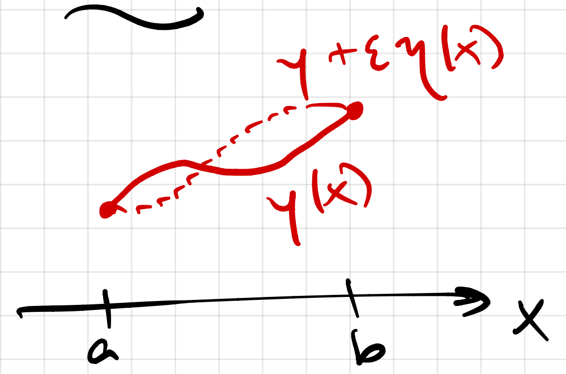
$$\Rightarrow (2H)^2 = \frac{1}{g^3}\left(h_{xx} + h_{yy} + O(\nabla h^3)\right)^2 = (h_{xx} + h_{yy})^2 \left(1 + O(\nabla h)^2\right)$$

$$\therefore \mathcal{E} = \int_\Sigma dS\,(\gamma + 2\kappa H^2) = \int_U dx\,dy\,\sqrt{g}\,(\gamma + 2\kappa H^2) = \int_U dx\,dy \left(1 + \tfrac{1}{2}(\nabla h)^2\right)\left(\gamma + 2\kappa(\nabla^2 h)^2\right) + \dots$$

$$= \int_U dx\,dy \left(\gamma + \tfrac{1}{2}\gamma(\nabla h)^2 + \tfrac{1}{2}\kappa(\nabla^2 h)^2\right) + \dots$$

const . so won't impact minimisation

Recall Calc. of variations $\quad \mathcal{E}[y(x)] = \int_a^b F(x, y(x), y'(x), \dots)\,dx$



$y(x)$ is an extremum of $\mathcal{E}$ if

$$\underbrace{\frac{d}{d\varepsilon}\mathcal{E}[y(x) + \varepsilon\eta(x)]}_{\text{Expand in } \varepsilon\text{, int. by parts}}\Bigg|_{\varepsilon \to 0} = 0 \quad \forall \eta(x)$$

Expand in ε, int. by parts $\longrightarrow$ $\int \Big(\underbrace{\qquad}_{\text{set } 0}\Big)\eta(x)\,dx + \overset{\text{also } 0}{BT} = 0$

Euler-Lagrange

Variation: $$\frac{d}{d\varepsilon}\mathcal{E}\left[h(x,y)+\varepsilon\eta(x,y)\right]\Big|_{\varepsilon=0} = \int_U dx\,dy\left(\gamma\underbrace{\nabla h\cdot\nabla\eta}_{=\,\nabla\cdot(\eta\nabla h)-\eta\nabla^2 h} + \kappa\underbrace{\nabla^2 h\,\nabla^2\eta}_{\nabla\cdot(\nabla\eta\nabla^2 h)\,-\,\nabla\eta\nabla\nabla^2 h}\right)$$

$$= \oint_{\partial U}\left(\gamma\eta\nabla h\cdot \underset{\sim}{N} + \kappa(\nabla\eta\nabla^2 h)\cdot \underset{\sim}{N}\right)ds - \int_U dx\,dy\left(\gamma\nabla^2 h\,\eta + \kappa\underbrace{\nabla\nabla^2 h\,\nabla\eta}_{=\,\nabla\cdot(\eta\nabla\nabla^2 h)\,-\,\eta\nabla^4 h}\right)$$

$$\boxed{= \oint_{\partial U}\Big[\eta\big(\underbrace{\gamma\nabla h - \kappa\nabla\nabla^2 h}_{A}\big) + \kappa\nabla\eta\underbrace{\nabla^2 h}_{B}\Big]\cdot \underset{\sim}{N}\,ds - \int_U dx\,dy\left(\gamma\nabla^2 h - \kappa\nabla^4 h\right)\eta} = 0$$

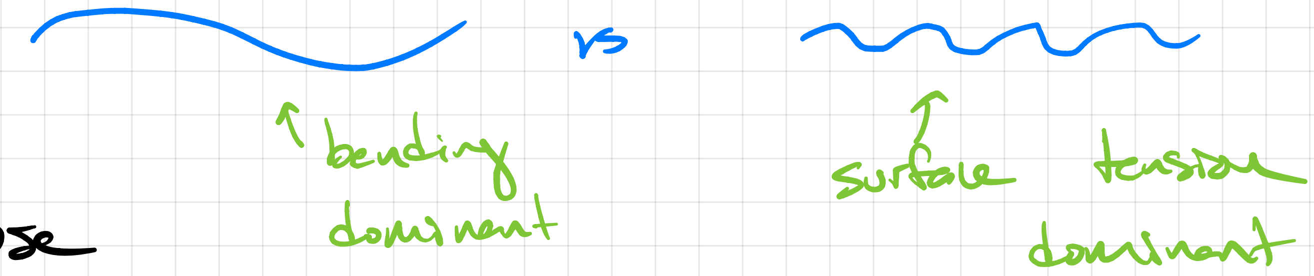
For extremum, we require $\boxed{\nabla^4 h - \frac{1}{\lambda^2}\nabla^2 h = 0}$, $\lambda := \sqrt{\frac{\kappa}{\gamma}}$

(Note: $\lambda \ll 1 \Rightarrow$ singular pert problem!)

Bdy cond: Need A and B to vanish.

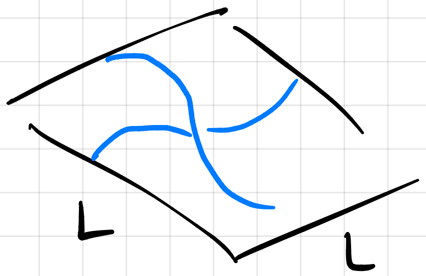
- If impose h at bdy $\Rightarrow$ must impose $\eta = 0$ at b'dy
  then A already 0, $\therefore$ require $\nabla^2 h = 0$ on b'dy
- If impose $\nabla h\cdot \underset{\sim}{N}$ on bdy $\Rightarrow$ must impose $\nabla\eta\cdot \underset{\sim}{N} = 0$ on b'dy
  so B already 0, require $\left(\nabla h - \frac{1}{\lambda^2}\nabla(\nabla^2 h)\right)\cdot \underset{\sim}{N} = 0$ on b'dy

Flicker spectroscopy: Study biomembrane mechanics by measuring spectrum of thermal undulations via light microscopy



- Small gradients — linear eqns, so can use Fourier methods

- Square membrane



seek $h(x,y) = h(\underline{r}) = \sum_{\underline{q}} h_{\underline{q}} e^{i\underline{q}\cdot\underline{r}}$

$\underline{q} = \frac{2\pi}{L}(n_x, n_y)$, $n_x, n_y \in \mathbb{Z}$ Fourier modes

$h_{\underline{q}} \in \mathbb{C}$ Fourier coeffs, h real $\rightarrow h_{-\underline{q}} = \overline{h_{\underline{q}}}$

Then $\nabla h = \sum_{\underline{q}} i\underline{q}\, h_{\underline{q}} e^{i\underline{q}\cdot\underline{r}} \Rightarrow (\nabla h)^2 = \sum_{\underline{q},\underline{q}'} -\underline{q}\cdot\underline{q}'\, h_{\underline{q}} h_{\underline{q}'} e^{i\underline{r}\cdot(\underline{q}+\underline{q}')}$

And $\nabla^2 h = \sum -\underline{q}^2 h_{\underline{q}} e^{i\underline{q}\cdot\underline{r}} \Rightarrow (\nabla^2 h)^2 = \sum_{\underline{q},\underline{q}'} \underline{q}^2 \underline{q}'^2 h_{\underline{q}} h_{\underline{q}'} e^{i\underline{r}\cdot(\underline{q}+\underline{q}')}$

$\therefore \mathcal{E} = \frac{1}{2}\int_0^L dx\, dy \sum_{\underline{q},\underline{q}'} h_{\underline{q}} h_{\underline{q}'} e^{i\underline{r}\cdot(\underline{q}+\underline{q}')} \left(\kappa \underline{q}^2 \underline{q}'^2 - \gamma \underline{q}\cdot\underline{q}'\right)$

Orthogonality: $\int_0^L dx\,dy\, e^{i\underline{r}\cdot(\underline{q}+\underline{q}')} = L^2 \delta_{0,\underline{q}+\underline{q}'}$

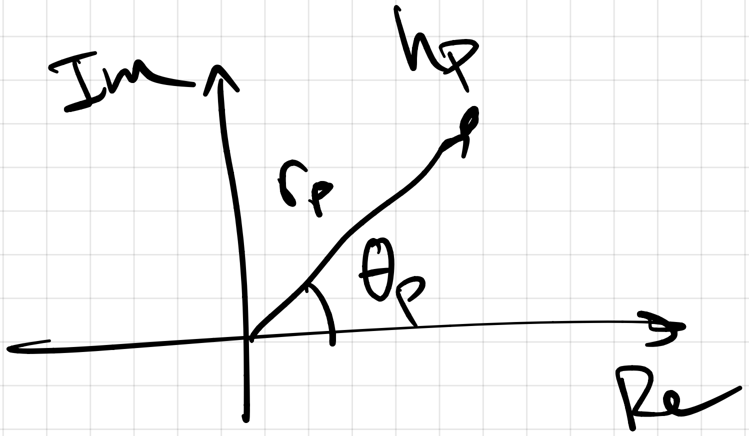
$$\Rightarrow \mathcal{E} = \frac{L^2}{2}\sum_{\underline{q}} h_{\underline{q}} h_{-\underline{q}} (\kappa q^4 - \gamma q^2) = \boxed{\frac{L^2}{2}\sum_{\underline{q}} |h_{\underline{q}}|^2 (\kappa q^4 - \gamma q^2)}$$

Goal: compute $\langle |h_p|^2 \rangle$ for a given mode $p = \frac{2\pi}{L}(p_x, p_y)$

state: $\{h_1, h_2, \dots\}$

$$= \int D[h]\, |h_p|^2\, p(\{h_1, h_2, \dots\})$$

$$p(\{h_1, h_2, \dots\}) = \frac{e^{-\beta \mathcal{E}(\{h_1, h_2 \dots\})}}{Z}, \quad \beta = \frac{1}{k_B T}$$



$h_p \in \mathbb{C}$, so can write $h_p = r_p e^{i\theta_p}$

$f(\underline{q}) := \kappa q^4 - \gamma q^2$

Then $$Z = \int \prod_m dr_m\, r_m\, d\theta_m\, e^{-\frac{\beta L^2}{2}\sum_{\underline{q}} r_{\underline{q}}^2 f(\underline{q})}$$

$$\langle |h_p|^2 \rangle = \frac{1}{Z} \underbrace{\int \prod_m dr_m\, d\theta_m\, r_m\, r_p^2\, e^{-\beta \frac{L^2}{2} \sum_{\underset{\sim}{q}} r_q^2 f(\underset{\sim}{q})}}$$

Key: has all same integrals as Z, except at $m=p$

$\therefore$ everything cancels w/ matching term in Z except:

$$\langle |h_p|^2 \rangle = \frac{\overbrace{\int_0^\infty dr_p\, r_p^3\, e^{-\beta \frac{L^2}{2} r_p^2 f(\underset{\sim}{p})}}^{N_p}}{\underbrace{\int_0^\infty dr_p\, r_p\, e^{-\beta \frac{L^2}{2} r_p^2 f(\underset{\sim}{p})}}_{\text{call } I_p}}$$

We observe

$$\frac{\partial}{\partial \beta}(I_p) = \frac{-1}{\frac{L^2}{2} f(\underset{\sim}{p})} \cdot N_p$$

$$\therefore \langle |h_p|^2 \rangle = \frac{-1}{\frac{L^2}{2} f(\underset{\sim}{p})} \frac{\frac{\partial}{\partial \beta}(I_p)}{I_p}$$

$$\frac{\partial}{\partial \beta} \ln I_p = \frac{-1}{\beta}$$

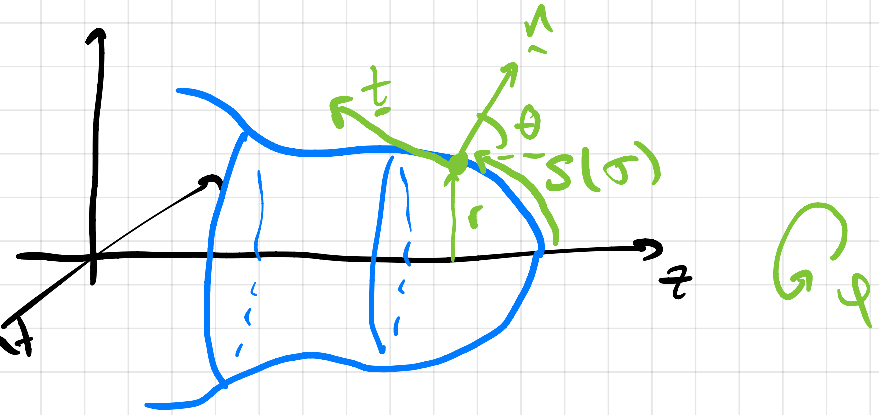
$$\therefore \boxed{\langle |h_p|^2 \rangle = \frac{2 k_b T}{L^2 (\kappa \hat{p}^4 - \gamma \underset{\sim}{p}^2)}}$$

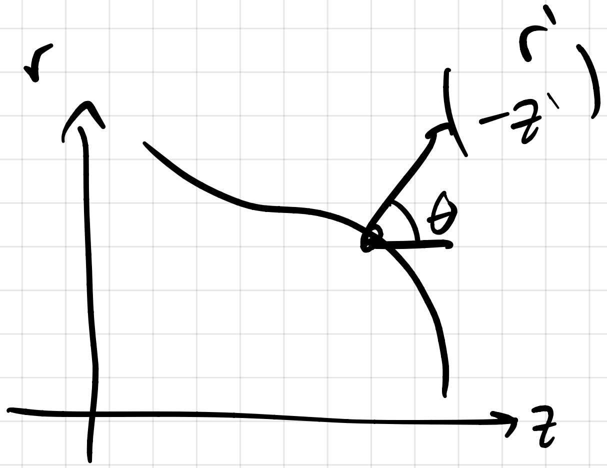
Axisymmetric Membranes & Shells

1. Kinematics - We take the membrane to be a surface of revolution

w/:

- $\underset{\sim}{n}$ unit normal
- $\underset{\sim}{t} = \underset{\sim}{e}_s$ unit tangent in dir. of increasing arclength s
- σ arclength before deformation - material parameter
- r radius, distance from z-axis

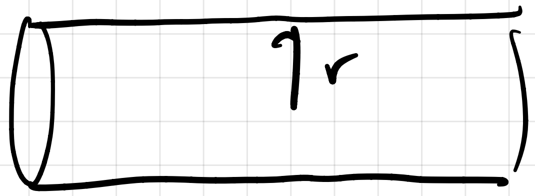




Trigonometry gives:

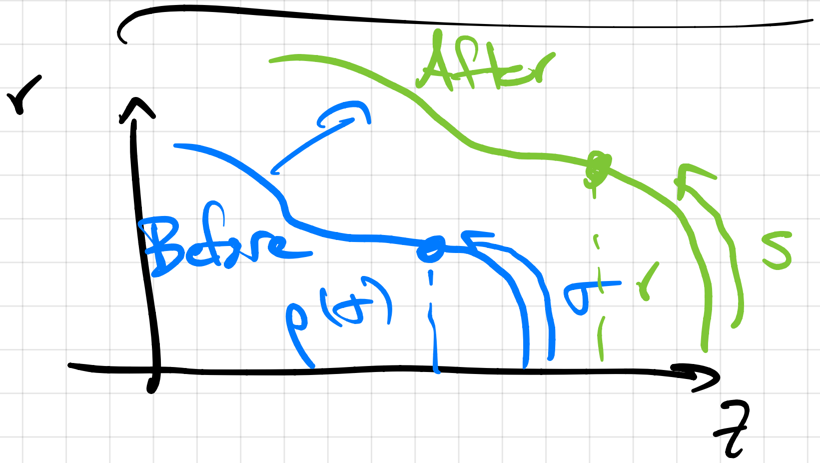
$$\frac{dr}{ds} = \cos\theta, \quad \frac{dz}{ds} = -\sin\theta$$

Principal curvatures:
$$\kappa_s = \frac{d\theta}{ds}, \quad \kappa_\varphi = \frac{\sin\theta}{r}$$



$\kappa_\varphi = \frac{1}{r}$

Stretch variables

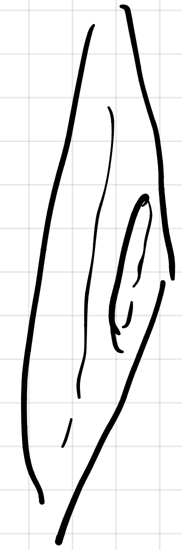


- stretch is s-direction

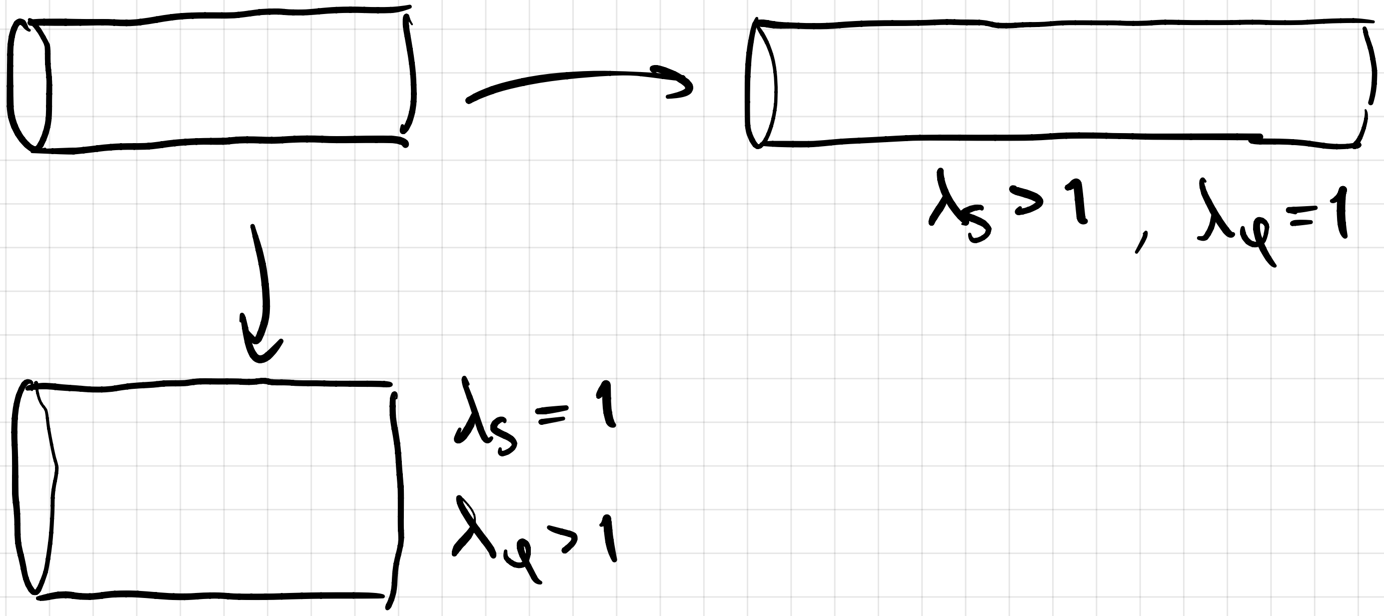
$$\lambda_s = \frac{\partial s}{\partial \sigma}$$

- radial stretch $\lambda_\varphi = \frac{r}{\rho}$

$\theta \approx 0$



$\kappa_\varphi \approx 0$



Mechanics - Consider an element of membrane associated w/ section $[s, s+\Delta s]$, $[\varphi, \varphi+\Delta\varphi]$ subject to:

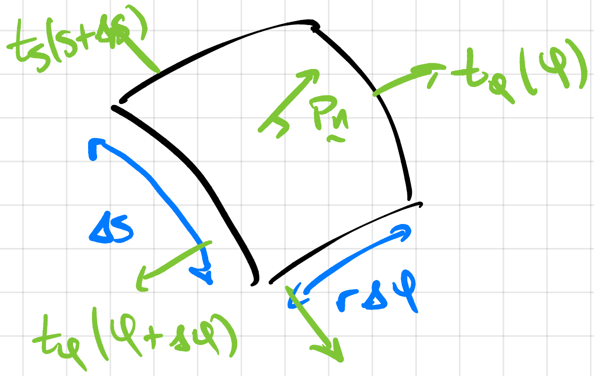
i) Pressure, $P\underset{\sim}{n}$, eg due to pressurized internal fluid

ii) Tension on surface $\underset{\sim}{T} = t_s \underset{\sim}{e}_s + t_\varphi \underset{\sim}{e}_\varphi$ unit vector in direction of increasing azimuthal angle φ

iii) A force per unit area $\underset{\sim}{F} = f \underset{\sim}{e}_s$ to maintain axisymmetry

eg due to external fluid movement

Zoom in on surface element:



(Goal: express ito $\underset{\sim}{e}_s$, $\underset{\sim}{n}$ components)

Force balance (ignore inertia)

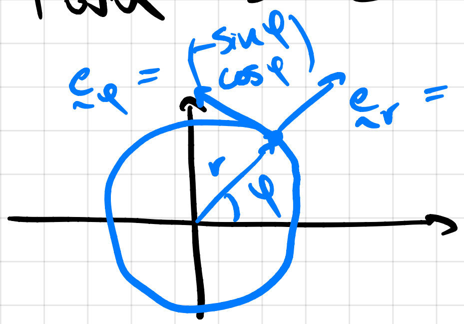
$$\Delta s\Delta\varphi\left[\frac{\partial}{\partial s}\left(r t_s \underset{\sim}{e}_s\right)+\frac{\partial}{\partial\varphi}\left(t_\varphi \underset{\sim}{e}_\varphi\right)+rP\underset{\sim}{n}+rf\underset{\sim}{e}_s\right]=\underset{\sim}{0}$$

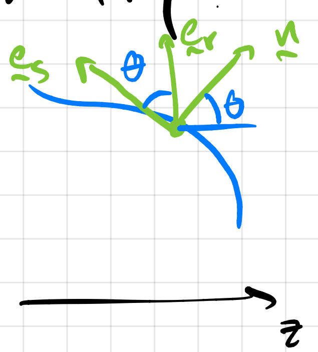
(neglecting higher order terms)

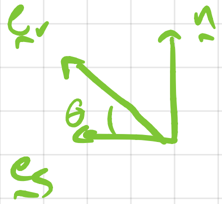
you'd have $r(s+\Delta s)t_s(s+\Delta s)\underset{\sim}{e}_s(s+\Delta s)\Delta\varphi - r(s)t_s(s)\underset{\sim}{e}_s(s)\Delta\varphi$

Turn back to geometry:

$\underset{\sim}{e}_\varphi=\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$ $\quad$ $\underset{\sim}{e}_r=\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$







$\underset{\sim}{e}_r=\cos\theta\,\underset{\sim}{e}_s+\sin\theta\,\underset{\sim}{n}$

$$\Rightarrow\quad \frac{\partial \underset{\sim}{e}_\varphi}{\partial\varphi}=-\underset{\sim}{e}_r=-\cos\theta\,\underset{\sim}{e}_s-\sin\theta\,\underset{\sim}{n}$$

Also require $\frac{\partial t_\varphi}{\partial\varphi}=0$ for axisymm.

Also, recall $\underset{\sim}{e}_s=\underset{\sim}{t}$, and so

$$\frac{\partial \underset{\sim}{e}_s}{\partial s}=\frac{\partial \underset{\sim}{t}}{\partial s}=-\kappa_s\underset{\sim}{n} \quad \text{by defn of curvature}$$

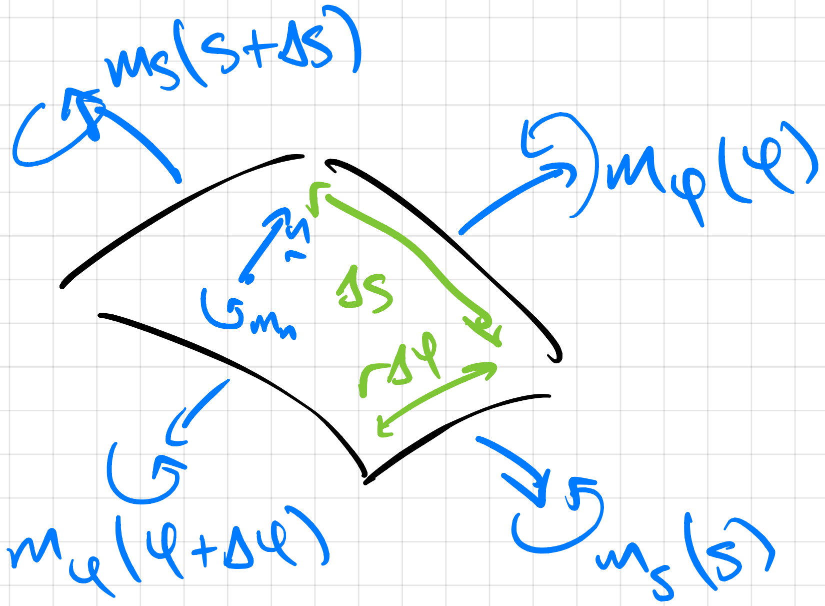
Observe: if $t_s=t_\varphi=t$, $\boxed{P=2tH}$

$\therefore$ Force balance becomes:

$$\underset{\sim}{0}=\underset{\sim}{e}_s\left[\frac{\partial}{\partial s}(rt_s)-t_\varphi\cos\theta+rf\right]+\underset{\sim}{n}\left[-rt_s\kappa_s-t_\varphi\sin\theta+rP\right]$$

$$\therefore\ \boxed{\begin{aligned}P&=t_s\kappa_s+t_\varphi\frac{\sin\theta}{r}=t_s\kappa_s+t_\varphi\kappa_\varphi\\ \frac{\partial}{\partial s}(rt_s)&=t_\varphi\cos\theta-rf=t_\varphi\frac{\partial r}{\partial s}-rf\end{aligned}}$$

Moment balance



Balancing moments on section gives:

$$\Delta s \Delta \varphi \left[\frac{\partial}{\partial s}(r m_s \underset{\sim}{e}_s) + \frac{\partial}{\partial \varphi}(m_\varphi \underset{\sim}{e}_\varphi) + m_n \underset{\sim}{n} \right] = \underset{\sim}{0}$$

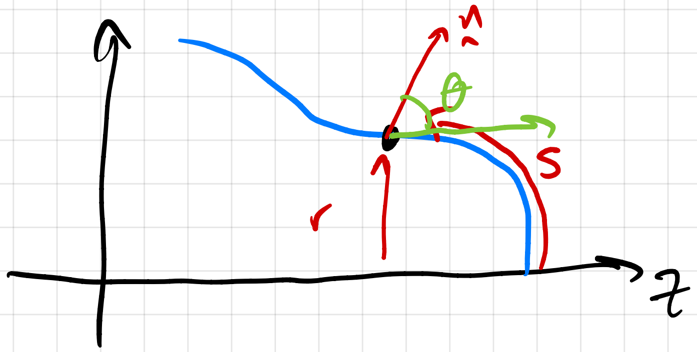
Again, use $\frac{\partial}{\partial \varphi} \underset{\sim}{e}_\varphi = -\cos\theta \, \underset{\sim}{e}_s - \sin\theta \, \underset{\sim}{n}$

$$\rightarrow \boxed{\frac{\partial}{\partial s}(r m_s) - m_\varphi \cos\theta = 0} \quad \underset{\sim}{e}_s \text{ comp.}$$

$\underset{\sim}{n}$ component gives m_n in terms of m_s, m_φ, but <u>not</u> needed, because m_n not imposed.

Constant Pressure Case $(f=0)$

Claim $P = \text{const} \Rightarrow r^2(2t_s\kappa_\varphi - P) = \text{const}$



Force balance: (i) $P = t_s\kappa_s + t_\varphi\kappa_\varphi$

$$= t_s\theta' + t_\varphi \frac{\sin\theta}{r}$$

ii) $(r t_s)' = t_\varphi r'$, $(r' = \cos\theta)$

Pf $(r^2P)' = 2rr'P \overset{(i)}{=} 2rr't_s\kappa_s + 2rr't_\varphi\kappa_\varphi = (2rt_s\sin\theta)'$

$\cos\theta\,\theta'$ $\qquad \sin\theta \qquad \overset{\|(ii)}{(rt_s)'}$

$$= 2(r^2\kappa_\varphi t_s)'$$

$\therefore\; r^2P - 2r^2\kappa_\varphi t_s = \text{const}$ ✓

Observe: if surface crosses z-axis, so $r=0$ at a point,

then $\text{const}=0 \Rightarrow \boxed{P = 2\kappa_\varphi t_s}$ (Young - Laplace law)

Constitutive laws. We need to relate t_s and t_φ to the deformation of the membrane, defined by the stretch ratios $\lambda_s = \frac{ds}{d\sigma}$, $\lambda_\varphi = \frac{r}{\rho}$

[there is a 3rd stretch, in normal direction, λ_3

. for incompressible material (volume doesn't change),

$$\lambda_3 = \frac{1}{\lambda_s \lambda_\varphi} \quad]$$

General form: $t_s = A f_s(\lambda_s, \lambda_\varphi)$, $t_\varphi = A f_\varphi(\lambda_s, \lambda_\varphi)$ such that $f_s(1,1) = 0$
$f_\varphi(1,1) = 0$

Standard moment constit relation

$$m_s = m_\varphi = B(\kappa_s + \kappa_\varphi - \kappa_0)$$

isotropic

sum of curvatures in stress-free config.

Recall: moment balance: $\frac{d}{ds}(r m_s) - \underset{m_s}{\underset{\shortparallel}{m_\varphi}} \underset{\frac{dr}{ds}}{\underset{\shortparallel}{\cos\theta}} = 0$

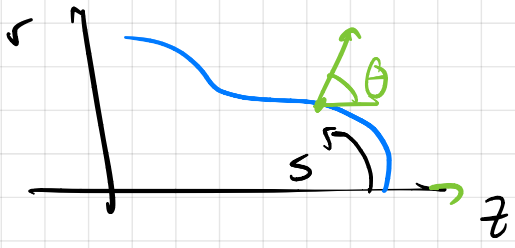
$$\rightarrow \frac{dm_s}{ds} = 0 \quad \Rightarrow \frac{d}{ds}\kappa_s + \frac{d}{ds}\kappa_\varphi = 0 \quad (\text{if } \kappa_0 = \text{const})$$

$$\Rightarrow \kappa_s + \kappa_\varphi = \kappa_1 \quad \text{const}$$

$$\text{ie} \quad \boxed{\kappa_s = \kappa_1 - \frac{\sin\theta}{r}}$$

Workspace

- Geometry



$r' = \cos\theta$

$z' = -\sin\theta$

$' = \frac{d}{ds}$

$\kappa_s = \theta'$, $\kappa_\varphi = \frac{\sin\theta}{r}$

- MB (isotropic) $\frac{dm_s}{ds} = 0$ • CL $m_s = B(\kappa_s + \kappa_\varphi - \kappa_0)$

- FB $P = t_s \kappa_s + t_\varphi \kappa_\varphi$ • CL : $t_s = A f_s(\lambda_s, \frac{r}{\rho})$

$t_\varphi = A f_\varphi(\lambda_s, \frac{r}{\rho})$

$\underbrace{(r t_s)' = r' t_\varphi}$

$\downarrow$

$t_s' = \frac{r'}{r}(t_\varphi - t_s)$

Closed System

- $\frac{ds}{d\sigma} = \lambda_s$
- $\frac{dr}{d\sigma} = \lambda_s \cos\theta$
- $\frac{dz}{d\sigma} = -\lambda_s \sin\theta$
- $\frac{dm_s}{d\sigma} = 0$
- $\frac{d\theta}{d\sigma} = \lambda_s \kappa_s$
- $\frac{dt_s}{d\sigma} = \frac{A\cos\theta}{r}\left(f_\varphi(\lambda_s, \frac{r}{\rho}) - f_s(\lambda_s, \frac{r}{\rho})\right)$
- $P = t_s \kappa_s + A f_\varphi(\lambda_s, \frac{r}{\rho}) \frac{\sin\theta}{r}$
- $m_s = B\left(\kappa_s + \frac{\sin\theta}{r} - \kappa_0\right)$

6 ODEs + 2 Algebraic for

$\{s, r, z, m_s, \theta, t_s, \lambda_s, \kappa_s\}$ functions of σ

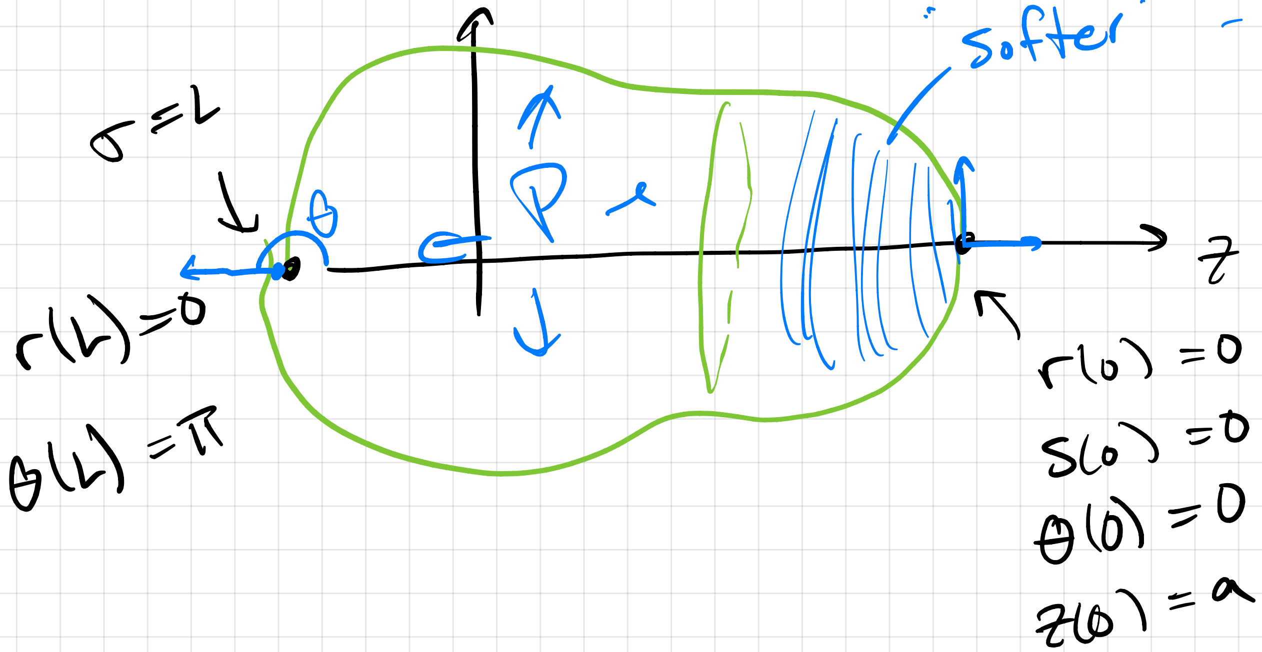
- Notes : - f_s, f_φ known fns
  - B, A, K_0 all material properties, assumed known

  $= K_{s_0} + K_{\varphi_0}$ could be worked out from $\{ \rho(\sigma), z_0(\sigma) \}$

- Typical problem : impose P (plus BC), determine deformed shape

Ex. Pressurised cell w/ variable stiffness — $A = A(\sigma)$

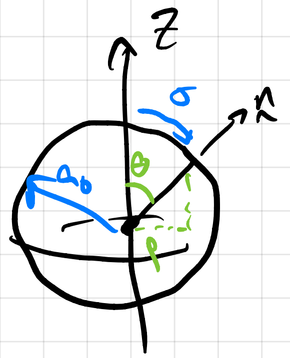
"softer" — $A(\sigma)$ smaller



Ex. - Inflation of a Sphere. Consider a sphere of radius a_0 in undeformed state. Suppose the pressure is increased to P^*. Find the deformed radius a.

Initial

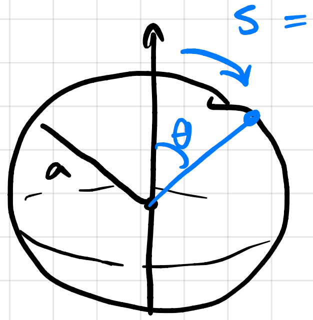
$$(\lambda_s = \lambda_\varphi = 1)$$



$\sigma = a_0\theta$ Geometry: $\left(' = \frac{d}{d\sigma} \right)$

$$\left.\begin{aligned} z_0' &= -\sin\theta = -\sin\left(\frac{\sigma}{a_0}\right) \\ \rho' &= \cos\theta \\ z_0(0) &= a_0 \\ \rho(0) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} z_0 &= a_0 \cos\left(\frac{\sigma}{a_0}\right) \\ \rho &= a_0 \sin\left(\frac{\sigma}{a_0}\right) \end{aligned}$$

Deformed



$$s = a\theta = \frac{a}{a_0}\sigma \Rightarrow \lambda_s = \frac{ds}{d\sigma} = \frac{a}{a_0}$$

$$\left.\begin{aligned} z' &= -\lambda_s \sin\theta \\ r' &= \lambda_s \cos\theta \end{aligned}\right\} \rightarrow \begin{aligned} z(\sigma) &= a\cos\left(\frac{\sigma}{a_0}\right) \\ r(\sigma) &= a\sin\left(\frac{\sigma}{a_0}\right) \end{aligned}$$

$$z(0) = a$$
$$r(0) = 0$$

$$\theta = \frac{\sigma}{a_0} \Rightarrow \frac{d\theta}{d\sigma} = \frac{1}{a_0} = \lambda_s \kappa_s = \frac{a}{a_0} \cdot \frac{1}{a} \checkmark$$

Suppose $f_s = f_\varphi = A(\lambda_s^2 + \lambda_\varphi^2 - 2)$

symmetry

$$\lambda_s^2 = \left(\frac{a}{a_0}\right)^2, \quad \lambda_\varphi^2 = \left(\frac{a}{a_0}\right)^2$$

$$\Rightarrow t_s = t_\varphi = 2A\left(\frac{a^2}{a_0^2} - 1\right)$$

$$\beta := \frac{P^* a_0}{4A}$$

$$P^* = t_s\kappa_s + t_\varphi\kappa_\varphi = 4A\left(\frac{a^2}{a_0^2} - 1\right)\cdot\frac{1}{a} \Rightarrow \boxed{a = \frac{a_0}{2}\left(\beta + \sqrt{\beta^2 + 4}\right)}$$

Mechanics

$$m_s = B(\kappa_s + \kappa_\varphi - \kappa_0)$$

$$\kappa_{s_0} = \frac{1}{a_0} = \kappa_{\varphi_0} \Rightarrow \kappa_0 = \frac{2}{a_0}$$

$$\kappa_s = \kappa_\varphi = \frac{1}{a}$$

$$\Rightarrow m_s = 2B\left(\frac{1}{a} - \frac{1}{a_0}\right)$$

satisfies $m_s' = 0$